# On the Convexity of the System Loss Function

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***Abstract*— We show that the system loss function in a power network is bounded below by any number of supporting hyper-planes in the space of generalized injections. A supporting hyper-plane is defined by the linear Taylor series expansion of the system loss function around a given operating point. The supporting hyper-plane property is valid provided that the expansion point is sufficiently near the flat-voltage profile (FVP). We have assessed experimentally the range of validity of this assumption, called the range of hyper-plane support (RHS), showing that for typical networks, the RHS is broad, particularly when the bus voltages are controlled near 1 per unit. The supporting hyper-plane model was also tested as part of an economic dispatch with transmission losses to demonstrate that this linear model provides the same results as when the losses are treated as a non-linear function.**

***Index Terms*—System loss, load flow, linear loss approximation, supporting hyper-planes, convexity, loss formulae.**

## I. Nomenclature

***Definition:*** Flat Voltage Profile (FVP) is the operating condition under which all complex bus voltages are equal to some arbitrary level, $e_{fvp} + jf_{fvp}$ in rectangular coordinates or $V_{fvp}\angle\delta_{fvp}$ in polar coordinates, where $e_{fvp} = V_{fvp}\cos\left(\delta_{fvp}\right)$ and $f_{fvp} = V_{fvp}\sin\left(\delta_{fvp}\right)$. Without loss of generality, we assume that bus $N$ is the reference with $\delta_N = 0$ or, equivalently, with $f_N = 0$. This then implies that $\delta_{fvp} = 0$ and that $f_{fvp} = 0$.

***Parameters:***

- $N$ Number of network buses;
- $n$ Number of load flow equations and unknowns = 2$N$-1;
- $\mathbf{G}$ Real part of the network admittance matrix;
- $\mathbf{G}'$ Sub-matrix of **G** without the $N$-th row and column;
- $\mathbf{L}$ Block diagonal matrix formed with **G** and $\mathbf{G}'$;
- $\mathbf{B}$ Imaginary part of the network admittance matrix;
- $\mathbf{B}_{ser}$ Component of **B** due to line series admittances;
- $\mathbf{B}_{sh}$ Component of **B** due to shunt admittances;
- $\mathbf{1}$ Vector of 1's of dimension $N$;
- $\mathbf{0}$ Vector or matrix of 0's;
- $\mathbf{1}_i$ $N$-dimensional vector of 0's with 1 at position $i$;
- $\Omega_i$ Set of nodes connected to node i.

***Variables:***

- $\bar{\mathbf{V}}$ Vector of complex nodal voltages;
- $\mathbf{e}$ Real part of $\bar{\mathbf{V}}$;
- $\mathbf{f}$ Imaginary part of $\bar{\mathbf{V}}$;
- $\mathbf{f}'$ **f** excluding the reference bus N;
- $\mathbf{V}^2$ Vector of magnitudes squared of $\bar{\mathbf{V}}$;
- $\mathbf{V}$ Vector of magnitudes of $\bar{\mathbf{V}}$;
- $\boldsymbol{\delta}$ Vector of phase angles of $\bar{\mathbf{V}}$ excluding the reference bus $N$;
- $\mathbf{x}$ $n$-dimensional vector of rectangular coordinate voltage components, $\begin{bmatrix}\mathbf{e}\\ \mathbf{f}'\end{bmatrix}$;
- $\mathbf{x}_{fvp}$ Value of **x** under FVP;
- $e_{fvp}\mathbf{1}$ Real part of $\mathbf{x}$ under FVP;
- $P_i$ Real power injection at bus $i$;
- $Q_i$ Reactive power injection at bus $i$;
- $\mathbf{P}$ Vector of real power injections at all $N$ buses;
- $\mathbf{P}'$ **P** excluding the slack bus;
- $\mathbf{Q}$ Vector of reactive power injections at all $N$ buses;
- $S_P$ Set of buses with specified real power injection; i.e., all buses except for the slack.
- $S_Q$ Set of buses with specified reactive power injection;
- $S_V$ Set of buses where the voltage magnitude is specified;
- $\mathbf{z}$ Vector of generalized injections of dimension $n$ comprised of all $P_i \in S_P$, all $Q_i \in S_Q$ and all $V_i^2 \in S_V$;
- $\mathbf{z}(\mathbf{x})$ Function of **x** characterizing the generalized injections;
- $P_{loss}$ System loss;
- $P_{loss}(\mathbf{x})$ Function of **x** characterizing the system loss.

## II. Introduction

IN problems such as unit commitment, resource scheduling, or network expansion, the power network is often modeled in an approximate manner or is altogether neglected. This is done in order to reduce the number of variables and the overall computational complexity, but also to make the problem linear thereby permitting the use of powerful mixed integer linear programming tools [1]. Today, such tools can reliably solve problems with very large numbers of constraints and variables, both integer and continuous, while non-linear mixed-integer programming tools are still unreliable and with limited applications.

The simplest network models completely ignore the transmission loss and describe the network as a single node. The next level of complexity models the system transmission loss, $P_{loss}$, in the power balance equation as an approximate explicit polynomial function of the generation levels, $\mathbf{P}_g$,

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$$\sum_{i=1}^{m} P_{gi} = P_d + P_{loss}(\mathbf{P}_g) \quad (1)$$

Loss approximation formulae were extensively studied as early as the 1960's, beginning with the B-coefficients approach [2] in which the system loss function was expressed as a constant coefficient quadratic function of the power generation levels, $\mathbf{P_g}$. In the early 1980's, taking advantage of specific analytic properties of the load flow equations in rectangular coordinates [3], more exact explicit loss approximation formulae up to third order were derived based on Taylor series expansions of $P_{loss}$ in terms of $\mathbf{z}$ around a specified load flow operating point [4]. A similar result to [4] was published recently with applications to voltage stability [5]. Other authors have examined explicit loss approximations based on polynomials in the generation levels [6, 7].

It is also possible to model the loss of every transmission line in a power network by a quadratic approximation of the corresponding line phase angle difference. This leads to a variation of the DC load flow model of the form,

$$\begin{aligned} &\mathbf{P} = [\mathbf{B}']\boldsymbol{\delta} \\ &\mathbf{P_f} = \mathbf{B_f}\boldsymbol{\delta} + \mathbf{q}(\boldsymbol{\delta}) \\ &-\overline{\mathbf{P}}_\mathbf{f} \le \mathbf{P_f} \le \overline{\mathbf{P}}_\mathbf{f} \end{aligned} \quad (2)$$

which includes the line power flows, $\mathbf{P_f}$, their limits $\overline{\mathbf{P}}_\mathbf{f}$, and the approximation of the individual line real transmission losses, $\mathbf{q}(\boldsymbol{\delta})$, [8, 9].

Notwithstanding the availability of the above mentioned loss approximation formulae, when solving problems that do not involve integer decisions, such as an optimal power flow, in our experience it is questionable whether such approximations offer any significant advantage over a non-linear optimization method that accounts for the full non-linear load flow model.

In problems involving integer decisions, such as unit commitment, one disadvantage of the more accurate quadratic or higher order loss approximation formulae is that in order to make use of mixed-integer linear programming solvers, it is necessary to further decompose each non-linear term into piece-wise linear components. This step requires the introduction of new continuous and possibly integer variables, therefore adding to the overall modeling and computational complexity. On the other hand, the use of linear loss approximation formulae, although readily compatible with mixed-integer linear programming tools, is subject to greater approximation inaccuracies. One drawback of all loss approximation formulae, whether linear or higher order, is that the errors introduced by the approximation in comparison with the exact loss model cannot be easily quantified or estimated.

The main contribution of this paper is a theoretical result that had been previously hypothesized [10], but not rigorously proven or tested, namely to demonstrate that, under certain relatively weak assumptions, the non-linear system loss function, $P_{loss}$, is convex in the generalized power flow injections, $\mathbf{z}$. As such, a first order Taylor series expansion can be shown to define a lower bound approximation of the system loss linear in $\mathbf{z}$, known as a supporting hyper-plane. Such representations are not approximations in the usual linearization sense, since they also constitute a lower bound on the non-linear loss function.

This property leads to two important results: (i) As each such inequality is a necessary condition, new hyper-planes can be added without invalidating the existing ones; (ii) The error between the non-linear function and the describing supporting hyper-planes is greater than or equal to zero. Thus, the non-linear system loss behaviour can be approached with arbitrary accuracy by increasing the number of linear necessary conditions. It is also important to note that the linearized loss model based on the well-known incremental transmission loss (ITL) coefficients is a special case of a supporting hyper-plane.

Another contribution of this paper is to quantify the range of validity of the convexity property of the loss function inside of which the system losses can be modeled by a set of linear inequalities. Numerical results on systems up to 118 buses suggest that the convexity property is valid over a wide range of operating conditions, including typical operating points.

The system loss bounding theorem has also been applied to solve a lossy economic dispatch problem to illustrate the accuracy of the hyper-plane approximation.

## III. System Loss Function Supporting hyper-planes

### A. Non-Linear Load Flow Model

In load flow models the decision variables and governing equations are as follows,

$$\begin{aligned} &P_i = P_i(\mathbf{x}); \;\; i \in S_P \\ &Q_i = Q_i(\mathbf{x}); \;\; i \in S_Q \\ &V_i^2 = V_i^2(\mathbf{x}); \;\; i \in S_V \\ &P_{loss} = \sum_{i=1}^{N} P_i = \sum_{i=1}^{N} P_i(\mathbf{x}) \equiv P_{loss}(\mathbf{x}) \end{aligned} \quad (3)$$

### B. Generalized Load Flow Injections

The vector of generalized load flow injections, $\mathbf{z} \in \Re^n$, is defined by the set of $P_i$, $Q_i$ and $V_i^2$ of equation (3). As shown in Appendix A, Result 12, the relation between $\mathbf{z}$ and $\mathbf{x}$ is denoted by,

$$\mathbf{z} = \mathbf{z}(\mathbf{x}) \equiv \frac{1}{2}\mathbf{J}(\mathbf{x})\mathbf{x} \quad (4)$$

### C. System Loss Sensitivity Vector

The system loss sensitivity vector around a given operating point $\mathbf{x}^0$, denoted by $\boldsymbol{\beta}$, is defined by the sensitivities of the system loss with respect to the generalized load flow injections, $\mathbf{z}$, that is,

$$\begin{aligned} \boldsymbol{\beta} = \frac{\partial P_{loss}\left(\mathbf{z}(\mathbf{x}^0)\right)}{\partial \mathbf{z}} &= \left[\left[\frac{\partial \mathbf{z}(\mathbf{x}^0)}{\partial \mathbf{x}}\right]^T\right]^{-1} \frac{\partial P_{loss}(\mathbf{x}^0)}{\partial \mathbf{x}} \\ &= \left[\mathbf{J}^T(\mathbf{x}^0)\right]^{-1}\mathbf{L}\mathbf{x}^0 \end{aligned} \quad (5)$$

where we have used the definitions of $\mathbf{z}(\mathbf{x})$ in equation (4) and of $P_{loss}(\mathbf{x})$ in Result 5 of Appendix A.

Note that the well-known incremental transmission loss (ITL) coefficients are the components of $\boldsymbol{\beta}$ corresponding to the real power injections.

The sensitivity vector, $\boldsymbol{\beta}$, characterizes the first order Taylor series expansion of the loss function around the operating point $\mathbf{x}^0$, that is,

$$P_{loss} = P_{loss}(\mathbf{x}^0) + \boldsymbol{\beta}^T\left(\mathbf{z} - \mathbf{z}(\mathbf{x}^0)\right) + \varepsilon \tag{6}$$

where the quantity $\varepsilon$ represents the error in this linear loss approximation due to higher order terms. One of the properties of the rectangular load flow formulation derived in Appendix B is that,

$$P_{loss}(\mathbf{x}^0) = \boldsymbol{\beta}^T\mathbf{z}(\mathbf{x}^0) \tag{7}$$

which when substituted into (6) reduces the first order expansion of the system loss to,

$$P_{loss} = \boldsymbol{\beta}^T\mathbf{z} + \varepsilon \tag{8}$$

Appendix B also shows that when $\mathbf{x}^0$ is sufficiently near the FVP, the approximation error $\varepsilon$ is non-negative for any feasible **z,** that is, for any **z** for which there exists a corresponding load flow solution, **x**. These results allow us to formally enunciate the following theorem.

### *D. System Loss Bounding Theorem*

Given any operating point $\mathbf{x}^0$ sufficiently near a flat-voltage-profile such that the load flow Jacobian matrix, $\mathbf{J}(\mathbf{x}^0)$, is non-singular, then, for any feasible vector of generalized load flow injections, **z**, the corresponding system loss satisfies,

$$P_{loss} \geq \boldsymbol{\beta}^T\mathbf{z} \tag{9}$$

The proof of this theorem is detailed in Appendix B.

### *E. Corollary to the System Loss Bounding Theorem*

Under the conditions of the system loss bounding theorem, for any feasible **z** not collinear with $\mathbf{z}(\mathbf{x}^0)$,

$$P_{loss} > \boldsymbol{\beta}^T\mathbf{z} \tag{10}$$

while for any feasible **z** collinear with $\mathbf{z}(\mathbf{x}^0)$,

$$P_{loss} = \boldsymbol{\beta}^T\mathbf{z} \tag{11}$$

The proof of this corollary follows directly from Result 9 in Appendix A.

The above linear equation (11) defines what is called a supporting hyper-plane of the unknown non-linear loss function.

Since power systems normally operate relatively near the FVP, the proximity condition requiring that the expansion point $\mathbf{x}^0$ be sufficiently near the FVP is generally not a severe limitation. Experimental evidence presented in the results section backs up this statement.

### *F. Linear System Loss Model Based on Supporting Hyper-Planes*

Since (9) is a necessary condition, any number of supporting hyper-planes based on different expansion points $\mathbf{x}^0$ can be merged into a more restrictive constraint set. Let $SH$ be a set of $\boldsymbol{\beta}'s$ of which each element corresponds to a supporting hyper-plane of the system loss function. Then, the power balance equation with losses can be approximated by the following linear model,

$$\sum_{i=1}^{N} P_i = P_{loss}$$
$$P_{loss} \geq \boldsymbol{\beta}^T\mathbf{z};\ \forall \boldsymbol{\beta} \in SH \tag{12}$$

In addition, operational constraints such as $P_i^{\min} \leq P_i \leq P_i^{\max}$ can easily be added to the model while still retaining its linear nature.

## IV. Numerical Results

All quantities in this section are in per unit except for the bus voltage phase angles which are in degrees.

### *A. Examples of Supporting and Non-Supporting Hyper-planes*

Consider a 5-bus network with the following line data,

TABLE I: TEST NETWORK LINE DATA; 100MVA, 100KV BASES.

| From Bus | To Bus | Series R | Series X | Shunt B |
|---|---|---|---|---|
| 1 | 2 | 0.042 | 0.168 | 0.030 |
| 2 | 3 | 0.031 | 0.126 | 0.020 |
| 3 | 5 | 0.053 | 0.210 | 0.015 |
| 3 | 4 | 0.084 | 0.336 | 0.012 |
| 4 | 5 | 0.063 | 0.252 | 0.011 |
| 5 | 1 | 0.031 | 0.126 | 0.010 |

The FVP condition in polar coordinates is given by $\boldsymbol{\delta} = [0,0,0,0]^T$ and $\mathbf{V} = [1,1,1,1,1]^T$, where the reference angle is $\delta(5) = 0$. In terms of generalized load flow injections, the FVP is described by $\mathbf{P}' = [0,0,0,0]^T$ and $\mathbf{V}^2 = [1,1,1,1,1]^T$.

Tables II, III and IV below describe the supporting hyper-plane corresponding to a specified operating point expressed in polar coordinates. The symbols $\boldsymbol{\beta}_P$ and $\boldsymbol{\beta}_V$ in these tables denote respectively the components of $\boldsymbol{\beta}$ corresponding to the generalized injections, $\mathbf{P}'$ and $\mathbf{V}^2$.

TABLE II: EXAMPLE A: SUPPORTING HYPER-PLANE. 5-BUS SYSTEM

| Node | $\boldsymbol{\delta}^0$ | $(\mathbf{P}')^0$ | $\beta_P$ |
|---|---|---|---|
| 1 | 2.7 | 1.88 | -.001 |
| 2 | -12.5 | -6.80 | -0.184 |
| 3 | 43.1 | 12.40 | 0.411 |
| 4 | 1.3 | -1.60 | 0.030 |

TABLE III: EXAMPLE A: SUPPORTING HYPER-PLANE. 5-BUS SYSTEM

| Node | $\mathbf{V}^0$ | $(\mathbf{V}^0)^2$ | $\beta_V$ |
|---|---|---|---|
| 1 | 1 | 1 | -0.205 |
| 2 | 1 | 1 | -0.532 |
| 3 | 1 | 1 | -2.645 |
| 4 | 1 | 1 | -0.099 |
| 5 | 1 | 1 | -0.140 |

TABLE IV: EXAMPLE A: SUPPORTING HYPER-PLANE. 5-BUS SYSTEM

| Eigenvalue Number | $\lambda_i(\mathbf{L})$ | $\lambda_i(\mathbf{L}-\mathbf{H}(\boldsymbol{\beta}))$ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0.861 | 0.914 |
| 3 | 1.685 | 1.801 |
| 4 | 1.686 | 1.806 |
| 5 | 2.935 | 3.307 |
| 6 | 3.491 | 3.888 |
| 7 | 4.440 | 5.081 |
| 8 | 5.752 | 7.705 |
| 9 | 6.634 | 8.169 |

In this case, the expansion point, $\mathbf{x}^0$, shown in Table II, expressed in polar coordinates, has phase angles that are not all close to zero, one of them being equal to 43 degrees. This suggests that the requirement that the expansion point be sufficiently near the FVP is not very stringent. Moreover, as called for by the theory, Table IV shows that the eigenvalues of $\mathbf{L}$ are all positive but one. Note, in addition, that the eigenvalues of the error matrix, $\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})$, are all close to those of $\mathbf{L}$, and that the zero eigenvalue of $\mathbf{L}$ remains at zero for the error matrix.

Table V, VI and VII show an example of an operating point that does not correspond to a supporting hyper-plane. Here, two of the eigenvalues of [ $\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})$ ] are negative. A linear expansion about this operating point would therefore not be a lower bound for the system loss over all $\mathbf{z}$. Note that in this non-supporting case, the expansion point is relatively far from the FVP as evidenced from the values of the phase angles.

TABLE V: EXAMPLE B: NON-SUPPORTING HYPER-PLANE. 5-BUS SYSTEM

| Node | $\boldsymbol{\delta}^0$ | $(\mathbf{P}')^0$ | $\beta_P$ |
|---|---|---|---|
| 1 | -56.6 | -9.72 | -1.706 |
| 2 | 28.7 | 1.91 | 1.649 |
| 3 | 76.8 | 13.19 | 1.161 |
| 4 | 40.8 | 1.16 | 0.663 |

TABLE VI: EXAMPLE B: NON-SUPPORTING HYPER-PLANE. 5-BUS SYSTEM

| Node | $\mathbf{V}^0$ | $(\mathbf{V}^0)^2$ | $\beta_V$ |
|---|---|---|---|
| 1 | 1 | 1 | -7.918 |
| 2 | 1 | 1 | -9.730 |
| 3 | 1 | 1 | -1.835 |
| 4 | 1 | 1 | -1.191 |
| 5 | 1 | 1 | -7.222 |

TABLE VII: EXAMPLE B: NON-SUPPORTING HYPER-PLANE. 5-BUS SYSTEM

| Eigenvalue Number | $\lambda_i(\mathbf{L})$ | $\lambda_i(\mathbf{L}-\mathbf{H}(\boldsymbol{\beta}))$ |
|---|---|---|
| 1 | 0 | -0.404 |
| 2 | 0.861 | -0.402 |
| 3 | 1.685 | 0 |
| 4 | 1.686 | 1.724 |
| 5 | 2.935 | 2.225 |
| 6 | 3.491 | 3.238 |
| 7 | 4.440 | 11.162 |
| 8 | 5.752 | 22.740 |
| 9 | 6.634 | 25.444 |

A network with data based on the IEEE-118-bus test system with all bus voltages held at 1 per unit is considered now. Due to the large data set, only some values are provided. Table VIII shows the operating point in polar coordinates, the generalized injections, and the values of $\boldsymbol{\beta}$ at two buses. Table IX shows a subset consisting of the smallest and largest eigenvalues of $\mathbf{L}$ and of the error matrix, all of which are positive except for one which is zero.

TABLE VIII: EXAMPLE C: SUPPORTING HYPER-PLANE. 118-BUS SYSTEM

| Node | $\bar{V}^0$ | $\mathbf{z}(\mathbf{x}^0)$ | $\boldsymbol{\beta}$ |
|---|---|---|---|
| 104 | $1\angle-4.229^\circ$ | $P_{104}=-0.975$ <br> $V_{104}^2=1$ | $\beta_{P-104}=-0.0633$ <br> $\beta_{V-104}=0.1442$ |
| 109 | $1\angle 4.461^\circ$ | $P_{109}=4.423$ <br> $V_{109}^2=1$ | $\beta_{P-109}=0.0487$ <br> $\beta_{V-109}=-0.679$ |

TABLE IX: EXAMPLE C: SUPPORTING HYPER-PLANE. 118-BUS SYSTEM

| $\lambda_i(\mathbf{L})$ | $\lambda_i(\mathbf{L}-\mathbf{H}(\boldsymbol{\beta}))$ |
|---|---|
| 0.0000 | 0.0000 |
| 0.0571 | 0.0567 |
| 0.1905 | 0.1924 |
| . | . |
| 194.50 | 186.18 |
| 387.72 | 386.67 |
| 387.72 | 386.67 |

### *B. Continuity and Range of Hyper-plane Support*

The range of hyper-plane support (*RHS*) describes the set of expansion points $\mathbf{x}^0$ for which the convexity property (9) is valid. Formally, the *RHS* is a set of $\mathbf{x}^0$ defined by,

$$RHS = \left\{ \begin{array}{c} \mathbf{x}^0 / P_{loss} \geq \boldsymbol{\beta}^T \mathbf{z}; \boldsymbol{\beta} = \left[ \mathbf{J}^T(\mathbf{x}^0) \right]^{-1} \mathbf{L}\mathbf{x}^0; \mathbf{L} - \mathbf{H}(\boldsymbol{\beta}) \geq \mathbf{0} \\ P_{loss} = P_{loss}(\mathbf{x}); \mathbf{z} = \mathbf{z}(\mathbf{x}); \mathbf{x} \in \Re^n \end{array} \right\} \quad (13)$$

Since the load flow Jacobian, $\mathbf{J}(\mathbf{x}^0)$, is non-singular when $\mathbf{x}^0$ is at or near $\mathbf{x}_{fvp}$, the *RHS* is a set that includes $\mathbf{x}_{fvp}$ and extends continuously outward from $\mathbf{x}_{fvp}$ in all directions. Its boundaries are reached when $\mathbf{x}^0$ satisfies either of: (i) $\mathbf{L} - \mathbf{H}(\boldsymbol{\beta})$ becomes indefinite; (ii) $\mathbf{J}(\mathbf{x}^0)$ becomes singular. Next, some experimental tests to estimate the "size" of the *RHS* are conducted.

### C. Numerical Estimates of the Range of Hyper-plane Support

In order to quantify the RHS, a neighborhood around the FVP condition is defined,

$$\left| \delta_i - \delta_j \right| \leq \delta_{line}^{\max}; \forall i, \forall j \in \Omega_i \quad (14)$$

This condition limits the angle differences across all lines. Table X shows the results of various experiments on the 5-bus network for different values of $\delta_{line}^{\max}$, each based on 10,000 values of the expansion point, $\mathbf{x}^0$, picked uniformly randomly within the set defined by (14). As Table X shows, the range of existence of loss supporting hyper-planes is quite broad, encompassing bus angles between $\pm 90$ degrees and line angle differences up to 76.5 degrees; the latter value being the most significant. Outside the region of support, a small but non-zero percentage of the randomly generated hyper-planes does not support the loss function. Moreover, the farther from that range that a hyper-plane lies, the higher the chances that it is non-supporting.

TABLE X: RANGE OF HYPER-PLANE SUPPORT. 5-BUS SYSTEM

| Ratio of Non-supporting to Supporting hyper-planes | $\delta_{line}^{\max}$ |
|---|---|
| 0/10,000 | 76.5 º |
| 31/10,000 | 81º |
| 202/10,000 | 90 º |

The results for the 118-bus network with 1000 randomly generated expansion points are shown in Table XI.

TABLE XI: RANGE OF HYPER-PLANE SUPPORT. 118-BUS SYSTEM

| Ratio of Non-supporting to Supporting hyper-planes | $\delta_{line}^{\max}$ |
|---|---|
| 0/1,000 | 68.04 º |
| 7/1,000 | 70 º |
| 73/1,000 | 80 º |

These results indicate that if the absolute line angle differences are less than 68.04º, then, the corresponding hyper-plane is likely to be supporting. Note that experiments indicate that the range of hyper-plane support remains essentially unaffected if the bus voltages are kept within plus or minus 5% of nominal.

## V. APPLICATION: ECONOMIC DISPATCH WITH LOSSES

The general theory presented in this paper is suitable for use in many different practical problems where losses are an important issue; examples being unit commitment, economic-dispatch and hydro-thermal coordination. Note however that the supporting hyper-plane approach is not intended to be used to solve problems with constraints on dependent variables such as line flows, load bus voltage magnitudes, or reactive generation levels. Such problems should be solved with an optimal power flow

For the sake of illustration, the proposed general theory is tested solving a lossy economic dispatch problem; the system used is the previously presented 118-bus power system; with 49 generating buses, 58 load buses and 179 lines.

### A. Solving the economic dispatch.

Note that the economic dispatch problem, when modeled with supporting hyper-planes, consists of the objective function, the bounds for the generation, one inequality constraint per hyper-plane and an additional equality constraint stating that the summation of generation must equal the summation of demand plus losses; thus, no explicit account for the network is needed. The lossy economic dispatch may be solved *exactly* using hyper-planes as follows **(Algorithm 1)**:

**Step 0.** Initialize a working set of hyper-planes with the trivial inequality constraint, that is, $P_{loss} \geq 0$.

**Step 1.** Solve the economic-dispatch problem using the working set of hyper-planes for the generator outputs.

**Step 2.** Solve a power flow problem whose input data are the generation levels of all units from Step 1, except for the slack. The solution provides the values of $P_{slack}$ and of the phase angles of all buses.

**Step 3.** With the data obtained in Step 2 compute vector $\boldsymbol{\beta}$ using (5). This vector defines a new hyper-plane.

**Step 4.** If the difference between the values obtained for $P_{slack}$ in steps 1 and 2 is under a certain threshold, **STOP**; otherwise, add the hyper-plane derived in Step 3 to the set of working hyper-planes and return to **Step 1.**

An alternative algorithm that is better suited for on-line operation is now presented **(Algorithm 2)**:

**Step 1:** Compute a good[1] set of hyper-planes off-line.

**Step 2:** When on-line, for any actual value of the vector of demands, solve an economic dispatch using the inequality constraints obtained in Step 1.

### B. Estimating the error of the method

It is clear that in Algorithm 2, the number of hyper-planes calculated off-line, $N_H$, is the most important parameter in determining the cost error. Thus, a procedure is now presented for estimating the minimum cost error in terms of the number of hyper-planes used to solve a lossy economic dispatch, *error(*$N_H$*)*. **(Algorithm 3)**:

[1] The concept of "good" depends on the precision desired; the more hyper-planes considered in Step 0, the more accurate the solution of Step 1.

**Step 0:** Using any suitable algorithm, such as Algorithm 1 above, solve for the exact solution of a large number[2], $N_T$ , of lossy economic dispatch problems with different demand levels, and calculate for each of them the optimal cost: $Cost_0(n)\ n: 1...N_T$ .

**Step 1:** Compute $N_H$ supporting hyper-planes for a set of $N_H$ operating conditions different from those found in Step 0.

**Step 2:** Solve each of the $N_T$ problems from Step 0 using only the $N_H$ inequality constraints from Step 1 using Algorithm 2 to obtain the approximate costs $Cost_1(n)\ n: 1...N_T$ . Comparing these costs with the optimal costs that were obtained in Step 0, compute the corresponding errors. Define $\varepsilon(n) = Cost_0(n) - Cost_1(n)$ [3] as the error obtained when trying to approximate the *n*-th problem with $N_H$ hyper-planes. Define the maximum of these errors as an estimate of the maximum total error that will be obtained whenever $N_H$ hyper-planes are used to solve the economic dispatch problem with Algorithm 2.

In Algorithm 3, note that $N_H$ is the number of hyper-planes used to create a model. Also, note that $N_T$ is the number of problems used to assess the accuracy of that model; the exact solution for those $N_T$ problems is calculated in Step 0.

Figure 1 shows the maximum total cost error as a percentage of the exact non-linear cost in terms of the number of hyper-planes used in Algorithm 2. Note that there are four plots in Figure 1 for two different values of NT (100 and 400) and two different values of *w* (20% and 30%). Parameter *w* indicates how near to the base-case demand were the $N_H$ random demands. For instance, w=30% means that, the randomly generated demand values lie inside an interval of ±30% around their original value.

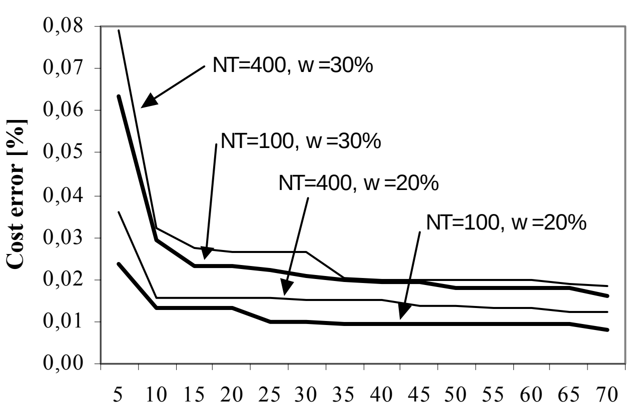


Figure 1. Cost error as a function of the number of hyper-planes used in Algorithm 2

Note that the effect of adding or removing hyper-planes as illustrated by Figure 1 of the paper is not very significant on the optimum dispatch cost, for example, as few as 10 hyper-planes give an error of around 0.03%.

[2] A suitable minimum number for the IEEE 118-bus system will be 100.

[3] This will always be a positive number, because the supporting hyper-planes are lower bounds of the actual function.

## VI. Conclusions

We have shown that the system loss function in a power network is bounded below by any number of supporting hyper-planes in the space of generalized injections. A supporting hyper-plane is defined by the linear Taylor series expansion of the system loss function with respect to the generalized injections around a given operating point. This supporting hyper-plane property is valid over a continuous range of expansion points provided that these are sufficiently near the flat-voltage profile (FVP). The required degree of proximity between an operating point defining a supporting hyper-plane and the FVP, called the range of hyper-plane support (RHS), was assessed experimentally. For typical networks, the RHS was found not to be very restrictive, particularly for networks where the bus voltages are closely controlled near 1 per unit.

The supporting hyper-plane model was also tested as part of an economic dispatch with transmission losses to demonstrate that this linear model provides the same results as when the losses are treated as a non-linear function.

Future research should look at integrating the supporting hyper-plane model into problems such as unit commitment with transmission losses. The system loss bounding theorem should also be examined with networks containing FACTS devices or phase-shifting transformers. In addition, it would be interesting to extend these results to load flow models with distributed slack generation.

## VII. Appendix A

For completeness, most of the relevant properties of the load flow equations in rectangular coordinates are presented and proven in this appendix. For these and other interesting properties of the quadratic formulation of the load flow problem, see also [11], [12] and [13].

For all the results that follow, we make the realistic assumption that all transmission lines have non-zero series resistance and zero shunt conductance.

**Result 1**: The system loss in a power transmission network is positive or zero.

**Proof:** This is self-evident from energy conservation and the fact that the transmission network is a passive RLC network.

**Result 2:** The system loss is zero if and only if all the series branch currents are zero.

**Proof:** Let $I_{ser-ij}$ be the series current magnitude of the line connecting buses $i$ and $j$ with non-zero resistance $R_{ij}$. The corresponding transmission loss component, $I_{ser-ij}^2 R_{ij}$ , is then zero if and only if the series current magnitude is zero. Under the assumption that the shunt branches are lossless, the system loss will also be zero if and only if all the series branch currents are zero.

**Result 3**: All the series branch currents are zero if and only if the complex bus voltages at all buses are equal.

**Proof:** The series branch current through the line connecting buses $i$ and $j$ with non-zero series impedance, $R_{ij}+jX_{ij}$ , is $\bar{I}_{ser-ij} = \dfrac{\bar{V}_i - \bar{V}_j}{R_{ij} + jX_{ij}}$, which is zero if and only if $\bar{V}_i = \bar{V}_j$ .

**Result 4**: The system loss is zero if and only if the network operates at FVP.
**Proof:** This follows directly from the definition of FVP, Results 2 and 3, and the zero shunt conductance assumption.

**Result 5:** In rectangular coordinates of the complex bus voltages, the system loss can be expressed by the following pure quadratic form,

$$P_{loss} = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{L}\mathbf{x} \equiv P_{loss}(\mathbf{x}) \qquad \text{(A1)}$$

where,

$$\mathbf{L} = 2\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{G'} \end{bmatrix} \qquad \text{(A2)}$$

**Proof:** Expressing the complex node voltages in rectangular coordinates, $\bar{\mathbf{V}} = \mathbf{e} + j\,\mathbf{f}$ , the total real power consumed by the network, which defines the system losses, is given by,

$$\begin{aligned} P_{loss} &= \sum_{i=1}^{N} P_i = \Re\left\{\sum_{i=1}^{N}\sum_{j=1}^{N} \bar{V}_i \bar{Y}_{ij}^{*} \bar{V}_j^{*}\right\} \\ &= \Re\left\{\bar{\mathbf{V}}^T \left[\bar{\mathbf{Y}}\bar{\mathbf{V}}\right]^{*}\right\} \\ &= \Re\left\{(\mathbf{e} + j\mathbf{f})^T \left[(\mathbf{G} + j\mathbf{B})(\mathbf{e} + j\mathbf{f})\right]^{*}\right\} \end{aligned} \qquad \text{(A3)}$$

If G is symmetric, then

$$\begin{aligned} P_{loss} &= \mathbf{e}^T\mathbf{G}\mathbf{e} + \mathbf{f}^T\mathbf{G}\mathbf{f} \\ &= \mathbf{e}^T\mathbf{G}\mathbf{e} + (\mathbf{f}')^T\,\mathbf{G}'\mathbf{f}' \end{aligned} \qquad \text{(A4)}$$

However, if **G** is not symmetric, as may occur in networks with phase-shifting transformers, then in equation (A4), **G** gets replaced, without loss of generality by the symmetric matrix $(\mathbf{G} + \mathbf{G}^T)/2$ . This is necessary since, if **G** is not symmetric, its eigenvalues are not related to those of $(\mathbf{G} + \mathbf{G}^T)/2$ . The latter is always a symmetric matrix whose eigenvalues are real and non-negative. We therefore assume throughout this appendix that **G** is symmetric.

Recall from the Nomenclature that, without loss of generality, $f_N$ is here set to zero, thus obtaining the second form in equation (A4).

**Result 6:** The symmetric matrix **G** is positive semi-definite.
**Proof:** From Result 1, $P_{loss} \geq 0$ for all **f** and **e**. In particular, by letting $\mathbf{f} = 0$ **,** it follows from equation (A2) in Result 5 that $P_{loss} = \mathbf{e}^T\mathbf{G}\mathbf{e} \geq 0$ for any $\mathbf{e}$ , which defines **G** as positive semi-definite. From matrix theory, the positive semi-definite condition also implies that all the eigenvalues of **G** are real and non-negative.

**Result 7:** The conductance matrix **G** has one and only one zero eigenvalue with eigenvector $\mathbf{e} = e_{fvp}\mathbf{1}$ where $e_{fvp}$ is an arbitrary constant. The remaining N-1 eigenvalues must therefore be real and positive.

**Proof:** To prove that **G** has one non-zero eigenvalue, let the network operate at FVP, that is, $\bar{\mathbf{V}} = e_{fvp}\mathbf{1}$ . From Result 3, all branch currents must be null, which implies that the component of the net bus current injections due to branch currents must also be null, in other words,

$$\bar{\mathbf{Y}}_{ser}\bar{\mathbf{V}} = (\mathbf{G} + j\mathbf{B}_{ser})e_{fvp}\mathbf{1} = \mathbf{0} \qquad \text{(A5)}$$

where $\bar{\mathbf{Y}}_{ser} = \mathbf{G} + j\mathbf{B}_{ser}$ is the component of the network admittance matrix due to series elements. Note that the shunt current vector components, $\bar{\mathbf{Y}}_{sh}\bar{\mathbf{V}} = (j\mathbf{B}_{sh})\bar{\mathbf{V}}$ is not zero at FVP but under the assumption of zero shunt conductances this component does not contribute to the system loss. Condition (A5) that the series current injections are null under FVP implies that $\mathbf{G}(e_{fvp}\mathbf{1}) = \mathbf{0}$ , which proves that **G** has a zero eigenvalue whose eigenvector is $e_{fvp}\mathbf{1}$ .

In order to prove that **G** has only one zero eigenvalue, suppose that there exists another non-zero vector $\hat{\mathbf{e}} \in \Re^N$ satisfying the zero eigenvalue condition, $\mathbf{G}\,\hat{\mathbf{e}} = \mathbf{0}$ , and not of the FVP form $e_{fvp}\mathbf{1}$ . Since $\bar{\mathbf{V}} = \hat{\mathbf{e}}$ is a vector of voltages that does not correspond to the FVP, according to Result 4, the system loss must be strictly positive, that is, $P_{loss} > 0$ . A contradiction then arises since (A4) states that $P_{loss} = \hat{\mathbf{e}}^T\mathbf{G}\hat{\mathbf{e}} = 0$ . Consequently, there cannot exist a vector $\hat{\mathbf{e}}$ with the above property and the only possible eigenvector corresponding to the zero eigenvalue is $e_{fvp}\mathbf{1}$ .

**Result 8:** The symmetric matrix $\mathbf{G}'$ is positive definite.
**Proof:** From Result 6, **G** is positive semi-definite. Thus for all **f**, $\mathbf{f}^T\mathbf{G}\mathbf{f} \geq 0$ . Moreover, from Result 7, $\mathbf{f}^T\mathbf{G}\mathbf{f} = 0$ for non-zero **f** only if **f** is proportional to **1**. Now, if $f_N$ = 0, $(\mathbf{f}')^T\mathbf{G}'\mathbf{f}' = \mathbf{f}^T\mathbf{G}\mathbf{f}$ . However, with $f_N$ = 0, it is impossible for **f** to be proportional to **1.** Thus, with $f_N$ = 0, $(\mathbf{f}')^T\mathbf{G}'\mathbf{f}' = \mathbf{f}^T\mathbf{G}\mathbf{f} > 0$ for all $\mathbf{f}'$ except for the trivial case when $\mathbf{f}' = \mathbf{0}$ , which is the condition for positive definiteness of $\mathbf{G}'$ .

**Result 9:** The matrix **L** defined in Result 5 is positive semi-definite with a single zero eigenvalue, whose eigenvector corresponds to the FVP.
**Proof:** From the properties of block-diagonal matrices, the set of eigenvalues of **L** is the union of the respective sets of eigenvalues of **G** and $\mathbf{G}'$ . As proven in Results 7 and 8, the combined eigenvalues of these two matrices are all positive except for one which is zero whose eigenvector corresponds to the FVP.

**Result 10:** The vectors of real and reactive power injections, **P** and **Q**, as well as the vector of bus voltage magnitudes, $\mathbf{V}^2$ , are quadratic functions of the real and imaginary complex voltage components, $\mathbf{x}$ .
**Proof:** The vector of complex power injections is,

$$\bar{\mathbf{S}} = \mathbf{P} + j\,\mathbf{Q} = diag\{\bar{\mathbf{V}}\}\bar{\mathbf{I}}^{*} = diag\{\bar{\mathbf{V}}\}\{\bar{\mathbf{Y}}\bar{\mathbf{V}}\}^{*} \qquad \text{(A6)}$$

which in rectangular form becomes,

$$\bar{\mathbf{S}} = diag\{\mathbf{e} + j\,\mathbf{f}\}\{(\mathbf{G} + j\,\mathbf{B})(\mathbf{e} + j\,\mathbf{f})\}^{*} \qquad \text{(A7)}$$

Separating $\overline{\mathbf{S}}$ into its real and imaginary parts, **P** and **Q,** gives the following quadratic equations in $\mathbf{x}$,

$$\begin{aligned}\mathbf{P} &= diag\{\mathbf{e}\}(\mathbf{Ge}-\mathbf{Bf}) + diag\{\mathbf{f}\}(\mathbf{Gf}+\mathbf{Be})\\ \mathbf{Q} &= diag\{\mathbf{f}\}(\mathbf{Ge}-\mathbf{Bf}) - diag\{\mathbf{e}\}(\mathbf{Gf}+\mathbf{Be})\end{aligned} \tag{A8}$$

Similarly, since $V_i^2 = e_i^2 + f_i^2$, the vector of voltage magnitudes squared can also be expressed as a quadratic in **x**,

$$\mathbf{V}^2 = diag\{\mathbf{e}\}\mathbf{e} + diag\{\mathbf{f}\}\mathbf{f} \tag{A9}$$

**Result 11**: Any scalar linear combination of elements of $\mathbf{P}, \mathbf{Q}$ and $\mathbf{V}^2$ can be expressed in the form $\frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x}$ where **H** is a real symmetric matrix dependent only on the network parameters.

**Proof:** From the definition of the N-dimensional vector $\mathbf{1}_i$ in the Nomenclature and from equation (A8) in Result 10 the real power injection at bus *i* is expressed as,

$$\begin{aligned}P_i &= (\mathbf{1}_i)^T\mathbf{P}\\ &= (\mathbf{1}_i)^T\left[diag\{\mathbf{e}\}(\mathbf{Ge}-\mathbf{Bf}) + diag\{\mathbf{f}\}(\mathbf{Gf}+\mathbf{Be})\right]\\ &= \mathbf{e}^T\left[diag\{\mathbf{1}_i\}(\mathbf{Ge}-\mathbf{Bf})\right] + \mathbf{f}^T\left[diag\{\mathbf{1}_i\}(\mathbf{Gf}+\mathbf{Be})\right]\end{aligned} \tag{A10}$$

Defining now the symmetric matrix,

$$\hat{\mathbf{H}}_i^P = \begin{bmatrix} diag\{\mathbf{1}_i\}\mathbf{G} + \mathbf{G}^T diag\{\mathbf{1}_i\} & -diag\{\mathbf{1}_i\}\mathbf{B} + \mathbf{B}^T diag\{\mathbf{1}_i\} \\ -\mathbf{B}^T diag\{\mathbf{1}_i\} + diag\{\mathbf{1}_i\}\mathbf{B} & diag\{\mathbf{1}_i\}\mathbf{G} + \mathbf{G}^T diag\{\mathbf{1}_i\}\end{bmatrix} \tag{A11}$$

it follows from (A10) that $P_i$ can be expressed as,

$$P_i = \frac{1}{2}\begin{bmatrix}\mathbf{e}\\ \mathbf{f}\end{bmatrix}^T \hat{\mathbf{H}}_i^P \begin{bmatrix}\mathbf{e}\\ \mathbf{f}\end{bmatrix} \tag{A12}$$

Now, defining $\mathbf{H}_i^P$ as $\hat{\mathbf{H}}_i^P$ without the *N*-th row and column,

$$P_i = \frac{1}{2}\mathbf{x}^T\mathbf{H}_i^P\mathbf{x} \tag{A13}$$

Similarly, for the reactive power injections we obtain,

$$Q_i = \frac{1}{2}\mathbf{x}^T\mathbf{H}_i^Q\mathbf{x} \tag{A14}$$

where $\mathbf{H}_i^Q$ is found by removing the *N*-th row and column from,

$$\hat{\mathbf{H}}_i^Q = \begin{bmatrix} -diag\{\mathbf{1}_i\}\mathbf{B} - \mathbf{B}^T diag\{\mathbf{1}_i\} & -diag\{\mathbf{1}_i\}\mathbf{G} + \mathbf{G}^T diag\{\mathbf{1}_i\} \\ -\mathbf{G}^T diag\{\mathbf{1}_i\} + diag\{\mathbf{1}_i\}\mathbf{G} & -diag\{\mathbf{1}_i\}\mathbf{B} - \mathbf{B}^T diag\{\mathbf{1}_i\}\end{bmatrix} \tag{A15}$$

Finally, for bus voltage magnitudes squared we obtain,

$$V_i^2 = \frac{1}{2}\mathbf{x}^T\mathbf{H}_i^V\mathbf{x} \tag{A16}$$

where $\mathbf{H}_i^V$ is found by removing the *N*-th column and row from,

$$\hat{\mathbf{H}}_i^V = 2\begin{bmatrix} diag\{\mathbf{1}_i\} & \mathbf{0} \\ \mathbf{0} & diag\{\mathbf{1}_i\}\end{bmatrix} \tag{A17}$$

Now, from equations (A13), (A14) and (A16), it follows that any individual generalized power flow injection, $\mathbf{z}_i$, can therefore be expressed as a pure quadratic of the form,

$$z_i = \frac{1}{2}\mathbf{x}^T\left[\mathbf{H}_i\right]\mathbf{x} \tag{A18}$$

Thus,

$$\sum_{i=1}^{n}\alpha_i z_i = \frac{1}{2}\mathbf{x}^T\mathbf{H}(\boldsymbol{\alpha})\mathbf{x} \tag{A19}$$

where we defined,

$$\mathbf{H}(\boldsymbol{\alpha}) = \sum_{i=1}^{n}\alpha_i\mathbf{H}_i \tag{A20}$$

**Result 12:** The load flow problem with *n* specified generalized injections, $z_i$; $i=1,\ldots,n$, can be expressed as a set of equations of the form,

$$\mathbf{z} = \frac{1}{2}\mathbf{J}(\mathbf{x})\mathbf{x} \tag{A21}$$

where $\mathbf{J}(\mathbf{x})$ is an *n* by *n* matrix defined by,

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix}\mathbf{x}^T\mathbf{H}_1\\ \vdots\\ \mathbf{x}^T\mathbf{H}_n\end{bmatrix} \tag{A22}$$

**Proof:** This result follows immediately from equation (A19) in Result 11.

**Result 13:** The matrix $\mathbf{J}(\mathbf{x})$ is the Jacobian matrix of the load flow equations in rectangular coordinates.

**Proof**: The load flow Jacobian matrix is defined as the sensitivity of **z** with respect to **x,** which from equations (A21) and (A22) in Result 12 gives,

$$\frac{\partial\mathbf{z}}{\partial\mathbf{x}} = \frac{\partial\left[\frac{1}{2}\begin{bmatrix}\mathbf{x}^T\mathbf{H}_1\mathbf{x}\\ \vdots\\ \mathbf{x}^T\mathbf{H}_n\mathbf{x}\end{bmatrix}\right]}{\partial\mathbf{x}} = \begin{bmatrix}\mathbf{x}^T\mathbf{H}_1\\ \vdots\\ \mathbf{x}^T\mathbf{H}_n\end{bmatrix} = \mathbf{J}(\mathbf{x}) \tag{A23}$$

**Result 14:** The Jacobian matrix $\mathbf{J}(\mathbf{x})$ is linear in **x.**

**Proof:** This result follows directly from (A23).

**Result 15**: For any $\Delta\mathbf{x}$, $\boldsymbol{\alpha}^T\mathbf{J}(\Delta\mathbf{x}) = \Delta\mathbf{x}^T\mathbf{H}(\boldsymbol{\alpha})$.

**Proof**: From Result 11,

$$\begin{aligned}\Delta\mathbf{x}^T\mathbf{H}(\boldsymbol{\alpha}) &= \Delta\mathbf{x}^T\left[\sum_{i=1}^{n}\alpha_i\mathbf{H}_i\right] = \left[\sum_{i=1}^{n}\alpha_i\Delta\mathbf{x}^T\mathbf{H}_i\right]\\ &= \boldsymbol{\alpha}^T\begin{bmatrix}\Delta\mathbf{x}^T\mathbf{H}_1\\ \vdots\\ \Delta\mathbf{x}^T\mathbf{H}_n\end{bmatrix} = \boldsymbol{\alpha}^T\mathbf{J}(\Delta\mathbf{x})\end{aligned} \tag{A24}$$

## VIII. Appendix B: Proof of System Loss Bounding Theorem

As shown in Result 5 of Appendix A, the system loss function can be written as a pure quadratic form in $\mathbf{x}$,

$$P_{loss} = \frac{1}{2}\mathbf{x}^T\mathbf{L}\mathbf{x} \tag{B1}$$

Furthermore, from Result 9 in Appendix A, **L** has only one zero eigenvalue whose corresponding eigenvector is the flat voltage profile.

We examine now the behavior of the load flow equations and the system loss function in terms of deviations from an arbitrary operating point, $\mathbf{x}^0$, so that $\mathbf{x} = \mathbf{x}^0 + \Delta\mathbf{x}$. Recalling the expression for the load flow equations in Result 12 and the fact that the load flow Jacobian matrix is linear in **x** from

Result 14, we have,

$$\begin{aligned}\mathbf{z} &= \frac{1}{2}\mathbf{J}(\mathbf{x}^0+\Delta\,\mathbf{x})(\mathbf{x}^0+\Delta\,\mathbf{x}) \\ &= \frac{1}{2}\mathbf{J}(\mathbf{x}^0)(\mathbf{x}^0)+\mathbf{J}(\mathbf{x}^0)\Delta\,\mathbf{x}+\frac{1}{2}\mathbf{J}(\Delta\,\mathbf{x})\Delta\,\mathbf{x}\end{aligned} \quad \text{(B2)}$$

Similarly, the system loss equation (B1) can be rewritten as,

$$P_{loss} = \frac{1}{2}(\mathbf{x}^0)^T\,\mathbf{L}\mathbf{x}^0+(\mathbf{x}^0)^T\,\mathbf{L}\,\Delta\,\mathbf{x}+\frac{1}{2}\Delta\,\mathbf{x}^T\mathbf{L}\,\Delta\,\mathbf{x} \quad \text{(B3)}$$

Since the load flow Jacobian at the operating point, $\mathbf{x}^0$, is non-singular, from (B2),

$$\Delta\mathbf{x} = \left[\mathbf{J}(\mathbf{x}^0)\right]^{-1}\left(\mathbf{z}-\frac{1}{2}\mathbf{J}(\mathbf{x}^0)(\mathbf{x}^0)-\frac{1}{2}\mathbf{J}(\Delta\mathbf{x})(\Delta\mathbf{x})\right) \quad \text{(B4)}$$

Substituting (B4) into the linear term of (B3) gives,

$$\begin{aligned}P_{loss} &= \frac{1}{2}(\mathbf{x}^0)^T\,\mathbf{L}\mathbf{x}^0+\frac{1}{2}\left[\Delta\,\mathbf{x}\right]^T\mathbf{L}\,\Delta\,\mathbf{x} \\ &\quad +(\mathbf{x}^0)^T\,\mathbf{L}\left[\left[\mathbf{J}(\mathbf{x}^0)\right]^{-1}\left(\mathbf{z}-\frac{1}{2}\mathbf{J}(\mathbf{x}^0)(\mathbf{x}^0)-\frac{1}{2}\mathbf{J}(\Delta\,\mathbf{x})(\Delta\,\mathbf{x})\right)\right] \quad \text{(B5)} \\ &= (\mathbf{x}^0)^T\,\mathbf{L}\left[\mathbf{J}(\mathbf{x}^0)\right]^{-1}\left(\mathbf{z}-\frac{1}{2}\mathbf{J}(\Delta\,\mathbf{x})(\Delta\,\mathbf{x})\right)+\frac{1}{2}\left[\Delta\,\mathbf{x}\right]^T\mathbf{L}\,\Delta\,\mathbf{x}\end{aligned}$$

From the definition of $\boldsymbol{\beta}$ in (5), and taking its transpose, the exact system loss equation in (B5) can now be expressed as,

$$\begin{aligned}P_{loss} &= \boldsymbol{\beta}^T\left(\mathbf{z}-\frac{1}{2}\mathbf{J}(\Delta\mathbf{x})(\Delta\mathbf{x})\right)+\frac{1}{2}\left[\Delta\mathbf{x}\right]^T\mathbf{L}\Delta\mathbf{x} \\ &= \boldsymbol{\beta}^T\mathbf{z}-\frac{1}{2}\boldsymbol{\beta}^T\mathbf{J}(\Delta\mathbf{x})(\Delta\mathbf{x})+\frac{1}{2}\left[\Delta\mathbf{x}\right]^T\mathbf{L}\Delta\mathbf{x}\end{aligned} \quad \text{(B6)}$$

Now, using Result 15 with $\boldsymbol{\alpha}$ taking the particular value of $\boldsymbol{\beta}$, it follows from (B6) that,

$$P_{loss} = \boldsymbol{\beta}^T\mathbf{z}+\frac{1}{2}\left[\Delta\mathbf{x}\right]^{\mathbf{T}}\left[\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})\right]\Delta\mathbf{x} \quad \text{(B7)}$$

Comparing (B7) with (8), we see that the approximation error, $\varepsilon$, is given by the quadratic form,

$$\varepsilon = \frac{1}{2}\left[\Delta\mathbf{x}\right]^{\mathbf{T}}\left[\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})\right]\Delta\mathbf{x} \quad \text{(B8)}$$

To complete the proof of the System Loss Bounding Theorem, we now show that if $\mathbf{x}^0$ lies sufficiently near a FVP, the real symmetric matrix $\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})$ is positive semi-definite. This will imply that for all $\Delta\,\mathbf{x}\in\Re^n$, $\Delta\,\mathbf{x}^T\left[\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})\right]\Delta\,\mathbf{x}\ge 0$ or equivalently that for all $P_{loss}$ and $\mathbf{z}$ satisfying respectively (B1) and (4),

$$P_{loss} \ge \boldsymbol{\beta}^T\mathbf{z} \quad \text{(B9)}$$

To prove that $\mathrm{L}-\mathrm{H}(\beta)$ is positive semi-definite first recall from Result 9 that $\mathbf{L}\mathbf{x}_{fvp}=0$. Thus, if $\mathbf{x}^0\cong\mathbf{x}_{fvp}$, then $\mathbf{L}\mathbf{x}^0\cong\mathbf{L}\mathbf{x}_{fvp}=0$. Similarly, given that the load flow Jacobian is non-singular at $\mathbf{x}^0$, then $\boldsymbol{\beta}=\left[\mathbf{J}(\mathbf{x}^0)\right]^{-1}\mathbf{L}\mathbf{x}^0\cong \left[\mathbf{J}(\mathbf{x}^0)\right]^{-1}\mathbf{L}\mathbf{x}_{fvp}=\mathbf{0}$. Now, from equation (A20) in Result 11, if $\boldsymbol{\beta}\cong\mathbf{0}$ then $\mathbf{H}(\boldsymbol{\beta})\cong\mathbf{0}$.

Since $\mathbf{L}$ and $\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})$ are arbitrarily close real symmetric matrices, their respective sets of eigenvalues are real and arbitrarily close. From Result 9, the eigenvalues of $\mathbf{L}$ are strictly positive except for $\lambda_n(\mathbf{L})=0$. This means that all the eigenvalues of $\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})$ are guaranteed to be positive except possibly $\lambda_n(\mathbf{L}-\mathbf{H}(\boldsymbol{\beta}))$ which although arbitrarily close to $\lambda_n(\mathbf{L})=0$, may have shifted to a negative value.

To discard this option, recall from equation (5) and from Result 15 in Appendix A that,

$$\mathbf{L}\mathbf{x}^0 = \left[\mathbf{J}(\mathbf{x}^0)\right]^T\boldsymbol{\beta} = \mathbf{H}(\boldsymbol{\beta})\,\mathbf{x}^0 \quad \text{(B10)}$$

Then,

$$\left[\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})\right]\mathbf{x}^0 = 0 \quad \text{(B11)}$$

This implies that $\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})$ still has a zero eigenvalue with eigenvector $\mathbf{x}^0$ and therefore that $\lambda_n(\mathbf{L}-\mathbf{H}(\boldsymbol{\beta}))=0$. This proves that $\mathbf{L}-\mathbf{H}(\boldsymbol{\beta})$ is positive semi-definite and concludes the proof of the system loss bounding theorem.

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## X. Biographies


**Sebastián de la Torre** (M'04) was born in Málaga, Spain. He received the Ingeniero Industrial degree from the Universidad de Málaga, Spain, in 1999 and the PhD degree from the Universidad de Castilla-La Mancha, Spain in 2003. His research interests include operations and economics of electrical energy systems, restructuring of electric systems and development of new electricity markets.

**Francisco D. Galiana** (F'91) received his B.Eng. (Hons.) from McGill University, Montreal, QC, Canada followed by the S.M. and the Ph.D. degrees from M.I.T. He spent several years at the Brown Boveri Research Center and at the University of Michigan. He is presently the Professor of Electrical Engineering at McGill University. His research interest have included system security, optimal power flow, expert systems, and recently, open access and competition in power networks.