ON THE MEMBERSHIP IN BERGMAN SPACES OF THE DERIVATIVE
OF A BLASCHKE PRODUCT WITH ZEROS IN A STOLZ DOMAIN

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Abstract. It is known that the derivative of a Blaschke product whose zero sequence lies in a Stolz angle belongs to all the Bergman spaces $A^p$ with $0 < p < 3/2$. The question of whether this result is best possible remained open. In this paper, for a large class of Blaschke products $B$ with zeros in a Stolz angle, we obtain a number of conditions which are equivalent to the membership of $B'$ in the space $A^p$ ($p > 1$). As a consequence, we prove that there exists a Blaschke product $B$ with zeros on a radius such that $B' \notin A^{3/2}$.

1. Introduction. We denote by $\mathbb{D}$ the unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$ and by $H^p$ $(0 < p \leq \infty)$ the classical Hardy spaces of analytic functions in $\mathbb{D}$ (see [3]). The Bergman space $A^p$ $(0 < p < \infty)$ consists of all functions $f$ analytic in $\mathbb{D}$ which belong to $L^p(\mathbb{D}, dA)$, where $dA(z) = \frac{1}{\pi} dx dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We mention [4] and [6] as general references for the theory of Bergman spaces.

A sequence $\{a_n\}$ of points in $\mathbb{D}$ is said to be a Blaschke sequence if $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$. The corresponding Blaschke product $B$ is defined as $B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n - z}$.

If $\xi \in \partial \mathbb{D}$ and $\sigma \in (1, \infty)$, we set $\Omega_\sigma(\xi) = \{ z \in \mathbb{D} : |1 - \xi z| \leq \sigma (1 - |z|) \}$. The domains $\Omega_\sigma(\xi)$ $(1 < \sigma < \infty)$ are called Stolz angles with vertex at $\xi$. The domain $\Omega_\sigma(1)$ will be simply denoted by $\Omega_\sigma$.

If a Blaschke product $B$ has zeros $a_n = r_n e^{i\theta_n}$, we define

$$f_B(t) = \sum_{a_n \neq 0} \frac{1 - |a_n|}{(1 - |a_n|)^2 + (t - t_n)^2}, \quad t \in (-\pi, \pi).$$

Ahern and Clark ([2], Lemma 1, p. 121) proved that

$$B' \in H^p \iff f_B \in L^p(-\pi, \pi), \quad 0 < p < \infty.$$  \hspace{1cm} (1)

Using this criterion we can deduce:

(i) If the zeros of a Blaschke product $B$ all lie in some Stolz angle, then $B' \in \cap_{0 < p < 1/2} H^p$.

(ii) If $B$ is the Blaschke product with zeros $a_n = 1 - 1/(n \log^2 n)$, $n \geq 2$, then $B' \notin H^{1/2}$.

2. The main results. Even though we do not have a Bergman space analogue of (1), using Theorem 6.1 of [1] (see also Theorem 3 of [5]), it follows that if the zeros of a Blaschke product $B$ all lie in some Stolz angle, then $B' \in A^p$ for all $p \in (0, 3/2)$. We shall prove that the exponent $3/2$ is sharp in this result even for Blaschke products with zeros on a radius.

Theorem 1. The Blaschke product $B$ with zeros $a_n = 1 - 1/(n \log^2 n)$, $n \geq 2$, has the property that $B' \notin A^{3/2}$.

For a large class of Blaschke products $B$ with zeros in a Stolz angle, we shall obtain a number of conditions which are equivalent to the membership of $B'$ in the space $A^p$ $(1 < p < \infty)$. Theorem 1 will follow from these results. We remark that if $B$ is an arbitrary
infinite Blaschke product, $B^p \not\in A^p$ for any $p \geq 2$ (see Theorem 1.1 of [7]). Hence, our coming results are really significant only for $3/2 \leq p < 2$.

Following Vinogradov [9], if $B$ is the Blaschke product with zeros $\{a_n\}_{n=1}^{\infty}$, we define

$$\varphi_B(\theta) = \sum_{n \neq 0} \frac{1 - |a_n|}{\theta + (1 - |a_n|)^2}, \quad \theta \in (0, \infty).$$

We shall prove the following result.

**Theorem 2.** Let $B$ be a Blaschke product whose sequence of zeros lies in a Stolz angle. If there exist a positive constant $C$ and $\theta_0 \in (0, \pi)$ such that

$$\theta \varphi_B(\theta) \geq C \quad \text{for all} \quad \theta \in (0, \theta_0),$$

then, for any given $p \in (1, \infty)$, we have that $B' \in A^p$ if and only if $\varphi_B \in L^{p-1}(0, 1)$.

Theorem 1 can be deduced from Corollary 2 below but here we give a direct proof using Theorem 2.

**Proof of Theorem 1.** If $B$ is the Blaschke product considered in Theorem 1 then

$$\varphi_B(\theta) = \sum_{n=2}^{\infty} \frac{1 - |a_n|}{\theta + (1 - |a_n|)^2} = \sum_{n=2}^{\infty} \frac{n \log^2 n}{(1 + \theta n \log^2 n)^2}. \quad \theta > 0.$$

For $0 < \theta < 1$, let $N_0$ be the unique number greater than 1 such that $\theta N_0 \log^2 N_0 = 1$. By a standard argument involving summation by parts, we have

$$\varphi_B(\theta) \geq \frac{1}{\theta} \sum_{2 \leq n \leq N_0} n \log^2 n \asymp N_0^2 \log^2 N_0 = \frac{N_0}{\theta} = \frac{1}{\theta^2 \log^2 N_0}.$$

Now, the definition of $N_0$ easily implies that $\log N_0 \sim \log \frac{1}{\theta}$, as $\theta \to 0$. Then it follows that there exist a positive constant $C$ and $\theta_0 \in (0, 1)$ such that

$$\varphi_B(\theta) \geq C \frac{1}{\theta^2 \log^2 \frac{1}{\theta}}, \quad 0 < \theta < \theta_0.$$

This implies that $\varphi_B \not\in L^{1/2}(0, 1)$. Then using Theorem 2 we deduce that $B' \not\in A^{3/2}$. $\square$

Theorem 2 follows immediately from Theorem 3.

**Theorem 3.** Suppose that $1 \leq p < \infty$ and $\sigma > 1$, and let $B$ be a Blaschke product whose zeros lie in a Stolz angle. Then there exist $C_1 > 0$, $C_2 > 0$, $M > 0$ and $\theta_0 \in (0, \pi)$ such that

$$C_1 \int_{0}^{2\pi} \varphi_B^{p-1}(\theta) d\theta \geq \int_{B} |B'(z)|^p dA(z) \geq C_2 \int_{0}^{\theta_0} \varphi_B^{p-1}(\theta) \left(1 - e^{-M \theta \log(\theta)}\right) d\theta.$$

A number of results which will be needed to prove Theorem 3. The pseudo-hyperbolic metric in the unit disc will be denoted by $\varrho$: $\varrho(z, w) = \left|\frac{z - w}{1 - \overline{w} z}\right|$, $z, w \in \mathbb{D}$. The following result, which is due to Marshall and Sarason, is proved in Proposition 4 of [8].

**Proposition A.** Let $K$ be a closed convex subset of $\overline{\mathbb{D}}$ with $0 \in K$. Let $B$ be a Blaschke product whose zeros $\{a_n\}$ are all contained in $K$. If $z \in \mathbb{D} \setminus K$ and $\varepsilon = \varrho(z, K)$, then

$$|B'(z)| \geq \frac{2\varepsilon}{1 + \varepsilon^2} \frac{|B(z)|}{1 - |z|^2} \sum_{n=1}^{\infty} (1 - \varrho^2(z, a_n)).$$

The following lemma can be proved using simple geometric arguments.

**Lemma 1.** Given $\sigma > 1$ and $0 < \delta < 1$ there exists $\overline{\sigma} > \sigma$ such that $\rho(z, \Omega_{\sigma}) \geq \delta$ for every $z \in \mathbb{D} \setminus \Omega_{\overline{\sigma}}$. 
LEMMA 2. Let $B$ be the Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^{\infty}$ and let $\delta \in (0,1)$. If $z \in \mathbb{D}$ satisfies that $\rho(z,a_n) \geq \delta$, for all $n$, then

\begin{equation}
|B(z)| \geq \exp \left( -\frac{1}{2\delta^2} \sum_{n=1}^{\infty} (1 - \rho^2(z,a_n)) \right) .
\end{equation}

Proof. Take $z \in \mathbb{D}$ such that $\rho(z,a_n) \geq \delta$ for all $n = 1, 2, \ldots$, then using the elementary inequality $\log x \leq x - 1$, for $x \geq 1$, we deduce that

$$
\log \frac{1}{|B(z)|} = \frac{1}{2} \sum_{n=1}^{\infty} \log \frac{1}{\rho^2(z,a_n)} \leq \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{\rho^2(z,a_n)} - 1 \right) \leq \frac{1}{2\delta^2} \sum_{n=1}^{\infty} (1 - \rho^2(z,a_n)),
$$

which implies (5). $\square$

We shall use also the following two elementary lemmas.

LEMMA 3. Given $R \in (0,1)$, there exists $C_R \in (0,1)$ such that

\begin{equation}
C_R [(1-r) + (1-\rho) + |t|] \leq (1-r) + (1-\rho) + |t| , \quad r, \rho \in [R, 1) \ t \in [-\pi, \pi].
\end{equation}

LEMMA 4. If $\sigma > 1$ then $\frac{1}{2\pi \sigma} \leq \frac{|1-\frac{x}{1-\rho}|}{1-|\rho|^2} \leq 2 + \sigma$, whenever $z \in \mathbb{D}$ and $\lambda \in \Omega_\sigma$.

Proof of Theorem 3. Take $p \geq 1$ and assume, without loss of generality, that $B$ is a Blaschke product with $B(0) \neq 0$ whose sequence of zeros $\{a_n\}_{n=1}^{\infty}$ lies in the Stolz angle $\Omega_\sigma$ ($\sigma > 1$). Write $\varphi$ for $\varphi_B$.

There exists $R \in (0,1)$ such that $|a_n| \geq R$, for all $n$. Let $C_R$ be the constant associated to $R$ by Lemma 3. Fix a number $\delta \in (0,1)$. Using Lemma 1, we can take $\overline{\sigma} > \sigma$ such that $\rho(z,\Omega_\overline{\sigma}) \geq \delta$, for all $z \in \mathbb{D} \setminus \Omega_\overline{\sigma}$. Using Proposition A with $K = \overline{\Omega_\sigma}$ and bearing in mind that the function $x \mapsto \frac{2\delta}{1+\delta x}$ is increasing in $(0,1)$, we obtain that, for every $z \in \mathbb{D} \setminus \Omega_\sigma$,

$$
|B'(z)| \geq \frac{2\delta}{1+\delta |z|^2} \frac{|B(z)|}{1-|z|^2} \sum_{n=1}^{\infty} (1 - \rho^2(z,a_n)) \geq \frac{2\delta}{1+\delta |z|^2} |B(z)| \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n}z|^2}.
$$

If $z \in \mathbb{D} \setminus \Omega_\sigma$, then $\rho(z,a_n) \geq \delta$ for all $n$. Lemma 2 and the above inequality yield

\begin{equation}
|B'(z)| \geq \frac{2\delta}{1+\delta^2} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n}z|^2} \exp \left( -\frac{1}{2\delta^2} \sum_{n=1}^{\infty} \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a_n}z|^2} \right) , \quad z \in \mathbb{D} \setminus \Omega_\sigma.
\end{equation}

Using (7), Lemma 4 and Lemma 3, we see that if $z = re^{it} \in \{z \in \mathbb{D} : |z| \geq R\} \setminus \Omega_\sigma$

\begin{align}
|B'(z)| & \geq \frac{2\delta}{1+\delta^2} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n}z|^2} \exp \left( -\frac{1}{2\delta^2} \sum_{n=1}^{\infty} \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a_n}z|^2} \right) \\
& \geq \frac{2\delta}{1+\delta^2} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n}z|^2} \exp \left( -\frac{2\delta}{2\delta^2} \sum_{n=1}^{\infty} \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a_n}z|^2} \right) \\
& \geq \frac{2\delta}{1+\delta^2} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n}z|^2} \exp \left( -\frac{2\delta}{2\delta^2} \sum_{n=1}^{\infty} \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a_n}z|^2} \right) \\
& \geq \frac{2\delta}{1+\delta^2} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n}z|^2} \exp \left( -\frac{2\delta}{2\delta^2} \sum_{n=1}^{\infty} \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a_n}z|^2} \right) \\
& = A \varphi ((1-r) + |t|) \exp (-K(1-r)\varphi ((1-r) + |t|)),
\end{align}

where $A$ and $K$ are two positive constants. Observe that there exists a positive constant $\beta$ such that

\begin{equation}
|t| \geq \beta (1-r), \quad \text{for } z = re^{it} \in \{z \in \mathbb{D} : |z| \geq R\} \setminus \Omega_\sigma.
\end{equation}

Take $R_0 \geq R$ such that $(\beta + 1)(1-R_0) \leq \pi$. Using (8), making three consecutive changes of variable: $\theta = \theta(t) = 1 - r + t$, $u = u(r) = 1 - r$, $x = x(u) = u \varphi(\theta)$ and using Fubini’s
where,

\[ \frac{A}{A + 1}. \]

Then the following conditions are equivalent:

1. \( A \leq 0 \)
2. \( \frac{A}{A + 1} \)
3. \( (1 - r) \varphi ((1 - r) + t) \exp (-Kp(1 - r) \varphi ((1 - r) + t)) dt dr \]
4. \( \int_0^1 \varphi^p(\theta) \exp (-K p \varphi(\theta)) d\theta d\theta \]
5. \( \int_0^{2\pi} \varphi^{p-1}(\theta) d\theta \left( \int_0^1 \varphi^{p}(\theta) d\theta \right) \leq \frac{2 A}{K p} \int_0^{2\pi} \varphi^{p-1}(\theta) d\theta. \]

Since \( \int_0^1 \varphi^p r^p dA(z) \) increases with \( r \), this implies the first inequality of (4). □

**Corollary 1.** Suppose that \( 1 < p < \infty \) and \( B \) is a Blaschke product whose zeros lie in a Stolz angle and with the property that there exist \( C > 0 \) and \( b_0 \in (0, \pi) \) such that (3) holds. Then the following conditions are equivalent:

(a) \( B' \in A^p \).
(b) \( \varphi_B \in L^{p-1}(0, \pi) \).
(c) \( B' \in H^{p-1} \).
(d) \( f_B \in L^{p-1}(-\pi, \pi) \).
Proof. Theorem 2 shows that (a) ⇔ (b). The equivalence (c) ⇔ (d) follows from Lemma 1 and, the implication (c) ⇒ (a) follows from Theorem 6.1 of [1].

To prove that (a) ⇒ (c), suppose that $B$ is a Blaschke product with $B(0) \neq 0$, $B' \in A^p$ and such that its zeros $\{a_n\}$ lie in $\Omega_\sigma$ for a certain $\sigma > 1$. Write $a_n = |a_n|e^{i\theta_n}$ with $|\theta_n| \leq \pi$. Since $\{a_n\} \subset \Omega_\sigma$, there exists a positive constant $\lambda$ such that $|\theta_n| \leq \lambda(1 - |a_n|)$, $n = 1, 2, \ldots$. We have $|(1 - |a_n|) + |\theta|^2 \leq 2|(1 - |a_n|)^2 + \theta^2|$, and $\theta^2 \leq 2 (|\theta - \theta_n|^2 + \theta_n^2) \leq 2 (|\theta - \theta_n|^2 + \theta_n^2)$ whenever $n \geq 1$ and $\theta \in [-\pi, \pi]$. Then it follows that there exists a constant $C > 0$ such that

\begin{equation}
(13) \quad [(1 - |a_n|) + |\theta|^2 \leq C \left((1 - |a_n|)^2 + (\theta - \theta_n)^2\right), \quad n \geq 1, \quad \theta \in [-\pi, \pi].
\end{equation}

Since (a) ⇔ (b), $\varphi_B \in L^{p-1}(0, 1)$. Then (13) gives $f_B \in L^{p-1}(-\pi, \pi)$ and $B' \in H^{p-1}$. □

Condition (3) is not a simple one. Next we find a simple condition which implies it.

**Corollary 2.** If the zeros $\{a_n\}$ of Blaschke product $B$ lie in a Stolz angle and there exist $\lambda > 0$ and $n_0 \geq 1$ such that $1 - |a_{n+1}| \geq \lambda(1 - |a_n|)$, if $n \geq n_0$, then there exist $C > 0$ and $\theta_0 \in (0, \pi)$ such that (3) holds. Hence, $B' \in A^p \Leftrightarrow \varphi_B \in L^{p-1}(0, \pi)$ ($p > 1$).

**Proof.** Given $\theta \in (0, 1 - |a_{n_0}|)$ take $n \geq n_0$ such that $1 - |a_{n+1}| < \theta \leq 1 - |a_n|$. Then

\[\theta \varphi_B(\theta) \geq \frac{\theta(1 - |a_n|)}{(\theta + (1 - |a_n|))^2} \geq \frac{(1 - |a_{n+1}|)(1 - |a_n|)}{4(1 - |a_n|)^2} = \frac{(1 - |a_{n+1}|)}{4(1 - |a_n|)} \geq \frac{\lambda}{4}.
\]

Hence, we have proved (3) with $C = \lambda/4$ and $\theta_0 = 1 - |a_{n_0}|$. □

**References**


