CYCLES IN LEAVITT PATH ALGEBRAS BY MEANS OF IDEMPOTENTS

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Abstract. We characterize, in terms of its idempotents, the Leavitt path algebras of an arbitrary graph that satisfies Condition (L) or Condition (NE). In the latter case, we also provide the structure of such algebras. Dual graph techniques are considered and demonstrated to be useful in the approach of the study of Leavitt path algebras of arbitrary graphs. A refining of the so-called Reduction Theorem is achieved and is used to prove that $I(P_{\ell}(E))$, the ideal of the vertices which are base of cycles without exits of the graph $E$, a construction with a clear parallelism to the socle, is a ring isomorphism invariant for arbitrary Leavitt path algebras. We also determine its structure in any case.

Introduction

Leavitt path algebras $L_K(E)$ of row-finite graphs were recently introduced in [2] and [10]. They have become a subject of significant interest, both for algebraists and for analysts working in $C^*$-algebras. The Cuntz-Krieger algebras $C^*(E)$ (the $C^*$-algebra counterpart of these Leavitt path algebras) are described in [26]. For a field $K$, the algebras $L_K(E)$ are generalizations of the algebras investigated by Leavitt in [25], and are generated by the quotients of the so-called (CK1) and (CK2) relations applied to path $K$-algebras associated to graphs $E$. Moreover, as established in [28], $L_K(E)$ is always an algebra of right quotients of $KE$. The family of algebras that can be obtained as the Leavitt path algebras of some graph includes, but is by no means limited to, matrix rings $M_n(K)$ for $n \in \mathbb{N} \cup \{\infty\}$ (where $M_\infty(K)$ denotes the ring of matrices of countable size with only a finite number of nonzero entries), the Laurent polynomial ring $K[x,x^{-1}]$, the algebraic Toeplitz algebra and the classical Leavitt algebras $L(1,n)$ for $n \geq 2$. Constructions such as direct sums, direct limits and matrices over the previous examples can also be realized in this setting.

Since Leavitt path algebras are constructed from graphs, it is natural to try to understand how the properties of the graph $E$ restrict and shape that of $L_K(E)$. In this approach, maybe the first noticeable restrictions are those related to the cardinality of the graph. In fact, the development of the theory of Leavitt path algebras (as well as that of graph $C^*$-algebras) has had several different stages marked by the cardinality of the graphs studied in them. At first, in the $C^*(E)$ case, only matrices (represented by finite graphs) were considered ([20]). The second step was to consider possibly infinite but countable row-finite graphs (that is, graphs with a countable number of vertices and edges such that every vertex emits a finite number of edges). As has already been mentioned, the study of Leavitt path algebras started directly at this point. The following breakthrough was to remove the hypothesis of row-finiteness in the underlying graphs. This was first done for Leavitt path algebras in [4] and [30].

Key words and phrases. Leavitt path algebra, graph $C^*$-algebra, Condition (L), Condition (NE), primitive idempotent, infinite idempotent, dual graph.
is often the case that the row-finite results are no longer valid for not necessarily row-finite graphs, and when they are, they may come up from totally different proofs, because the existence of infinite emitters (vertices that emit an infinite number of edges) disrupts the application of the (CK2) condition, fundamental to many established proofs, and therefore it causes new phenomena and forces the necessity of finding new tools to circumvent either the application of (CK2) or the appearance of infinite emitters. For example, in [4] it is shown that from a row-infinite countable graph $E$ one can construct a row-finite countable graph $F$, a desingularization of $E$, in such a way that $L_K(E)$ and $L_K(F)$ are Morita equivalent and there is a monomorphism of $K$-algebras from $L_K(E)$ to $L_K(F)$. Finally and very recently, Leavitt path algebras have entered their final stage in terms of cardinality restrictions: by dropping also the countability assumption, arbitrary graphs are now the subject of study, a road started in [22] and [8].

But cardinality is by no means the only graph property relevant to the study of Leavitt path algebras. Because of the handy pictorial representation that the graph provides, a great deal of effort has been focused on trying to figure out the algebraic structure of $L_K(E)$ in terms of the graphical nature of $E$. Concretely, necessary and sufficient conditions on a graph $E$ have been given so that the corresponding Leavitt path algebra $L_K(E)$ is simple [2], purely infinite simple [3], exchange [14], finite dimensional [6], locally finite (equivalently noetherian) [7], semisimple [5], prime or primitive [15] and von Neumann regular (equivalently $\pi$-regular) [8]. Reciprocally, there is some interest on finding ring theoretic characterizations for the Leavitt path algebras of graphs that satisfy properties that are recognizable just by visual inspection, since this implies that if $L_K(E) \cong L_K(F)$ as rings for two graphs $E$ and $F$, then those graph features are to be satisfied by either or none of the graphs. For example, as was established recently, acyclic graphs are precisely those whose Leavitt path algebras are von Neumann regular rings ([8]); also, graphs whose closed simple paths are never found alone, a graph property known as Condition (K) (formulated in [23]), were characterized in the row-finite [14] and the general case [22] as those whose Leavitt path algebras are exchange rings. In this paper we are interested on two graph properties known as Condition (NE) and Condition (L). The first of them asks for all the cycles of the graph to have no exits, while the second one, in full contrast, demands that every cycle has an exit (an exit for a cycle being an edge that allows us to get “untrapped” from the cycle). Both conditions showed up, jointly with other graph properties, in the characterization of locally noetherian [7] and simple Leavitt path algebras [2], respectively; and necessary and sufficient conditions on $L_K(E)$ in order for Condition (L) to be satisfied by $E$ are known too, but they involve some relation between vertices and ideals (e.g., [15, Proposition 2.8 (ii)]). We present ring theoretic characterizations of both conditions for arbitrary graphs, in terms of idempotents. Concretely, in Theorem 3.3 we establish that $E$ satisfies Condition (NE) if and only if $L_K(E)$ does not present infinite idempotents (and identify the algebraic structure of $L_K(E)$), while in Theorem 4.10 we show that $E$ satisfies Condition (L) if and only if $L_K(E)$ has no truly primitive idempotents (idempotents which are primitive but not minimal).

Conditions (NE) and (L) can be seen as two particular aspects of a more general setting. Consider the set of vertices which are in cycles without exits, $P_c(E)$, and the ideal it generates in $L_K(E)$, $I(P_c(E))$. Then it is to be expected that if $E$ satisfies Condition (NE), the main algebraic features of $L_K(E)$ will be comprised in this ideal, while it is easy to see that $E$ satisfies Condition (L) if and only if $I(P_c(E)) = 0$. So, we study this ideal in Section 5 with
To achieve our main results we develop two different tools: Condition (L) is determined thanks to a study of primitive idempotents, Condition (NE) by dual graph techniques. $I(P_c(E))$ is studied by combining both and by refining, in Theorem 5.7, a useful result about Leavitt path algebras.

In Section 4 we establish, in Proposition 4.3, that the primitive vertices of any Leavitt path algebra are precisely those whose tree does not contain any bifurcations. This contrasts with the path algebras setting even in the finite context, where any vertex is automatically primitive ([19, page 4, (7)]). Since minimal vertices are known to be those whose tree does not contain any bifurcations or cycles without exits, truly primitive vertices allow us to detect the presence of cycles without exits in the Leavitt path algebra.

In Section 2 we generalize some definitions and ideas of I. Raeburn and W. Szymański, presented in [27] for $C^*$-algebras, by following some other ideas introduced by G. Abrams and K. M. Rangaswamy in [8] in the setting of Leavitt path algebras, slightly changing the latter’s notation and definitions. Given a graph, we define the dual graph of any of its subgraphs and study its properties. The dual construction acts as a localization technique, allowing to isolate any subset $X$ of $L_K(E)$, with $E$ an arbitrary graph, into another Leavitt path algebra $L_K(D_X)$ which shares much of the relevant structure of $L_K(E)$ while having $D_X$ row-finite or even finite (in particular, the behavior of the cycles without exits is respected, as shown in Lemma 5.6). Moreover, the Leavitt path algebra of any graph can be seen as the direct limit of the Leavitt path algebras of the duals of its finite subgraphs. This one belongs to a series of results which serve to study the Leavitt path algebra of a “complex” graph by studying a sequence of Leavitt path algebras of “simpler” graphs. This series was started in [10], where it was shown that a row-finite graph (resp. its Leavitt path algebra) is the direct limit of its finite complete subgraphs (resp. their Leavitt path algebras), and continued in [22], where it is shown that any arbitrary graph (resp. its Leavitt path algebra) is the direct limit of its countable $CK$-subgraphs (resp. their Leavitt path algebras).

1. Preliminaries

We present the graph-theoretic notation that will be needed in what follows, together with the Leavitt path algebra definition and some basic results about it. Our notation coincides with the standard one encountered along the literature.

Definitions 1.1 (Graph concepts). A graph $E = (E^0, E^1, r, s)$ consists of two (disjoint) sets $E^0, E^1$ of arbitrary cardinal and maps $r, s : E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. For each edge $e$, $s(e)$ is called the source of $e$ and $r(e)$ is called the range of $e$. If $v := s(e)$ and $w := r(e)$, we say that $v$ emits $e$ and $w$ receives $e$. 
If $F$ is a set of edges and $V$ is a set of vertices of $E$, we denote $s(F) = \{v \in E^0 \mid v = s(e), e \in F\}$ and $s^{-1}(V) = \{e \in E^1 \mid s(e) = v, v \in V\}$. We define $r(F)$ and $r^{-1}(V)$ analogously.

We say that $E$ is countable if $E^0, E^1$ are both countable; if both are finite, we say that $E$ is finite. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called row-finite. This amounts to saying that each vertex in $E$ emits only a finite number of edges.

A vertex which emits no edges is called a sink. A vertex which emits an infinite number of edges is called an infinite emitter. A vertex that is neither a sink nor an infinite emitter is said to be regular. A vertex which receives no edges is called a source. An isolated vertex is at the same time a source and a sink.

A path $\mu$ in a graph $E$ is either a vertex or a sequence of edges $\mu = e_1 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n - 1$. In this case, $s(\mu) := s(e_1)$ is the source of $\mu$, $r(\mu) := r(e_n)$ is the range of $\mu$, and $l(\mu) := n$ is the length of $\mu$ (being 0 the length of a vertex by definition). If $s(\mu) = v$ and $r(\mu) = w$ we say that $\mu$ starts at $v$ and ends in $w$. We denote by $\mu^0$ the set of its vertices and by $\mu^1$ the set of its edges, that is: $\mu^0 = \{s(e_i), r(e_i) \mid i = 1, \ldots, n\}$, $\mu^1 = \{e_i\}_{i=1}^n$.

A bifurcation for a path $\mu$ is a vertex $v \in \mu^0$ such that $|s^{-1}(v)| > 1$. The set of paths of $E$ of length $n$ is denoted by $E^n$. The set of all paths of $E$ is denoted as $\text{Path}(E)$.

A cycle is a path $c = e_1 \ldots e_n$ ($e_i \in E^1$) such that $s(c) = r(c) \neq s(e_i)$ for $i \in \{2, \ldots, n\}$, i.e., such that it starts at and ends in the same vertex and does not go twice through the same vertex. A loop is a cycle of length one. If $v := s(c)$, we say that $v$ is the base vertex of $c$ or, equivalently, that $c$ is based at $v$. If $E$ is a graph such that $\text{Path}(E)$ does not contain any cycles, we say that $E$ is acyclic. Note that for every cycle $c = e_1 \ldots e_n \in \text{Path}(E)$ there are other $n - 1$ “equivalent” cycles in $\text{Path}(E)$ formed by cyclic permutation of the edges of $c$: $e_2 \ldots e_n e_1, e_3 \ldots e_n e_1 e_2, \text{etc.}$, and that these cycles are all based at different vertices. An edge $e$ is an exit for a cycle $c = e_1 \ldots e_n$ if there exists $i$ such that $s(e) = s(e_i)$ and $e \neq e_i$. The subset of vertices of $E$ which are base of cycles without exits is denoted by $P_c(E)$.

Recall that E satisfies Condition (NE) if no cycle of E has exits, while it satisfies Condition (L) if everyone of its cycles has an exit.

Given two vertices $v, w \in E^0$, if there is a path $\mu \in \text{Path}(E)$ such that $s(\mu) = v$ and $r(\mu) = w$, we say that $v$ connects to $w$ and denote it by $v \geq w$ (this relation is a preorder, but not a partial order if there are cycles).

A subset $H$ of $E^0$ is called hereditary if $v \geq w$ and $v \in H$ imply $w \in H$. The tree of a vertex $v$ is the set $T(v) = \{w \in E^0 \mid v \geq w\}$, which is the smallest hereditary set of $E^0$ containing $v$. A hereditary set is saturated if every vertex which (finitely) “feeds” into $H$ and only into $H$ is again in $H$, that is, if $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. The set of all hereditary and saturated subsets of $E$ is denoted $\mathcal{H}_E$. The hereditary saturated closure of $X \subseteq E^0$ is defined as the smallest hereditary and saturated subset of $E^0$ containing $X$, and is denoted as $\overline{X}$. It is shown in [17, Remark 3.1] that $\overline{X} = \bigcup_{n=0}^\infty \Lambda_n(X)$, where

$$
\Lambda_0(X) = T(X) := \{v \in E^0 \mid x \geq v \text{ for some } x \in X\},
$$

$$
\Lambda_n(X) := \Lambda_{n-1}(X) \cup \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\}, \text{ for } n \geq 1.
$$

**Definition 1.2** (Leavitt path algebra). For a graph $E$ and a field $K$ we define the Leavitt path $K$-algebra of $E$, denoted $L_K(E)$, to be the $K$-algebra generated by a set $\{v \mid v \in E^0\}$ of
pairwise orthogonal idempotents, together with a set of variables \( \{ e \mid e \in E^1 \} \cup \{ e^* \mid e \in E^1 \} \) which satisfy the following relations:

1. \( s(e)e = e = er(e) \) for all \( e \in E^1 \).
2. \( r(e)e^* = e^* = e^*s(e) \) for all \( e \in E^1 \).
3. \( e^*e' = \delta_{e,e'}r(e) \) for all \( e, e' \in E^1 \).
4. \( v = \sum_{\{ e \in E^1 \mid s(e) = v \}} ee^* \) for every \( v \in E^0 \) which is regular.

Relations (3) and (4) are called, respectively, (CK1) and (CK2) (CK stands for Cuntz-Krieger).

The elements of \( E^1 \) are called real edges while for \( e \in E^1 \) we will call \( e^* \) a ghost edge. We let \( r(e^*) \) denote \( s(e) \) and \( s(e^*) \) denote \( r(e) \). If \( \mu = e_1 \ldots e_n \) is a path, by \( \mu^* \) we denote the element \( e_n^* \ldots e_1^* \) of \( L_K(E) \) and call it a ghost path. For any subset \( X \subseteq L_K(E) \), we will denote by \( I(X) \) the (two-sided) ideal of \( L_K(E) \) generated by \( X \).

The following constitute “small”, interesting examples of Leavitt path algebras.

(i) The loop is the following graph:

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•  
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It represents the simplest graph (nontrivially) satisfying Condition (NE). Its associated Leavitt path algebra is isomorphic to \( K[x, x^{-1}] \), the ring of Laurent polynomials in one variable.

(ii) The (algebraic) Toeplitz algebra is the Leavitt path algebra associated to the following graph:

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• → •
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This very simple graph satisfies Condition (L) nontrivially.

We recollect now two fundamental facts about Leavitt path algebras, which are valid in the general case.

Any element in a Leavitt path algebra can be written as a sum of monomials of a specific form ([2, Lemma 1.5]): If \( x \in L_K(E) \), then \( x = \sum_{i=1}^{n} k_i p_i q_i^* \), where \( n \in \mathbb{N} \), \( k_i \in K \) and \( p_i, q_i \in \text{Path}(E) \) with \( r(p_i) = r(q_i) \) for every \( i \in \{1, \ldots, n\} \). (†)

Note that this kind of expression is usually not unique (e.g., apply (CK2) to a vertex \( v \) with \( 0 < |s^{-1}(v)| < \infty \)).

Leavitt path algebras are also \( \mathbb{Z} \)-graded algebras ([2, Lemma 1.7]), with grading induced by \( \text{deg}(v) = 0 \) for all \( v \in E^0 \), \( \text{deg}(e) = 1 \) and \( \text{deg}(e^*) = -1 \) for all \( e \in E^1 \). That is, \( L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n \), where \( L_K(E)_n = \{ \sum kpq^* \mid k \in K, p, q \in \text{Path}(E), l(p) - l(q) = n \} \) (note that \( E^0 \subseteq L_K(E)_0 \)).

Later, it will be of importance to know the ring structure of the corner generated by a vertex which is base of a cycle without exits ([13, Lemma 1.5], which is actually valid in full generality):
Proposition 1.3. Let $E$ be an arbitrary graph and let $v \in P_c(E)$ be the base of the cycle without exits $c$. Then $vL_K(E)v \cong K[x, x^{-1}]$ as $K$-algebras (via the identification $x \equiv c, x^{-1} \equiv c^*$).

Finally, we recall the useful result that follows, which will appear thoroughly in this paper, and whose proof, as done in [12, Proposition 3.1], is also valid in full generality:

Theorem 1.4 (Reduction Theorem). Let $E$ be an arbitrary graph. Then for every nonzero element $z \in L_K(E)$ there exist $\mu, \nu \in \text{Path}(E)$ such that:

(i) $\mu^*z\nu = kv$ for some $k \in K \setminus \{0\}$ and $v \in E^0$, or

(ii) there exists a vertex $w \in P_c(E)$ such that $\mu^*z\nu$ is a nonzero element in $wL_K(E)w \cong K[x, x^{-1}]$.

Both cases are not mutually exclusive.

2. Dual graphs

We present the notion of dual of a subgraph in a graph, which is a generalization of the usual notion of dual graph found in the literature, and explore some of its properties. We also propose a new definition of the dual of a graph, which extends the well-behaved properties of the usual one to a wider class of Leavitt path algebras.

Definition 2.1 (Usual dual). Let $E$ be an arbitrary graph. The usual dual of $E$, $D(E)$, is the graph formed from $E$ by taking its length-one paths as the vertices and its length-two paths as the edges; that is,

$$
D(E)^0 = \{e \mid e \in E^1\} \\
D(E)^1 = \{ef \mid ef \in E^2\} \\
s_{D(E)}(ef) = e, \quad r_{D(E)}(ef) = f \text{ for all } ef \in E^2.
$$

The interest on the usual dual graph notion in the context of Leavitt path algebras lies on the fact that, if $E$ is a row-finite graph without sinks, then there is an algebra isomorphism $L_K(E) \cong L_K(D(E))$ ([1, Proposition 2.11]). The same is true in the context of graph $C^*$-algebras; i.e., we have $C^*(E) \cong C^*(D(E))$ for $E$ row-finite and with no sinks ([18, Remark 3.3]). Unfortunately, these statements are untrue for the usual dual of a graph with sinks. In what follows, we will propose a new definition of dual graph which generalizes this important property to row-finite graphs with sinks.

Definition 2.2 (Dual of $F$ in $E$). Let $E$ be a graph and let $F$ be a subgraph of $E$. Denote $F_1^0 = \{v \in F^0 \mid s_F^{-1}(v) = \emptyset\}, F_1^1 = r_F^{-1}(F_1^0)$ and $F_2^0 = s(F^1) \cap s(E^1 \setminus F^1), F_2^1 = r_F^{-1}(F_2^0)$. We define the graph $D_E(F)$, the dual of $F$ in $E$, as follows:

$$
D_E(F)^0 = D(F)^0 \cup F_1^0 \cup F_2^0 \\
D_E(F)^1 = D(F)^1 \cup F_1^1 \cup F_2^1 \\
s_{D_E(F)}|_{D(F)} = s_{D(F)}, \quad r_{D_E(F)}|_{D(F)} = r_{D(F)}
$$

For all $e \in F_i^1$ with $i \in \{1, 2\}$, $s_{D_E(F)}(e) = e \in D(F)^0, \quad r_{D_E(F)}(e) = r_{F}(e) \in F_i^0$. 
That is, the dual of $F$ in $E$ extends the usual dual of $F$, by adding to it two kinds of vertices that we will collectively call vertex-vertices, together with some edges that we will call vertex-edges; concretely, we add:

(i) The sinks of $F$ with their natural connections. That is, for every vertex $v$ which is a sink of $F$ and every edge $e$ which arrives at $v$ in $F$, we have in $D_E(F)$ a vertex $v$ and an edge $(e, v)$ starting at the vertex $e$ and ending in $v$.

(ii) The “non-full emitters” of $F$, also with their natural connections. That is, for every vertex $v$ of $F$ which is not a sink of $F$ and emits more edges in $E$ than it does in $F$, and every edge $e$ which arrives at $v$ in $F$, we have in $D_E(F)$ a vertex $v$ and an edge $(e, v)$ starting at the vertex $e$ and ending in $v$. We will call any vertex of this kind, in $F$, an intermediate vertex.

In addition, we will refer as edge-vertices and edge-edges, respectively, to the vertices and the edges of the dual which come from the usual dual.

**Example 2.3 (Dual of a subgraph in graph).** Consider the following graph $E$:

```
   v1  e1  e2  v2  e3  v3  e4  v4
     ↓   ↓   ↓   ↓   ↓   ↓   ↓   ↓
   e1  e2  e3  e4  e5
```

Let $F$ be the subgraph of $E$ formed by the vertices $\{v_i\}_{i=1}^4$ and the edges $\{e_i\}_{i=1}^5$. Then the dual graph $D_E(F)$ is

```
   v1  e1  v2  e2  e3  v3  e4  v4  e5
     ↓   ↓   ↓   ↓   ↓   ↓   ↓   ↓   ↓
   e1  e2  e3  e4  e5
```

We expose, without proof, some elementary properties of the dual graph of a subgraph:

**Lemma 2.4.** Let $F$ be a subgraph of a graph $E$. Then:

(i) If $F$ is finite, so is $D_E(F)$.

(ii) If $F$ is row-finite, so is $D_E(F)$.

(iii) Every loop $e$ in $F$ generates a loop $ee$ (with base $e$) in $D_E(F)$.

(iv) All the vertex-vertices of $D_E(F)$ are sinks, and those are its only sinks.

(v) The isolated vertices of $F$ remain isolated in $D_E(F)$.

(vi) The intermediate vertices which are also sources in $F$ are isolated in $D_E(F)$.

Now we can define what we will call the dual of a graph (redefining thus the notion of usual dual) by taking the graph as a subgraph of itself:

**Definition 2.5 (Dual graph).** Given a graph $E$, we define $d(E) = D_E(E)$ and call it the dual graph of $E$.

When $E$ has no sinks, the usual dual $D(E)$ and the dual $d(E)$ coincide, but they do not when there are sinks present. The advantage of the latter definition has already been stated:
Proposition 2.6 (Isomorphism with the dual’s graph algebra). Let $E$ be a row-finite graph. Then:

(i) $L_K(d(E)) \cong L_K(E)$ as graded algebras.
(ii) $C^*(d(E)) \cong C^*(E)$ as $*$-algebras.

Proof. We will show that $E$ be a row-finite graph and let $\mathcal{P}$ be the partition of $E^1$ having $m(v) = |s^{-1}(v)|$ for every $v$ that is not a sink (i.e., the partition of $E^1$ which admits no refinements). Let $E_s(\mathcal{P})$ be the outsplit graph formed from $E$ using the partition $\mathcal{P}$. Since $\mathcal{P}$ is maximal, we have

$$E_s(\mathcal{P})^0 = \{v^e \mid s(e) = v\} \cup \{v \mid v \text{ is a sink}\}$$

$$E_s(\mathcal{P})^1 = \{e^f \mid s(f) = r(e)\} \cup \{e \mid r(e) \text{ is not a sink}\},$$

while

$$d(E)^0 = \{e \mid e \in E^1\} \cup \{v \mid v \text{ is a sink}\}$$

$$d(E)^1 = \{ef \mid e, f \in E^1, r(e) = s(f)\} \cup \{e \mid r(e) \text{ is not a sink}\}.$$

The maps $\phi^0 : E_s(\mathcal{P})^0 \to d(E)^0$ such that $\phi^0(v^e) = e$, $\phi^0(v) = v$ and $\phi^1 : E_s(\mathcal{P})^1 \to d(E)^1$ such that $\phi^1(e^f) = ef$, $\phi^1(e) = e$ are easily shown to commute with the source and range maps, whence they induce a graph isomorphism from $E_s(\mathcal{P})$ to $d(E)$.

Remark 2.7. Note that this proof provides us with another way to compute the maximal outsplit of a graph $E$ (other than by definition), namely by constructing its dual $d(E)$, what perhaps is easier and clearer to do, since we just have to put a vertex for every edge of $E$, an edge for every length-two path of $E$, a vertex $v$ for every sink of $E$ and an edge $(e, v)$ starting at $e$ and ending in $v$ for every edge $e$ with range $v$ in $E$ (observe that there are no intermediate vertices to consider, since $d(E) = D_E(E)$).

The result above generalizes to any row-finite subgraph, in the sense that the Leavitt path algebra of the dual of the subgraph is a subalgebra of the Leavitt path algebra of the graph:

Proposition 2.8. Let $E$ be a graph and $F$ be a row-finite subgraph of $E$. Then there is a graded monomorphism $\theta : L_K(D_E(F)) \to L_K(E)$. In addition, $F^0 \cup F^1 \subseteq \theta(L_K(D_E(F)))$.

Proof. The proof is essentially a rewriting, in the dual graphs language, of the proofs for [8, Proposition 1] and [8, items (1),(2) of Proposition 2]. The specific construction of the monomorphism will be of use in later sections and for that reason we include it here.

For clarity, denote $G = D_E(F)$. For any vertex-vertex $v \in G^0$, denote

$$u_v = v - \sum_{\{e \in F^1 \mid s(e) = v\}} ee^* \in L_K(E),$$

understanding an empty sum to be 0. Note that $u_v \neq 0$ because either $v$ is a sink, or it is an intermediate vertex and thus there exists $f \in E^1$ such that $s(f) = v$ and $f \notin F^1$. Note also that $\{u_v \mid v \in G^0 \setminus F^1\}$ is a set of pairwise orthogonal idempotents.
Now, define on the generators and extend to an algebra homomorphism the map \( \theta : L_K(G) \to L_K(E) \) in the following way:

- If \( e \) is an edge-vertex, then \( \theta(e) = ee^* \).
- If \( v \) is a vertex-vertex, then \( \theta(v) = vu = u_v \).
- If \( ef \) is an edge-edge, then \( \theta(ef) = \theta(e)f = ef f^* \).
- If \((e, v)\) is a vertex-edge, then \( \theta((e, v)) = e\theta(v) = eu_v = \sum_{\{f \in F^1 \mid s_{DE}(f) = v\}} \theta(ef) \).

If \( \mu \in G^1 \) then \( \theta(\mu^*) = (\theta(\mu))^* \).

Now, to check that \( \theta \) is compatible with the Leavitt path algebra relations is a matter of simple algebraic manipulations. Moreover, it is not difficult to see that \( \theta(v)^* = \theta(v) \) and \( \theta(e)^* = \theta(e) \). Note that every generator is mapped to an element of its same degree, so that the homomorphism will be a graded homomorphism. Note also that all the vertices have nonzero images and thus, by the Graded Uniqueness Theorem ([30, Theorem 4.8]), \( \theta \) will in fact be a graded monomorphism. \( \square \)

**Remark 2.9.** It may prove useful for the future to write down, defined on generators, the inverse isomorphism of \( \theta \), \( \Phi : \theta(L_K(\mathcal{D}_E(F))) \to L_K(\mathcal{D}_E(F)) \), in order to know explicitly where are mapped, in its dual, the vertices and edges of \( F \):

- If \( v \in F^0 \), then \( \Phi(v) = \delta_v + \sum_{\{e \in F^1 \mid s_{DE}(e) = v\}} e \), where an empty sum is 0 and \( \delta_v = \begin{cases} 1, & \text{if } v \text{ is a sink or an intermediate vertex of } F \\ 0, & \text{otherwise.} \end{cases} \)
- If \( e \in F^1 \), then \( \Phi(e) = \sum_{\{f \in \mathcal{D}_E(F)^1 \mid s_{DE}(f) = e\}} f \).
- If \( e \in F^1 \), then \( \Phi(e^*) = (\Phi(e))^* \).

A subgraph \( F \) of a graph \( E \) is said to be *complete* when for each regular vertex \( v \) of \( F \) we have \( |s_{\mathcal{D}_E}(v)| = |s_{\mathcal{D}_E}(v)| \). Complete subgraphs are precisely those subgraphs that naturally induce a subalgebra \( L_K(F) \) of \( L_K(E) \). In the same spirit, given a set \( X \) of elements of \( L_K(E) \) satisfying suitable conditions, Proposition 2.8 allows us to find a subalgebra \( A \) of \( L_K(E) \) such that \( X \subseteq A \) and which is a Leavitt path algebra that inherits several important properties from \( L_K(E) \), in the following manner: we decompose every element of \( X \) as an expression on some generators, find a subgraph \( F \) of \( E \) which contains all those generators and, if \( F \) is row-finite, we conclude that \( X \subseteq \theta(L_K(\mathcal{D}_E(F))) \). The mentioned ‘suitable conditions’ are precisely those which allow the existence of such a row-finite subgraph \( F \). For example, this is trivially achieved if \( X \) is finite. Thus, this result is useful for graphs which do not contain “nontrivial” finite complete subgraphs enveloping the generators of our set \( X \) of interest, as happens with any subset containing edges of the infinite clock (\( \aleph \) being an infinite cardinal):
It is clear that the only finite complete subgraphs of the infinite clock are the empty graph, the graph consisting just of the central vertex, and any subset of the set of sinks.

The following notation will be useful:

**Definition 2.10.** Given a graph $E$, and given a subset $X \subseteq L_K(E)$, we express every $x \in X$ in a convenient form (as in (†)). Then we define $F_X$, an enveloping subgraph for $X$, as the subgraph of $E$ formed by taking all the vertices and all the edges appearing in those expressions, as well as all the sources and ranges of these edges. Concretely, if $X = \{x_i\}_{i \in \Lambda}$, write

$$x_i = \sum n_i k^l_i v^l_i + \sum m_i k^l_m p^l_m + \sum j_i k^l_j p^l_j + \sum i_i k^l_i p^l_i q^l_i,$$

where for every $l \in \Lambda$ we have $k^l_n, k^l_m, k^l_j, k^l_i \in K \setminus \{0\}$, $v^l_i \in E_0^0$ and $p^l_m, p^l_j, p^l_i, q^l_i \in \text{Path}(E)$, which are such that $r(p^l_i) = r(q^l_i)$, and such that we have $p^l_i = e^l_{i,1} \ldots e^l_{i,r^l_i}$ and $q^l_i = f^l_{i,1} \ldots f^l_{i,s^l_i}$, with $e^l_{i,m}, f^l_{i,n} \in E^1$ and $r^l_i, s^l_i \geq 1$ for every $i$ (i.e., we can assure that these paths are not vertices).

Then $F_X$ is formed by $F^1_X = \{e^l_{i,m} \mid l \in \Lambda, i, m \in \{1, \ldots, r^l_i\}\}$ and $F^0_X = \{v^l_i \mid l \in \Lambda\} \cup s(F^1_X)$ and $F^0_X = \{v^l_i \mid l \in \Lambda\} \cup s(F^1_X)$.

Note that the enveloping subgraph for $X$ is not unique, as its structure depends heavily on the selected expressions (as in (†)) of the elements of $X$, which are not unique themselves.

In the same spirit, we denote $D_X := E^D(F_X)$ and $A_X := \theta(L_K(D_X))$, as constructed in Proposition 2.8. In particular, if $X = \{x\}$ is a singleton set, we will forget the braces and simply write $F_x$, $D_x$ and $A_x$.

Our last result about dual graphs is the following proposition (the adaptation of [8, Proposition 2, (4)]), which states that the Leavitt path algebra of a graph can be viewed as the direct union of the Leavitt path algebras of the duals of its finite subgraphs.

**Proposition 2.11.** Let $E$ be an arbitrary graph. Then $L_K(E) = \bigsqcup_{\{X \subseteq L_K(E) \mid |X| < \infty\}} A_X$.

*Proof.* Let $X \subseteq L_K(E)$ be finite, say $X = \{x_n\}_{n=1}^N$, and assume a convenient expression as a sum of monomials (as in (†)) for every $x_n$. By construction, $x_n$ is in the subalgebra of $L_K(E)$ generated by $F_X$ for every $n \in \{1, \ldots, N\}$, which implies by Proposition 2.8 that $x_n \in A_X$ for every $n \in \{1, \ldots, N\}$. In addition, since $F_X$ is finite, $D_X$ is finite as well; in particular, $L_K(D_X)$ and thus $A_X$ are finitely generated $K$-algebras.

Now let $X_1, X_2$ be two finite subsets of $L_K(E)$ and let $T_1, T_2$ denote respective finite sets of generators for $A_{X_1}$ and $A_{X_2}$. Then $T = T_1 \cup T_2$ is, by construction, such that $A_{X_1} \cup A_{X_2} \subseteq A_T$. This proves that the collection $\{A_X \mid X \subseteq L_K(E), |X| < \infty\}$ is an upward directed set of subalgebras of $L_K(E)$. The claim follows now taking into account that $X \subseteq A_X$ for any finite subset $X$ of $L_K(E)$. \qed
3. Infinite idempotents and Condition (NE)

We characterize in full generality the Leavitt path algebras associated to graphs that satisfy Condition (NE) in terms of idempotents, and establish their structure via dual graph techniques.

The following lemma establishes that the cycles without exits of $D_E(F)$ cannot come from cycles with exits of $E$, even from those whose exits are “hidden” to $F$:

**Lemma 3.1.** Let $E$ be an arbitrary graph and $F$ be a row-finite subgraph of $E$. Then there is an injective map from the set of cycles without exits of $D_E(F)$ to the set of cycles without exits of $E$.

**Proof.** Let $c = (e_1, e_2)(e_2, e_3)\ldots(e_n, e_1)$ be a cycle without exits of $D_E(F)$, where $c^0 = \{e_1, \ldots, e_n\}$ and $(e_i, e_j)$ denotes the edge joining the vertices $e_i$ and $e_j$. By construction of the dual, any vertex $e_i$ must come from an edge $e_i$ of $F$. Also by construction, in $F$, $e_n$ connects (directly) to $e_1$, $e_i$ connects to $e_{i+1}$ for $i \in \{1, \ldots, n-1\}$ and $r(e_i) \neq r(e_j)$ for $i \neq j$, so that $c' = e_1 \ldots e_n$ is a cycle of $F$. Suppose that $c'$ has an exit $e \in E^1$ at a vertex $v = r(e_j)$; then either $e$ is in $F^1$, which is impossible because in $D_E(F)$ the cycle $c$ would have an exit $(e_j, e)$ at the vertex $e_j$, or $e \in E^1 \setminus F^1$, in which case $v$ would be an intermediate vertex of $F$ and, in $D_E(F)$, the vertex $e_j$ would have an exit $(e_j, v)$ (with range $v$), which is also impossible. Thus, every cycle without exits of $D_E(F)$ comes from a cycle without exits of $E$. That no two of these cycles of $D_E(F)$ come from the same one of $E$ is again clear from the construction of the dual. \hfill \Box

Recall that, given a ring $R$, an idempotent $e \in R$ is an infinite idempotent if $eR$ is isomorphic as a right $R$-module to a proper direct summand of itself (equivalently, if $Re$ is isomorphic as a left $R$-module to a proper direct summand of itself).

We remember also that two idempotents $p, q \in R$ are (Murray-von Neumann) equivalent, and denote it by $p \sim q$, if there exist $x, y \in R$ such that $p = xy$ and $yx = q$ or, equivalently, if $pR$ and $qR$ are isomorphic as right $R$-modules (equivalently, if $Rp$ and $Rq$ are isomorphic as left $R$-modules).

The following characterization of infinite idempotents in terms of elements of the ring is well known: $e \in R$ is an infinite idempotent if and only if there exists a pair of nonzero orthogonal idempotents $x, y \in R$ such that $e = x + y$ and $e \sim x$.

**Lemma 3.2.** If $\{R_i\}_{i=1}^m$ is a finite family of integral domains and $A = \bigoplus_{i=1}^m M_{n_i}(R_i)$ with $n_i \in \mathbb{N}$ for all $i$, then $A$ has no infinite idempotents.

**Proof.** Let $Q_i$ be the field of fractions of $R_i$, $i \in \{1, \ldots, m\}$, and let $Q(A)$ denote the classical ring of quotients of $A$. It is well-known that $Q(A) = \bigoplus_{i=1}^m M_{n_i}(Q_i)$, which is an artinian ring by the Wedderburn-Artin Theorem, and that we can see $A \subseteq Q(A)$. If $e \in A \subseteq Q(A)$ were an infinite idempotent, then there would exist $x, y, a, b \in A \subseteq Q(A)$ such that $x, y$ would be nonzero orthogonal idempotents, $e = x + y$, $e = ab$ and $x = ba$. But this would mean that $e$ is an infinite idempotent in $Q(A)$, which is impossible since $Q(A)$ is artinian. \hfill \Box
Theorem 3.3. (Structure Theorem for the Leavitt path algebra of an (NE) graph)

Let $E$ be any graph. The following conditions are equivalent:

(i) $E$ satisfies Condition (NE).

(ii) $L_K(E) \cong \bigoplus_{t \in T} \left( \left( \bigoplus_{i=1}^{r_t} \mathbb{M}_{n_i^t}(K) \right) \oplus \left( \bigoplus_{j=1}^{s_t} \mathbb{M}_{m_j^t}(K[x, x^{-1}]) \right) \right)$, where $r_t, s_t, n_i^t, m_j^t \in \mathbb{N}$.

(iii) $L_K(E)$ has no infinite idempotents.

Proof. (i) $\Rightarrow$ (ii). Let $E$ be a graph satisfying Condition (NE). By Proposition 2.11, $L_K(E) = \bigcup_{X \subseteq L_K(E)} A_X$. Since $E$ satisfies Condition (NE), any of its enveloping subgraphs $F_X$ satisfies it too, and so does $D_X$ by Lemma 3.1. In addition, $F_X$ is finite, which implies that its dual $D_X$ is also finite. Therefore, by the Structure Theorem of noetherian Leavitt path algebras ([7, Theorem 3.8]),

$$L_K(D_X) \cong \left( \bigoplus_{i=1}^{r_X} \mathbb{M}_{n_i^X}(K) \right) \oplus \left( \bigoplus_{j=1}^{s_X} \mathbb{M}_{m_j^X}(K[x, x^{-1}]) \right),$$

what implies the claim, as $A_X \cong L_K(D_X)$.

(ii) $\Rightarrow$ (iii). Suppose that

$$L_K(E) \cong \bigoplus_{t \in T} \left( \left( \bigoplus_{i=1}^{r_t} \mathbb{M}_{n_i^t}(K) \right) \oplus \left( \bigoplus_{j=1}^{s_t} \mathbb{M}_{m_j^t}(K[x, x^{-1}]) \right) \right)$$

contained an infinite idempotent. Then there should exist $e, x, y, a, b \in L_K(E)$ such that $e$ is an infinite idempotent, $x, y$ are nonzero orthogonal idempotents, $e = x + y$, $e = ab$ and $x = ba$. But then there should exist, for a $t_0$ big enough, similar elements in $\left( \bigoplus_{t=1}^{r_{t_0}} \mathbb{M}_{n_i^{t_0}}(K) \right) \oplus \left( \bigoplus_{j=1}^{s_{t_0}} \mathbb{M}_{m_j^{t_0}}(K[x, x^{-1}]) \right)$, what is impossible, since otherwise this algebra would contain an infinite idempotent, a contradiction to Lemma 3.2.

(iii) $\Rightarrow$ (i). Suppose that $L_K(E)$ does not satisfy Condition (NE) and that it has no infinite idempotents. Then there exists a cycle $c \in L_K(E)$ with exits based at a vertex $v$, and this implies that $v$ is an infinite idempotent since $v = cc^* + (v - cc^*)$ with $v - cc^* \neq 0$ (because $c$ has exits), $cc^* (v - cc^*) = 0$ and $v = c^* c \sim cc^*$. Thus we get a contradiction. \qed

Remark 3.4. In particular, we have shown that Condition (NE) is a ring isomorphism invariant for Leavitt path algebras; that is, if $E, F$ are graphs such that $L_K(E) \cong L_K(F)$ as rings and $E$ satisfies Condition (NE), then $F$ satisfies Condition (NE) too.

4. PRIMITIVE IDEMPOTENTS AND CONDITION (L)

Truly primitive idempotents are introduced and an algebraic characterization for Condition (L) is achieved consequently. In this manner, we add the characterization of Condition (L) alone to the already known ones of Condition (L) plus Condition (MT3) in the row-finite context (which give rise to primitive Leavitt path algebras, see [15, Theorem 4.3]) and Condition (L) plus cofinality (which give rise to simple Leavitt path algebras, see [22, Theorem 3.11]). As a corollary, we give a new algebraic characterization of simple Leavitt path algebras.
Proposition 4.1. Let $e$ be an idempotent in a ring $R$ (not necessarily unital). The following conditions are equivalent:

(i) $eR$ is an indecomposable right $R$-module (equivalently, $Re$ is an indecomposable left $R$-module).

(ii) $eRe$ is a ring without nontrivial idempotents.

(iii) $e$ has no decomposition into $a + b$, where $a, b$ are nonzero orthogonal idempotents in $R$.

Proof. As in [24, Proposition 21.8]. □

Definition 4.2. If an idempotent $0 \neq e \in R$ satisfies any of these conditions, we say that $e$ is a primitive idempotent.

We recall that a vertex $v$ is called a line point if there are neither cycles nor bifurcations at any vertex $w \in T(v)$. We denote, as usual, the set of all line points of $E$ by $P_l(E)$.

Proposition 4.3. Let $E$ be an arbitrary graph and let $v \in E^0$. Then $v$ is a primitive idempotent of $L_K(E)$ if and only if its tree $T(v)$ has no bifurcations.

Proof. Suppose that $T(v)$ has its first bifurcation at $w$, with $\mu$ being the path which connects $v$ to $w$. Let $e$ and $f$ be two different edges emitted by $w$; then $ee^* \neq w$ and therefore $0 \neq \mu ee^* \mu^* \neq v$. It is easy to verify that $\mu ee^* \mu^*$ is a (nontrivial) idempotent living in $vL_K(E)v$ and thus, by item (ii) of Proposition 4.1, $v$ cannot be a primitive idempotent.

Now let $v$ be a vertex of $E$ such that $T(v)$ has no bifurcations. Two cases can occur:

Case 1: $T(v)$ does not contain vertices in cycles. In this case, $v \in P_l(E)$, what means that $v$ is minimal ([9, Theorem 1.9]) and therefore primitive.

Case 2: $T(v) \cap P_c(E) \neq \emptyset$. Since $T(v)$ has no bifurcations, there can be only one cycle $c \in L_K(E)$ such that $T(v) \cap c^0 \neq \emptyset$, which in addition has no exits. Furthermore, every vertex of $T(v)$ is either in $c^0$ or connects to another vertex in $c^0$ via a path without bifurcations. Thus, by [12, Lemma 2.2] (which is valid in our context), there exists $w \in c^0$ such that $L_K(E)v \cong L_K(E)w$ as left $L_K(E)$-modules. Since $w$ is in a cycle without exits, by Proposition 1.3 we have $wL_K(E)w \cong K[x, x^{-1}]$, which is a ring without nontrivial idempotents. Now Proposition 4.1 gives that $w$ and $v$ are both primitive and finalizes the proof. □

Remark 4.4. The proof above would no longer be valid if we considered Leavitt path algebras $L_R(E)$ over a ring $R$ with (nontrivial) idempotents, because $\mathbb{R}[x, x^{-1}]$ would have them too. We note that Leavitt path algebras over more general rings (and not just over fields) have begun to be studied recently (e.g., [29]).

Remark 4.5. If $vL_K(E)v$ is a ring with no nontrivial idempotents (e.g., a domain) then $v$ is a primitive idempotent and, as seen as a consequence of the proof above, we have either $vL_K(E)v \cong K$ (if $v$ is minimal) or $vL_K(E)v \cong K[x, x^{-1}]$ (if it is not).

We have found a close relationship between the primitive and the minimal vertices of the Leavitt path algebra of any graph: the minimal vertices are those whose trees do not contain bifurcations nor bases of cycles, while the primitive vertices see this second condition suppressed. Thus, the following definition is of interest:
Definition 4.6. Let $R$ be any ring and $e \in R$ an idempotent. We say that $e$ is a truly primitive idempotent if it is primitive but not minimal.

Remark 4.7. Hence, $v \in E^0$ is a truly primitive vertex of $L_K(E)$ if and only if $vL_K(E)v \cong K[x, x^{-1}]$. In particular, the vertices in $P_e(E)$ are truly primitive.

Proposition 4.3 provides us with a tool to distinguish between cycles with and without exits in a graph, giving us a characterization of Condition (L) in terms of primitive vertices:

Corollary 4.8. Let $E$ be any graph. The following conditions are equivalent:

(i) $E$ satisfies Condition (L).

(ii) $L_K(E)$ has no truly primitive vertices.

Proof. By Proposition 4.3, $L_K(E)$ contains a truly primitive vertex if and only if $E$ contains a cycle without exits.

As far as we know, a ring-theoretic characterization of Condition (L) lacked in the literature. We provide one below, extending Corollary 4.8 from the truly primitive vertices to the truly primitive idempotents of the Leavitt path algebra. Hence, we show that Condition (L) is an invariant of ring isomorphisms, in the sense that if $E, F$ are two graphs such that $L_K(E) \cong L_K(F)$ as rings and $E$ satisfies Condition (L), then $F$ satisfies it too.

Proposition 4.9. If $z \in L_K(E)$ is a primitive idempotent such that we can write $\alpha z \beta = kv$ for $\alpha, \beta \in L_K(E)$ with $\alpha$ or $\beta$ a monomial, $k \in K \setminus \{0\}$, and some vertex $v \in E^0$, then $L_K(E)z \cong L_K(E)v$. If, moreover, $z$ is truly primitive, then $zL_K(E)z \cong K[x, x^{-1}]$.

Proof. Consider $a = \frac{1}{k} \alpha z, b = z \beta$ (note that either $va = a$ or $bv = b$ because either $\alpha$ or $\beta$ is a monomial). Then $ab = v$, and $e := ba = \frac{1}{k} z \beta \alpha z$ is in $zL_K(E)z$. Moreover, $e^2 = b \alpha b \alpha = b \alpha = e$ and thus $e \sim v$. Since $z$ is a primitive idempotent, $zL_K(E)z$ is a ring without nontrivial idempotents, so that $e \in \{0, z\}$; and since $ae \beta = kv \neq 0$ implies $e \neq 0$, we have $z = e \sim v$, what means, as desired, that $L_K(E)z \cong L_K(E)v$. If in addition $z$ is truly primitive, so is $v$, and hence $zL_K(E)z \cong vL_K(E)v \cong K[x, x^{-1}]$.

Theorem 4.10. Let $E$ be any graph. The following conditions are equivalent:

(i) $E$ satisfies Condition (L).

(ii) $L_K(E)$ has no truly primitive idempotents.

Proof. If $L_K(E)$ has no truly primitive idempotents, in particular it has no truly primitive vertices, so that by Corollary 4.8, $E$ satisfies Condition (L).

Now suppose $E$ satisfies Condition (L) and let $x$ be a truly primitive idempotent of $L_K(E)$. By the Reduction Theorem there exist a vertex $v$, a nonzero scalar $k$ and elements $\mu, \nu \in \text{Path}(E)$ such that $\mu^* x \nu = kv$. Note that, by Corollary 4.8, $v$ cannot be truly primitive. But this is a contradiction since by Proposition 4.9, $L_K(E)v \cong L_K(E)x$.

The tools developed above will allow us to reformulate, in terms of idempotents, the known simplicity and purely infinite simplicity results for Leavitt path algebras.

In [22, Theorem 3.11], arbitrary Leavitt path algebras $L_K(E)$ which are simple are characterized as those whose graphs simultaneously satisfy these two conditions:

(i) $\mathcal{H}_E = \{\emptyset, E^0\}$.
(ii) $E$ satisfies Condition (L).

Since condition (i) above happens to be equivalent to saying that there are no (two-sided) ideals generated by idempotents in $L_K(E)$ ([10, Proof of Theorem 5.3]), Theorem 4.10 allows us to state the following:

**Corollary 4.11.** Let $E$ be an arbitrary graph. Then $L_K(E)$ is simple if and only if it has no truly primitive idempotents and no nontrivial two-sided ideals generated by idempotents.

If we add a third condition to the characterization of simplicity exposed before the former corollary, we characterize the purely infinite simple Leavitt path algebras; namely, as those whose graphs $E$ satisfy ([4, Theorem 4.3]):

(i) $H_E = \{\emptyset, E^0\}$.

(ii) $E$ satisfies Condition (L).

(iii) Every vertex of $E$ connects to a cycle.

Note that if $E$ is a finite graph, then condition (iii) can be changed by the condition that there are no minimal idempotents: on the one hand, if every vertex connects to a cycle, there are no minimal vertices and, on the other hand, if there are no minimal vertices then there are no sinks, and since $E$ is finite, every vertex must connect to a cycle. Now, $Soc(L_K(E)) = I(P_l(E))$ ([12, Theorem 4.2]) and $P_l(E) = \emptyset$ (because $E$ is finite and there are no sinks) imply that if there are no minimal vertices, then there are no minimal idempotents at all (the converse is obvious). Therefore, we can establish a result similar to the corollary given above:

**Remark 4.12.** Let $E$ be a finite graph. Then $L_K(E)$ is purely infinite simple if and only if it has no primitive idempotents and no ideals generated by idempotents.

5. **The Reduction Theorem and $I(P_c(E))$**

$I(P_c(E))$, the ideal generated by the vertices in cycles without exits, bears a clear parallelism with $I(P_l(E))$, the ideal generated by vertices in line points, which is known to be the socle of the Leavitt path algebra $L_K(E)$. That makes this ideal interesting to be studied on its own. In this section we do so by relating truly primitive idempotents with $I(P_c(E))$, what serves us to prove that this ideal is invariant under ring isomorphisms between Leavitt path algebras. We achieve this by a refinement of the Reduction Theorem (see Section 1). First we will work in the row-finite case; next we will build on this one, using again dual graph techniques, to achieve the general case. In the last part we will also provide the structure of $I(P_c(E))$, revealing a bit more of its parallelism with $Soc(L_K(E))$.

5.1. **The row-finite case.** Under certain conditions, we can construct ‘quotient’ Leavitt path algebras by means of quotient graphs, what will be of use in our next proposition. We recollect their definition:

Let $E$ be a row-finite graph and consider $H \in H_E$. The quotient graph $E/H$ is defined as

$$(E/H)^0 = E^0 \setminus H$$

$$(E/H)^1 = \{e \in E^1 \mid r(e) \not\in H\}$$

$r|_{(E/H)^1}, s|_{(E/H)^1}$.

We note that $I(P_c(E))$ cannot contain any polynomials in cycles with exits:
Lemma 5.1. If $E$ is an arbitrary graph and $c$ is a cycle with exits of $L_K(E)$, then $p(c,c^*) \notin I(P_c(E))$ for any polynomial $p$.

Proof. Suppose on the contrary that there exists a cycle with exits, $c$, such that $p(c,c^*) \in I(P_c(E))$ for some polynomial $p$. Write $p(c,c^*) = \sum_n k_i c^i + \sum_i k'_j (c^*)^j$. As $I(P_c(E)) = I(P_c(E))$ (by the first part of [14, Lemma 2.1], which is valid in full generality), $I(P_c(E))$ is a graded ideal by [30, Lemma 5.6] (taking $S = \emptyset$), and therefore every monomial of $p(c,c^*)$ is in $I(P_c(E))$. In particular, for some $i$, we either have $c^i \in I(P_c(E))$ or $(c^*)^i \in I(P_c(E))$. In any case we get $(c^*)^i c^i = r(c) \in I(P_c(E))$, so that $r(c)$ is in $P_c(E)$ (because by [30, Proof of Theorem 5.7 (1)], which is valid in general, $I(H) \cap E^0 = H$ for $H \in \mathcal{H}$, taking $S = \emptyset$). Let $n$ be the smallest nonnegative integer having $\Lambda_n(P_c(E)) \cap \mathcal{E} = \emptyset$. Choose $v$ in this intersection. If $n > 0$ then $\Lambda_n-1(P_c(E)) \cap \mathcal{E} = \emptyset$ and therefore $\emptyset = r(s^{-1}(v)) \subseteq \Lambda_n-1(P_c(E))$. In particular $\Lambda_n-1(P_c(E)) \cap \mathcal{E} = \emptyset$, a contradiction, so $n = 0$ and consequently $T(P_c(E)) \cap \mathcal{E} = P_c(E) \cap \mathcal{E} = \emptyset$ (note that $P_c(E)$ is hereditary). But this is another contradiction, because no vertex can be simultaneously a base for a cycle with exits and for a cycle without exits. \qed

Recall that, when using the Reduction Theorem for an element $x$, we get in $I(\{x\})$ either a vertex or a polynomial in a cycle without exits; the following proposition gives a sufficient condition to guarantee that we can actually get a vertex:

Proposition 5.2. If $E$ is a row-finite graph and $x \in L_K(E) \setminus I(P_c(E))$ then there exist $k \in K \setminus \{0\}$, $\mu, \nu \in \text{Path}(E)$ and $v \in E^0$ such that $\mu^* xv = kv$.

Proof. Denote $I = I(P_c(E))$. Since $P_c(E)$ is a hereditary subset of $E^0$, $I$ is a graded ideal, and since $E$ is a row-finite graph, by [14, Lemma 2.3 (i)], $L_K(E)/I$ is graded isomorphic to the Leavitt path algebra with associated graph $F = E/P_c(E)$. Denote by $[x]$ the class of the element $x \in L_K(E)$ in $L_K(E)/I$. Suppose $x \notin I$. Then, $0 \neq [x] \in L_K(E)/I$ and by the Reduction Theorem we can find $k \in K \setminus \{0\}$ and two paths $[[\mu]], [[\nu]]$ in $L_K(E)/I$ (coming from paths $\mu, \nu$ of $L_K(E)$) such that either $[[\mu]^* x [[\nu]] = k [[v]]$ for some vertex $[v] \in F^0$ (coming from a vertex $v \in E^0)$ or $[[\mu]^* x [[\nu]] = [p]$ for some polynomial $[p]$ based at a cycle without exits $[c]$ of $F$. But then $[[\mu]^* x v] = [kv]$ or $[[\mu]^* x v] = [p]$ (the conjugation can go inside the class because of the specific construction of the epimorphism between $L_K(E)$ and $L_K(E)/I$) and thus $\mu^* xv = kv + y$ or $\mu^* xv = p + y'$ in $L_K(E)$, for some $y, y' \in I$. We can write $y$ and $y'$ in the form

$$\sum_{n} k_n v_n + \sum_{m} k_m p_m (r(p_m) + \sum_{j} k_j r(p_j) p_j^* + \sum_{i} i_k p_i r(p_i) q_i^*)$$

where the set of vertices $V = \{v_n, r(p_m), r(p_j), r(p_i)\}$ is contained in $I$ and all the paths have nonzero lengths. We will study these two cases separately.

Case 1: $[kv] \neq 0$ implies $v \notin I$, so either $v$ is a sink (in which case $vvv = 0$) or there exists $e_1 \in s^{-1}(v)$ such that $e_1 \notin I$. Moreover, $u_1 = r(e_1) \notin I$ either (because $e_1 = e_1 u_1$). Consider $z_1 = vv; v = 0$ we are done ($v\mu^* x x v = k v$). If $z_1 \neq 0$ it must be $z_1 = \sum_{i=1}^{N} k_i p_i q_i^*$ with $l(p_i), l(q_i) \geq 1$, since $V \subseteq I$ while $v \notin I$. Now consider $z_1' = e_1 z_1 e_1$: if $z_1' = 0$ we are done ($e_1^* \mu^* x v e_1 = k u_1$); if not, we can rearrange $z_1$ in the form $z_1 = \sum_{i=1}^{N_1} k_i e_1 p_i q_i^* e_1 + z_1''$, with $N_1 = N$, $l(p_i') = l(p_i) - 1$, $l(q_i') = l(q_i) - 1$ and $e_1^* z_1'' e_1 = 0$, so that $z_1'' = \sum_{i=1}^{N_1} k_i p_i q_i^*$ and $e_1^* \mu^* x v e_1 = k u_1 + z_1'$. But $z_1' \in I$, $u_1 \notin I$; then either $u_1$ is a sink (and we are finished) or there exists $e_2 \in s^{-1}(u_1)$ such that $e_2 \notin I$ and therefore we can consider $z_2 = u_1 z_1' u_1$ and
we arrive to a vertex $u_n$ and $z_n \in I$ which is either 0 or a linear combination of vertices (when $l(p_i) = l(q_i)$), paths (when $l(p_i) > l(q_i)$) and ghost paths (when $l(p_i) < l(q_i)$), and whence $u_n z_n u_n = 0$ because $u_n \not\in I$. Then $e_n^* \cdots e_1^* \mu^* x v e_1 \cdots e_n = k u_n$.

Case 2: Since $[p]$ is a polynomial in a cycle without exits $[c]$ of $F$, $c$ must be a cycle with exits in $E$ such that all of its exits belong to $I$ (otherwise $[c]$ would be 0 or have an exit in $L_K(E)/I$). Fix one of these exits $e$ and denote $v := s(e), w := r(e)$. We can suppose that $c$ is based at $v$, because we can cyclically permute $[c] = [e_1 \cdots e]$ in $L_K(E)/I$ as we please, by sandwiching it between $[e_n^* \cdots e_1^*]$ and $[e_1 \cdots e_j]$.

First suppose that $z := vy'v = 0$. Write $p = \sum_{i=1}^{N} (a_i e_i^2 + b_i (c^*)^i)$ where $t_i \in \mathbb{N}$ ($t_i \neq t_j$ whenever $i \neq j$) and $a_i$ or $b_i$ is nonzero (or both). Take $\mu' = c^{i_1}, \nu' = v$ if $a_1 \neq 0$ and $\mu' = v, \nu' = c^{i_j}$ otherwise, to get, respectively, $\mu'^* \nu' = a_1 v + p'$ or $b_1 v + p''$, where $p', p''$ are polynomials in $\{c, c^*\}$ without independent term. Now, $e^* \mu'^* \nu' e = a_1 w$ or $b_1 w$ implies that $e^* \mu'^* \nu' e$ is a nonzero multiple of a vertex, as desired.

If $z \neq 0$, as before, it must be of the form $z = \sum_{j=1}^{n} k_j e_j^* p_j' q_j^*$ with $l(p_j), l(q_j) \geq 1$. Write $z = \sum_{j=1}^{n} k_j e_j^* p_j' q_j^* (c^*)^{s_j}$, where $r_j, s_j \in \mathbb{N} \cup \{0\}$ and $c$ is not a left factor of any $p_j'$ or $q_j'$. Denote $r := \max\{r_1, \ldots, r_n\} + 1$ and $s := \max\{s_1, \ldots, s_n\} + 1$ (note that $r > r_j$ and $s > s_j$).

Then
\[
(c^*)^r z c^s = \sum_{j=1}^{n} k_j (c^*)^{r-r_j} p_j' q_j^* (c^{s-s_j}).
\]

We claim that, in fact, this sum equals to zero. To see it, write $c = e_1 \cdots e_l$ and consider how must $p_j' q_j^*$ be in order for the $j$-th term to be nonzero. Since $r - r_j > 0, s - s_j > 0$ and $p_j, q_j$ cannot be of the form $p_j' = c p_j''$, $q_j' = c q_j''$ (with $p_j'', q_j'' \in \text{Path}(E)$) because $c$ is not a left factor of them, $c$ must be of the form $c = c_1' c_2', c = c_3' c_4''$ (with $c', c'' \in \text{Path}(E)$), what implies that $p_j' = e_1 \cdots e_{l_1}, q_j' = e_1 \cdots e_{l_2}$ ($l_1, l_2 < l$, with $l_1 = 0$ denoting that the path involved is actually the vertex $v$). The $j$-th term of the sum would then read
\[
k_j (c^*)^{r-r_j-1} c^* p_j' q_j^* c (c^{s-s_j-1}) = k_j (c^*)^{r-r_j-1} e^*_1 e_1 \cdots e^*_l e_l c^{s-s_j-1},
\]
which equals $k_j (c^*)^{r-r_j-1} e^*_1 \cdots e^*_{l_1+1} e_{l_1+2} \cdots e_{l_2} c^{s-s_j-1}$ due to (CK1) cancelations. Since it is not 0, it is necessary that $l_1 = l_2$. This also applies in the special case $l_1 = 0$ or $l_2 = 0$, in which case the $j$-th summand simplifies to $k_j (c^*)^{r-r_j-1} c^{s-s_j-1}$; and this is a multiple of a power of either $c$ or $c^*$ (depending on the sizes of $r, s, r_j$ and $s_j$). But then $(c^*)^r z c^s$ would be a nonzero polynomial in $\{c, c^*\}$ with $c$ a cycle with exits, what contradicts the fact that $z \in I(P_r(E))$, because of Lemma 5.1. Hence, $(c^*)^r z c^s = 0$.

To finish the proof, consider the polynomial $(c^*)^r (v \mu^* x v e) c^s = (c^*)^r (p + z) c^s = (c^*)^r p c^s$ and apply it the process defined for the case when $z = 0$.

\[\square\]

**Examples 5.3.** We illustrate the two situations of the previous result:

(i) Let $E$ be the graph represented by:

```
  u  \
  \--f--\  \
     |     \  \
     v  \  \
       \  \
         \  \
   w
```


Consider the idempotent \( x = ff^* + w \) of \( L_K(E) \setminus I(P_c(E)) \), whose class in \( L_K(E) / I(P_c(E)) \) is \([x] = [ff^*] = [v] \). Now, \([v][x][v] = [v] \) means that we can take \( \mu = v = v \). In \( L_K(E) \), 
\[ \mu^* x v = v x v = v (f f^* + w) v = f f^* + v - v v^* \neq v \ (v v^* \in I(P_c(E))) \]. Finally, sandwiching by \( f^* \) and \( f \), we get that 
\[ f^* \mu^* x v f = f^* f f^* f = u \in \mathbb{E}^0. \]

(ii) Now let \( E \) be the graph represented by

and consider the element

\[ x = c^5 + c^2 + ee^* + c^3 e f e^* (c^*)^5 + ce_1 e_2 + ce_1 e_2 e^*_2 e_1^* (c^*)^2 \]

of \( L_K(E) \setminus I(P_c(E)) \), where \( c = e_1 e_2 e_3 \). Its class in \( L_K(E) / I(P_c(E)) \) is \([x] = [c^5 + c^2 + c e_1 e_2 + c^2] \), since \([e_1 e_2^*] = [v_1], [e_2 e_2^*] = [v_2] \). We can take \( \mu = v = v \) to get
\[ \mu^* x v = [v x v] = [c^5 + c^2 + c + c^*] \], which is a polynomial \([p] \) in \([c^5 + c^2 + c] \) without exits. In \( L_K(E) \),
\[ \mu^* x v = c^5 + c^2 + ee_1 e^*_1 + ce_1 e_2 e^*_2 e_1^* (c^*)^2 + c^3 e f e^* (c^*)^5 + ee^* = \]
\[ = p + y + c^3 e f e^* (c^*)^5 + ee^*, \]
where \( p \) is the already mentioned polynomial \( c^5 + c^2 + c + c^* \) and \( y \in I(P_c(E)) \) is such that \( p + y = c^5 + c^2 + c e_1 e_2 + c^2 + c e_1 e_2 e_2^* e_1^* (c^*)^2 \), the monomials which give rise to \([p] \) in \( L_K(E) \setminus I(P_c(E)) \). So, we must have \( y = c^5 + c^2 + c e_1 e_2 + c e_1 e_2 e_2^* e_1^* (c^*)^2 - p = (c e_1 e_1^* - c) + (c e_1 e_2 e_2^* e_1^* (c^*)^2 - c^*). \) By an application of (CK2), we see, on the one hand, that \( c = cc = c e_1 e_1^* + c e_1 e_2 e_2^* e_1^* \) so that \( A = c e_1 e_1^* - c = -c e_1 e_1^* \).

On the other hand, by adding \( c (c^2)^2 - c (c^2)^2 \) to \( B \), we get \( B = c e_1 e_2 e_2^* e_1^* (c^*)^2 - c^* = c e_1 e_2 e_2^* e_1^* (c^*)^2 - c^* \). Taking into account that, by (CK2), \( v = e_1 e_1^* + e e^* = e_1 e_2 e_2^* + e c e_1 e_2 e_2^* e_1^* (c^*)^2 - c^* \). Taking into account that, by (CK2), \( v = e_1 e_1^* + e e^* = e_1 e_2 e_2^* + e e^* \), we get:

(i) \( c (c^2)^2 = v (c^2)^2 = c e_1 e_2 e_2^* e_1^* (c^*)^2 + c e_1 e_2 e_2^* e_1^* (c^*)^2 + c e_1 e_2 e_2^* e_1^* (c^*)^2 = c e_1 e_2 e_2^* e_1^* (c^*)^2 \).

(ii) \( c^* = v c^* = e_1 e_2 e_2^* e_1^* (c^*)^2 + c e_1 e_2 e_2^* e_1^* (c^*)^2 = c e_1 e_2 e_2^* e_1^* (c^*)^2 \), that is \( c (c^*)^2 - c^* = e_1 e_2 e_2^* e_1^* (c^*)^2 - e e^* \).

(iii) Therefore, \( B = -c e_1 e_2 e_2^* e_1^* (c^*)^2 + c e_1 e_2 e_2^* e_1^* (c^*)^2 - c^* \), and \( c (c^*)^2 = e_1 e_2 e_2^* e_1^* (c^*)^2 - c^* \).

Then, \( v x v = p + A + B + c^3 e f e^* (c^*)^5 + e e^* = p + c e e^* - c e_1 e_2 e_2^* e_1^* (c^*)^2 - c e_1 e_2 e_2^* e_1^* (c^*)^2 - c e_1 e_2 e_2^* e_1^* (c^*)^2 - c^* = p - c e_1 e_2 e_2^* e_1^* (c^*)^2 - c e_1 e_2 e_2^* e_1^* (c^*)^5 + z^* \). 

We do not really have to take into account \( z^* = -c^* + e_1 e_2 e_2^* e_1^* (c^*)^2 + c^* \). Because \( c^* z^* = 0 \). Now we take \( r = max\{1, 2, 3\} = 4 \) and \( s = max\{2, 2, 5\} = 6 \), so that
\[ (c^*)^r v u x v v c^* = \]
\[ = (c^*)^4 p c^6 + (c^*)^4 c e_1 e_2 e_2^* e_1^* (c^*)^2 c^6 = (c^*)^4 c e_1 e_2 e_2^* e_1^* (c^*)^2 c^6 \]
\[ = (c^*)^4 p c^6 = c + c^2 + c^4 + c^7, \]

because \( c e e = 0 \) and \( (c^* c^2 e_1 g = c e_1 e_2 e_2^* e_1 g = c e_1 e_2 e_2^* g = 0 \). Hence, \( t_1 = 1 \) for the monomial \( c \) and we just have to multiply by \( (c^*)^4 = c e e^* \) on the left to get the polynomial
that for one of them, say \( v \), Proposition 5.4 above, \( \phi \) Lemma 3.1, \( \text{vertex, since those are sinks; thus } v \).

Proof. Since \( \phi \) Remark 4.7. Thus, \( \phi \)

Proposition 5.4. Let \( L_K(E) \) be a Leavitt path algebra:

(i) If \( E \) is an arbitrary graph and \( v \in E^0 \) is a truly primitive vertex of \( L_K(E) \) then \( v \in I(P_c(E)) \).

(ii) If \( E \) is a row-finite graph and \( x \in L_K(E) \) is a truly primitive idempotent then \( x \in I(P_c(E)) \).

Proof. Since \( I(P_c(E)) \) has no bifurcations and ends in a cycle without exits (Proposition 4.3) with base, say, \( w \in E^0 \). Then \( w \in I(P_c(E)) \). The saturated condition on \( I(P_c(E)) \cap E^0 = \bar{P}_c(E) \) and the fact that \( T(v) \) has no bifurcations give \( v \in I(P_c(E)) \).

(ii) Suppose \( x \not\in I(P_c(E)) \) is a truly primitive idempotent of \( L_K(E) \). Then by Proposition 5.2 there exist \( \mu, \nu \in \text{Path}(E) \), \( v \in E^0 \) and \( k \in K \setminus \{0\} \) such that \( \mu^*x\nu = kv \). By Proposition 4.9 \( L_K(E)v \cong L_K(E)x \), so that \( v \) is a truly primitive idempotent and whence \( v \in I(P_c(E)) \) by (i). Defining, as in Proposition 4.9, \( a = \frac{1}{k}\mu^*x, b = x\nu \), we see that \( x = e = bva \in I(P_c(E)) \) is a contradiction that finishes our proof.

This proposition implies that \( I(P_c(E)) \) is an invariant of ring isomorphisms between Leavitt path algebras of row-finite graphs:

Proposition 5.5. If \( E, F \) are row-finite graphs and \( \phi : L_K(E) \to L_K(F) \) is an isomorphism of rings, then \( \phi(I(P_c(E))) = I(P_c(F)) \).

Proof. \( I(P_c(E)) \) is generated by the vertices of \( P_c(E) \), which are truly primitive idempotents by Remark 4.7. Thus, \( \phi(P_c(E)) \) is a set of truly primitive idempotents of \( L_K(F) \) as well. By Proposition 5.4 above, \( \phi(P_c(E)) \subseteq I(P_c(F)) \) and so \( \phi(I(P_c(E))) \subseteq I(P_c(F)) \). By symmetry, we get \( \phi(I(P_c(E))) = I(P_c(F)) \) as desired.

5.2. The general case by means of dual graphs. The jump to the general case passes through the dual graph techniques introduced in Section 2 (in particular, \( \theta \) will denote the monomorphism defined in Proposition 2.8).

To start, we will see that the ideal generated by the vertices in cycles without exits of the dual of a subgraph goes, via \( \theta \), into the ideal generated by the vertices in cycles without exits of the whole graph:

Lemma 5.6. If \( E \) is an arbitrary graph and \( F \) is a row-finite subgraph of \( E \), then \( \theta(I(P_c(D_E(F)))) \subseteq I(P_c(E)) \).

Proof. Since \( I(P_c(D_E(F))) \) is generated by the set of vertices \( P_c(D_E(F)) \), it suffices to show that for one of them, say \( v \), we have \( \theta(v) \in I(P_c(E)) \). Note that \( v \) cannot be a vertex-vertex, since those are sinks; thus \( v \equiv e \) must be an edge-vertex. So, \( \theta(e) = ee^* \) where, by Lemma 3.1, \( e \) is an edge in a cycle without exits of \( E \) and therefore \( r(e) \in P_c(E) \). Now, \( ee^* = er(e)e^* \in I(P_c(E)) \) implies the desired result.
Theorem 5.7 (Refinement of Reduction Theorem). If $E$ is an arbitrary graph and $x \in L_K(E) \setminus I(P_c(E))$ then there exist $k \in K \setminus \{0\}$, $\mu, \nu \in \text{Path}(E)$ and $v \in E^0$ such that $\mu^*xv = kv$.

Proof. Take $x \in L_K(E) \setminus I(P_c(E))$ and express it as a convenient sum of monomials (as in (†)). Observe that $F_x$ and $D_x$ are finite and that $x \in A_x \setminus I(P_c(E))$. An application of Lemma 5.6 shows that $\theta^{-1}(x) \not\in I(P_c(D_x))$. Now, since $D_x$ is finite and $\theta^{-1}(x) \in L_K(D_x) \setminus I(P_c(D_x))$, by the refinement of the Reduction Theorem for the row-finite case (Proposition 5.2), there exist $k \in K \setminus \{0\}$, $\mu, \nu \in \text{Path}(D_x)$ and $v \in D^0_x$ such that $\mu^*\theta^{-1}(x)v = kv$ (†). We will see that this relation in $L_K(D_x)$ provides us with a similar one in $L_K(E)$.

Note that $r(\mu) = v = r(\nu)$. Write $\mu = (\mu_1, \mu_2)(\mu_2, \mu_3) \ldots (\mu_n, y)$ and $\nu = (\nu_1, \nu_2)(\nu_2, \nu_3) \ldots (\nu_m, y)$, with $(\mu_i, \mu_{i+1}), (\nu_j, \nu_{j+1})$ and $(\mu_n, y), (\nu_m, y)$ edges of $D_x$, $\mu, \nu$ edges of $F_x$ and $v \in F^0_x \cup F^1_x$. There exist three possible cases for $y$:

(i) $v$ comes from a sink of $F_x$. In this case $y = v$ with $\theta(y) = u_v = v$ and $\theta(\mu) = \mu_1\mu_2\mu_2\mu_3\mu_3 \ldots \mu_nv = \mu_1\mu_2 \ldots \mu_nv$, $\theta(\nu) = \nu_1 \ldots \nu_m$. Thus, the application of $\theta$ to $v$ gives

$$(\mu_1 \ldots \mu_n)^*x(\nu_1 \ldots \nu_n) = kv.$$  

(ii) $v$ comes from an intermediate vertex of $F_x$. In this case $y = v$ with $\theta(y) = u_v = v - \sum_{s(e) = \nu, e \in F_x} ee^* \text{ and } \theta(\mu) = \mu_1\mu_2 \ldots \mu_nv$, $\theta(\nu) = \nu_1\nu_2 \ldots \nu_m$, so that

$$(\mu_1\mu_2 \ldots \mu_nv)^*x(\nu_1\nu_2 \ldots \nu_nv) = kv.$$  

Since $v$ comes from an intermediate vertex of $F_x$, there exist $f \in E^1 \setminus F^1_x$ such that $s(f) = v$; this means that $u_vf = f, f^*u_v = f^*$ and $f^*u_vf = r(f)$, which implies that

$$f^*(\mu_1\mu_2 \ldots \mu_nv)^*x(\nu_1\nu_2 \ldots \nu_nv)f = kf^*u_vf$$

gives us finally

$$(\mu_1\mu_2 \ldots \mu_nv)^*x(\nu_1\nu_2 \ldots \nu_nv)f = kr(f).$$

(iii) $v \equiv e$ comes from an edge of $F_x$. In this case $y = e$ with $\theta(y) = ee^*$ and $\theta(\mu) = \mu_1 \ldots \mu_nee^*$, $\theta(\nu) = \nu_1 \ldots \nu_me^*$, so that

$$(\mu_1 \ldots \mu_nee^*)^*x(\nu_1 \ldots \nu_me^*) = kee^*.$$  

Now, sandwiching with $e^*, e$ and knowing that $e^*e = r(e)$ we get

$$(\mu_1\mu_2 \ldots \mu_nv)^*x(\nu_1\nu_2 \ldots \nu_nv) = kr(e).$$

We are now in condition to prove that no truly primitive idempotent of any Leavitt path algebra can live outside the ideal generated by the vertices in cycles without exits. Moreover, we will see immediately that this ideal can be used for classification purposes, as it remains invariant under isomorphisms of rings:

Corollary 5.8. If $E$ is an arbitrary graph and $x \in L_K(E)$ is a truly primitive idempotent then $x \in I(P_c(E))$.

Proof. As in Proposition 5.4 but building on Theorem 5.7.

Theorem 5.9. If $E, F$ are arbitrary graphs and $\phi : L_K(E) \to L_K(F)$ is an isomorphism of rings, then $\phi(I(P_c(E))) = I(P_c(F))$.

Proof. As in Proposition 5.5 but building on Theorem 5.7.
The structure of $I(P_c(E))$ was determined for the Leavitt path algebras of row-finite graphs in [5, Proposition 3.5 (iii)]. We show that this structure is essentially the same for any Leavitt path algebra. We will need the definition which follows ([21, Definition 1.3]). We note that in spite of it coming from the row-finite context, it is a particular case of the general definition (specifically, the case with $S = \emptyset$) as stated in [16], where it is used to study the structure of the graded ideals of an arbitrary Leavitt path algebra.

Let $E$ be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Define

$$F_E(H) = \{ \mu = \mu_1 \ldots \mu_n \mid \mu_i \in E^1, s(\mu_i) \in E^0 \setminus H \text{ for } i \leq n, r(\mu_n) \in H \}.$$ 

Denote by $\overline{F}_E(H)$ another copy of $F_E(H)$. For $\mu \in F_E(H)$, we write $\overline{\mu}$ to denote a copy of $\mu$ in $\overline{F}_E(H)$. Then, we define the graph $H E = (\mu E^0, H E^1, s', r')$ as follows:

$$(H E)^0 = H \cup F_E(H)$$

$$(H E)^1 = \{ e \in E^1 \mid s(e) \in H \} \cup \overline{F}_E(H).$$

For every $e \in E^1$ with $s(e) \in H$, $s'(e) = s(e)$ and $r'(e) = r(e)$.

For every $\overline{\mu} \in \overline{F}_E(H)$, $s'(\overline{\mu}) = \mu$ and $r'(\overline{\mu}) = r(\mu)$.

We also need to know that an infinite path in a graph $E$ is an infinite sequence of edges $\mu = e_1 e_2 \ldots$ such that $s(e_{i+1}) = r(e_i)$ for every $i \in \mathbb{N}$, and that an infinite path is said to end in a cycle if there exist a cycle $c$ and an $n \in \mathbb{N}$ such that $\mu = e_1 \ldots e_n c c c \ldots$.

**Theorem 5.10.** Let $E$ be an arbitrary graph. Then $I(P_c(E)) \cong \bigoplus_{j \in J} \mathbb{M}_{n_j}(K[x, x^{-1}])$, where $n_j \in \mathbb{N} \cup \{\infty\}$ and $J$ is an arbitrary set.

**Proof.** Suppose $I(P_c(E)) \neq 0$, otherwise the result becomes trivial. By [14, Lemma 2.1] (which first part is valid in full generality), $I(P_c(E)) = I(H)$, where $H = \overline{P}_c(E)$, and by [16, Proposition 3.7] (taking $S = \emptyset$ in that proposition), $I(H) \cong L_K(H E)$. Thus, we can reduce the problem to study the structure of $L_K(H E)$. We claim that it is a locally noetherian Leavitt path algebra with zero socle, and hence the result follows from [5, Theorem 3.7 (iv)] (dropping the countability result on the index sets, which comes exclusively as a result of the authors restricting their context to countable graphs).

To prove this, we will show that $H E$ is row-finite with no sinks, satisfies Condition (NE) and that any of its possible infinite paths must end in a cycle (whence $L_K(H E)$ satisfies [5, condition (iii) of Theorem 3.7]). The latter condition is easily seen to be true by construction. Also by construction, $H E$ contains no sinks, as any vertex in $H E$ must connect to some vertex in $P_c(E)$, and hence $\text{Soc}(L_K(H E)) = I(P_c(H E)) = 0$. That $H E$ is row-finite is proved as follows: as $H = \overline{P}_c(E)$, by the inductive construction of the hereditary saturated closure, $H$ (and subsequently $H E$) could contain an infinite emitter $v$ if and only if $v \in \Lambda_0(P_c(E)) = T(P_c(E))$. But since $P_c(E)$ is hereditary, $T(P_c(E)) = P_c(E)$, and since it contains no bifurcations (recall that it is formed by the vertices which are base of cycles without exits), it cannot contain any infinite emitter.

It remains to show that $H E$ satisfies Condition (NE). Suppose on the contrary that there exists a cycle with exits, $c$, in $H E$. By the definition of $H E$, $c$ must be a cycle with vertices in $H$. But this is impossible by Lemma 5.1. \qed
Note that this structure theorem keeps further the parallelism between $I(P_c(E))$ and $Soc(L_K(E))$, since by [13, Theorem 5.6] (dropping again the countability assumption),

$$Soc(L_K(E)) = I(P_c(E)) \cong \bigoplus_{j \in J} \mathbb{M}_{n_j}(K),$$

where $n_j \in \mathbb{N} \cup \{\infty\}$ and $J$ is an arbitrary set.

It is well-known that if $x \in Soc(R)$ for a semiprime ring $R$, then the right (resp. left) $R$-module $xR$ (resp. $Rx$) is semisimple. To finish, we use again the facts showed in the proof above to establish that the $L_K(E)$-modules generated by vertices of $I(P_c(E))$ can be written as a direct sum of indecomposable modules:

**Proposition 5.11.** Let $E$ be an arbitrary graph, and let $v \in I(P_c(E))$. Then $vL_K(E)$ (resp. $L_K(E)v$) is completely decomposable as a right (resp. left) $L_K(E)$-module.

**Proof.** Since $P_c(E)$ has no sinks and no infinite emitters, every $v \in I(P_c(E))$ is regular. Let $s^{-1}(v) = \{e_i\}_{i=0}^n$. By [11, Proof of Lemma 7.3], $vL_K(E) \cong \bigoplus_{i=0}^n r(e_i)L_K(E)$. Knowing that the only infinite paths in $I(P_c(E))$ end in a cycle, we can repeat this process (now with every $r(e_i)$) until every vertex in our sum is the base of a cycle without exits, what happens in a finite number of steps, to get $vL_K(E) \cong \bigoplus_{i=0}^n u_i L_K(E)$ with $u_i \in P_c(E)$. Now, by Proposition 4.3, every $u_i$ is primitive ($T(u_i)$ has no bifurcations) and hence every $u_iL_K(E)$ is an indecomposable left $L_K(E)$-module, proving the claim. The argument for $L_K(E)v$ is analogous. \[\square\]

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