Finding Multiperiod Nash Equilibria in Pool-Based Electricity Markets

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Abstract-Pool-based electricity markets can be simulated with various degrees of accuracy. When compared to actual markets, most of the simulators produce outcomes than cannot be extrapolated beyond the specific scenario analyzed. This is most critical for regulators and market participants which need tools to analyze market power and bidding strategies, respectively, for a broad range of scenarios. Both objectives can be tackled if the possible equilibria of a pool-based multiperiod market are determined. This paper presents a three-step methodology to find these equilibria. First, a detailed model of an electricity market is presented, considering multiperiod bidding, price elasticity, and network modeling. Second, an iterative simulation process is run to detect participants' bidding strategies implicit in the optimized production resulting from the simulation. Finally, output data from the simulator are analyzed to obtain Nash equilibria. Iterated deletion is used in the last step to remove strategies that are dominated by others that generate higher profits. A realistic case study illustrates the proposed technique.

Index Terms-Electricity markets, market power, market simulation, Nash equilibrium.

NOMENCLATURE

i	GENCOs in the market.
t	Time periods considered in the time horizon.
n,k	Network nodes.
$_{j}$	Generating units.
d	Demands.
h	Power blocks for each demand.
b	Power blocks for each generating unit.

- **B.** Functions
- $\lambda_t^i(q_t^i)$ Stepwise monotonically decreasing discontinuous function that expresses the market-clearing price as a function of the quota of GENCO i for hour t. This function is known as price-quota curve.
- Production cost for hour t of the *j*th generating unit. c_{jt}

C. Constants

Number of blocks of the cost function for every unit. N_B

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N_D	Number of demands in the system.
N_H	Number of price-blocks for every demand.
N_J	Number of units in the system.
N_N	Total number of nodes in the system.
λ_t^M	Market-clearing price corresponding to hour t. Note
-	that λ_t^M is not strictly a constant, its value is derived
	from the solution of problem (6) – (13) .
λ_{dth}^D	Price corresponding to the h th block of demand d in
	hour t.
λ_{itb}^G	Price corresponding to the b th block of unit j in hour
J.00	t.
p_{dth}^{Dbid}	Size of the h th quantity block offered by demand d
	in hour t.
p_{itb}^{Gbid}	Size of the b th quantity block offered by unit j in
J 00	hour t.
Ω_i	Feasible operating region for unit j .
$\bar{B_{nk}}$	Susceptance of the line between nodes n and k .
G_{nk}	Conductance of the line between nodes n and k .

D. Variables	
p_{it}^G	Power produced by unit j in hour t .
$p_{itb}^{\tilde{G}}$	Power produced with the b th block of unit j
J 00	in hour t.
p_{dt}^D	Power consumed by demand d in hour t .
$p_{dth}^{\tilde{D}}$	Power consumed by the h th block of demand
	d in hour t .
q_t^i	Quota of GENCO <i>i</i> in hour <i>t</i> .
\overline{p}_{nk}^F	Maximum capacity of the line between nodes
	n and k.
$p_{nt}^{i,\text{others}}$	Summation of power injected in node n in
	hour t by all the participants except GENCO
	i.
θ_{nt}	Phase angle of node n in hour t .
s_i	Strategy of GENCO <i>i</i> .
s_{-i}	Strategies of GENCOs other than GENCO <i>i</i> .
s =	Strategy vector of all the GENCOs.
(s_i, s_{-i})	
$u_i(s_i, s_{-i})$	Payoff function of GENCO <i>i</i> .

E. Sets

 Ψ_r

 Φ_r

Γ_i	Set of generating units belonging to GENCO <i>i</i> .
Ψ_n	Set of units connected to node n.
Φ_n	Set of nodes directly connected to node n .
Δ_n	Set of demands connected to node n .
H_i	GENCO <i>i</i> information set.
S_i	GENCO <i>i</i> strategy set.
A	Set of possible actions of GENCO <i>i</i> .

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I. INTRODUCTION

ARKET simulators are used increasingly to replicate the behavior of actual electricity markets. Regulators can use these tools to monitor and detect market power. Similarly, buyers and sellers can use them to refine their bids. Market power is dependent on many factors, namely, the number of market players, the bidding behavior, and the restrictions imposed. It is desirable for a simulator to reproduce as close as possible the actual market functioning to make the simulator results comparable to actual figures. Thus, a good simulator design must contain all of the rules of the market.

In an auction-based day-ahead market [1]–[3], the market operator processes the bid information provided by the producers and consumers and aggregates this information creating hourly offer and demand curves, respectively. Both producers and consumers bid with the target of maximizing their profits, respectively [4]. Once the bids are submitted, a market-clearing algorithm matches the production and demand curves producing a series of hourly prices and accepted quantities [5], [6]. Through the simulator, this process is repeated many times so that some patterns of behavior can be detected and studied.

Modeling a pool-based electric market is a complicated task. The need for integer variables, the nonconvex and nondifferentiable nature of the bid functions, different time spans, and transmission network modeling [7], just to name a few examples, make the modeling complex. Current market models lack some or all of these features, although they provide a valuable qualitative insight. A detailed study is provided in [8].

Searching for market equilibrium is a desirable objective both for market participants and regulators: for participants, because an equilibrium shows long-term bidding strategies of their rivals; for regulators, because market power monitoring and corrective measures are possible. To find equilibria, the methodology presented in this paper proposes: 1) create a realistic model of the market, 2) simulate how participants generate their bids iteratively and the market operator clears the market, and 3) identify plausible equilibria. Step 1 is possible by means of sophisticated optimization techniques, step 2 is possible given the procedure provided in this paper, and step 3 relies on the concept of Nash equilibrium.

Nash equilibrium can be defined as a set of strategies, one per player, so that each strategy is the best response to the other players' [9]. In a game there can be none, single, or multiple equilibria. In addition, equilibria can be pure or mixed [9], [10]. The Nash equilibrium concept has been mainly applied to Cournot models of electric markets with network constraints, as seen in [11]–[15]. In particular, [15] presents a dc transmission network model, but it does not include nonlinear losses. In addition, market power in simple networks is studied in [16], using a three-node system. Also, games with incomplete information, where participants do not have full knowledge of other participant's parameters, are shown in [17]. Finally, [18] presents a Nash bargaining game for transmission analysis, where power exchanges in a two-area system are analyzed; this is a full ac model. Other game theory methods to obtain an equilibrium, such as the supply function equilibrium model [19], [20], or the Stackelberg leader-follower model [21], are outside the scope of this paper. In this paper, the concept of Nash equilibrium is applied to interpret sensible outcomes of a very realistic simulator [6], in terms of produced quantities and obtained profits. Simplifications to the structure of the problem are not considered, except for a limitation in the number of strategies in the game.

The paper is organized as follows. Section II provides an introduction to the basic definitions and concepts in game theory and Nash equilibrium. Section III presents the simulator structure and the iterative process of bidding using the simulator. Section IV shows a representative case study where several Nash equilibria are found and interpreted. Conclusions and future work are outlined in Section V.

II. GAME THEORY AND NASH EQUILIBRIUM: BACKGROUND REVIEW

A game is a "formal representation of a situation in which a number of individuals interact in a setting of strategic interdependence" [10]. This means that the welfare of an individual depends upon its own action and the actions of the other participants in the game. To describe a game, there are four things to consider: 1) the players, 2) the rules of the game, 3) the outcomes and 4) the payoffs and the preferences (utility functions) of the players. It is usually assumed that the player's utility function is its payoff function. A game can be either *cooperative*, where the players collaborate to achieve a common goal, or *noncooperative*, where they act on their own. Also, a game can be either of *perfect* or *imperfect information*, and *sequential* or *simultaneous* (the players play at the same time).

A player plays a game through actions. An *action* is a choice or election that a player takes, according to his (or her) own *strategy*. Since a game sets a framework of strategic interdependence, a participant should be able to have enough information about its own and other players' past actions. This is called the *information set*. Note that there is one information set per player and per stage of the game. A *strategy* is a rule that tells the player which action(s) it should take, according to its own information set at any particular stage of a game. Finally, a *payoff* function expresses the utility that a player obtains given a strategy profile for all players.

More formally stated, assume that there is a finite set of players $\{1, \ldots, I\}$ participating in a game. If H_i is the collection of player *i*'s information sets, *A* is the set of possible actions, and $u_i(s_1, \ldots, s_I)$ is the payoff function of player *i*, then:

A (*pure*) strategy for player *i* is a function $s_i: H_i \to A$ such that s_i belongs to the strategy vector $s = (s_1, \ldots, s_I)$ that contains the strategies of all the players [10]. It is useful to write the strategy vector *s* as (s_i, s_{-i}) where s_{-i} is the (I-1) vector of strategies for players other than *i*. Since for every profile of strategies there is an outcome of the game, the payoffs received by each player can be deduced accordingly. Thus, the game can be specified in terms of strategies and associated payoffs. This is called the *normal* (or *strategic*) form of a game.

A normal form representation specifies for each player ia set of strategies S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1, \ldots, s_I)$. In this way, player i's strategy set can be expressed as $S_i = \{s_{1i}, s_{2i}, \ldots\}$ referring to each strategy of player i by its number. Whenever a player detects that one of its strategies is the best strategy regardless of what other players do, it has found a *strictly dominant strategy*. Thus, this strategy maximizes the player's payoff against any strategy that the rivals would choose. More formally, a strategy $s_i \in S_i$ is a *strictly dominant strategy* for player *i* in a game if $\forall s'_i \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \forall s_{-i} \in S_{-i}$.

Although it seems a good idea for a player to play dominant strategies always, it is rare that they exist. But it is plausible that the player does not play dominated strategies. A strategy $s_i \in S_i$ is *strictly dominated* for player *i* in a game if there exists another strategy $s'_i \in S_i$ such that $\forall s_{-i} \in S_{-i}, u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$. In that case, strategy s'_i strictly dominates strategy s_i .

Assuming that the same game can be played in several rounds (iterations), in each round the players may delete simultaneously the strategies that are strictly dominated. This is called *iterated deletion*. It is easy to see that only the strategies that survive iterated deletion will be played in the long term, and they are candidates to be equilibria of the game [10]. Among all definitions of equilibrium constitutes a profile of strategies such that each player's strategy is the best response to the other players' strategies that are actually played. Therefore, neither player has an incentive to change its strategy. More formally, a strategy vector $s = (s_1, \ldots, s_I)$ is a Nash equilibrium of a game if for every player $i = 1, \ldots, I, u_i(s_i, s_{-i}) \ge u_i(s'_i, s_{-i}) \forall s'_i \in S_i$.

III. MARKET SIMULATOR

This section presents a detailed explanation of the internal mechanism of the market simulator whose results are used to construct representative bidding strategies in a power pool. First, a detailed model of the generating and consuming companies is presented, then the market-clearing algorithm is described, and, finally, the iterative simulation algorithm is explained [6].

A. Generating Companies Model

Depending on its relative size and generating mix, a GENCO behaves either as a price-maker [4] or as a price-taker; however, the formulation presented next is valid for both price-makers and price-takers.

The formulation of the problem faced by GENCO i is as follows [4]:

$$\begin{array}{l} \underset{p_{jt}^{G}, q_{t}^{i}}{\text{maximize}} \sum_{t=1}^{N_{T}} \left[q_{t}^{i} \lambda_{t}^{i} \left(q_{t}^{i} \right) - \sum_{j \in \Gamma_{i}} c_{jt} \right] \\ \text{subject to : } p_{i}^{G} \in \Omega_{i}; \qquad \forall j \in \Gamma_{i}; t = 1, \dots, N_{T}$$
(2)

$$q_t^i = \sum_{j \in \Gamma_i} p_{jt}^G; \qquad t = 1, \dots, N_T \quad (3)$$

$$\sum_{j \in (\Psi_n \cap \Gamma_i)} p_{jt}^G + p_{nt}^{i, \text{others}} + \sum_{k \in \Phi_n} B_{nk}(\theta_{nt} - \theta_{kt}) - \frac{1}{2} \sum_{k \in \Phi_n} G_{nk}(\theta_{nt} - \theta_{kt})^2 = 0;$$

$$t = 1, \dots, N_T; n = 1, \dots, N_N \quad (4)$$

$$- \overline{p}_{nk}^F \leq B_{nk}(\theta_{nt} - \theta_{kt}) \leq \overline{p}_{nk}^F;$$

$$n = 1, \dots, N_N; \forall k \in \Phi_n; t = 1, \dots, N_T.$$
 (5)

The objective function (1) expresses the profit of the pricemaker: total revenue minus total costs. Taking advantage of the stepwise nature of price-quota curves (also known as effective demand curves or price-production curves), the total revenue can be expressed linearly using positive real variables and binary variables [4]. For a detailed formulation of the cost function, see [4] and [22].

Set of constraints (2) enforces that every unit works within its feasible operating region over the whole planning horizon. A precise mixed-integer linear description of this feasibility region can be found in [22] and [23].

Set of constraints (3) expresses for every hour the price-maker quota as the sum of the power production of its units.

Block of (4) defines power balance at every node, stating that the difference between power reaching any node and power leaving that node must equal zero. The first term in this equation expresses the power injected by the GENCO. The second term $p_{nt}^{i,\text{others}}$ expresses the total power injected by other generators in the node; it comprises the power injected by other generators minus the power demanded at the node. Note that $p_{nt}^{i,\text{others}}$ is considered publicly available data known by the GENCO before solving its optimization problem. The third term is the net power reaching the node through adjacent lines. Note that 50% of the losses incurred in each of the lines connected to the considered node are introduced as an artificial demand in that node. This mechanism allows formulating a simple yet accurate linear model for the losses. For more details regarding loss modeling, see [5].

Block of (5) imposes the restrictions related to the capacity of transmission lines.

For a given hour, the quota of a price-maker is the amount of power it contributes to serve the demand in that hour. The function that expresses how the market-clearing price changes as the quota of a given price-maker changes is called pricequota curve [4]. Note that different price-makers competing in the same electricity market present different price-quota curves. The price-quota curve for a given hour corresponding to a pricemaker is a stepwise monotonically decreasing curve because (producer/consumer) bids are assumed to be blocks of power at given prices. The 24 hourly day-ahead price-quota curves of a given price-maker provide all of the market information it needs to self-schedule optimally (i.e., to maximize its benefits).

The day-ahead price-quota curves of a price-maker can be obtained: 1) by market simulation or 2) using forecasting procedures, however, both techniques are outside the scope of this paper. For the case studies presented in this paper, a direct method was used to obtain the price-quota curves. For any given hour, the price-quota curve for a certain GENCO is equal to the aggregated demand curve minus the aggregated offer curve of the rest of the GENCOs. As a result of this subtraction, the market-clearing price is a function of the GENCOs own production. Note that the above two curves are assumed to be publicly available from the market operator. For the sake of illustration, Fig. 1 shows a typical price-quota curve.

The solution of problem (1)–(5) provides any GENCO with its optimal self-scheduling (i.e., the power blocks the GENCO should get accepted in the market to maximize its profit). To that end, the bidding strategy for all GENCOs for any given hour is defined as follows:



Fig. 1. Price-quota curve.

- only power blocks with optimal self-scheduling values different from zero are offered at their corresponding marginal costs;
- 2) the remaining blocks are offered at price infinity.

Although GENCOs do take into account the network to compute their bids, they do not try to use the network as an instrument to exert market power; in other words, no GENCO is trying to produce a saturation in order to take advantage of the resulting higher prices. The modeling of such a behavior is complicated and out of the scope of this paper.

B. Consuming Companies Model

CONCOs are modeled in a simple fashion because the main purpose of the procedure presented in this paper is to analyze the behavior of GENCOs. Each demand is considered a fixed set of price-quantity values. As in the real world, stepwise elastic demands are considered distributed over the nodes of the network (see Fig. 2).

C. Market-Clearing Algorithm

0

A network-constrained multiperiod auction to maximize social welfare is used to clear the market. It is based on mixedinteger linear programming. The complete formulation of the problem is as follows:

$$\underset{p_{j\text{tb}}^{G}, p_{d\text{th}}^{D}}{\text{maximize}} \sum_{t=1}^{N_{T}} \left[\sum_{d=1}^{N_{D}} \sum_{h=1}^{N_{H}} p_{d\text{th}}^{D} \lambda_{d\text{th}}^{D\text{bid}} - \sum_{j=1}^{N_{J}} \sum_{b=1}^{N_{B}} p_{jtb}^{G} \lambda_{jtb}^{G\text{bid}} \right]$$
(6)
$$p_{jt}^{G} \in \Omega_{j}; \quad j = 1, \dots, N_{J}; t = 1, \dots, N_{T}$$
(7)

$$\leq p_{dth}^D \leq p_{dth}^{Dbid}$$

$$d = 1, \dots, N_D; h = 1, \dots, N_H;$$

$$t = 1, \dots, N_T \tag{8}$$
$$0 \le p_{jtb}^G \le p_{jtb}^{Gbid}$$

$$j = 1, \dots, N_J; b = 1, \dots, N_B;$$

 $t = 1, \dots, N_T$ (9)

$$\sum_{b=1}^{N_B} p_{jtb}^G = p_{jt}^G;$$

 $j = 1, \dots, N_J; t = 1, \dots, N_T$ (10)



Blocks of offered power consumption for demand d at hour t [MWh]

Fig. 2. Example of stepwise elastic demand.

$$\sum_{h=1}^{N_H} p_{dth}^D = p_{dt}^D;$$

$$d = 1, \dots, N_D; \ t = 1, \dots, N_T \qquad (11)$$

$$\sum_{j \in \Psi_n} p_{jt}^G + \sum_{k \in \Phi_n} B_{nk}(\theta_{nt} - \theta_{kt}) = \sum_{d \in \Delta_n} p_{dt}^D$$

$$+ \frac{1}{2} \sum_{k \in \Phi_n} G_{nk}(\theta_{nt} - \theta_{kt})^2;$$

$$t = 1, \dots, N_T; n = 1, \dots, N_N \qquad (12)$$

$$- \overline{n}_{F_1}^F \leq B_{nt}(\theta_{nt} - \theta_{lt}) \leq \overline{n}_{F_1}^F;$$

$$n = 1, \dots, N_N; \forall k \in \Phi_n;$$

$$t = 1, \dots, N_T.$$
(13)

Equation (6) is the objective function; it expresses total social welfare as the summation of the social welfare for every hour. Social welfare is computed as the difference of two terms: the first term is the sum of accepted demand bids times their corresponding bidding prices; the second term is the sum of accepted production bids times their corresponding bidding prices. A block of (7) is equivalent to block (2) but extended to all of the units in the system. Blocks of (8)–(9) state the limits for the main variables of the problem. Block of (10) defines the power generated by any generator in any given hour as the summation of its corresponding production blocks. The block of (11) defines the power consumed by any demand in any given hour as the summation of its corresponding consumption blocks. Block of constraints (12) defines power balance at every node, stating that the total generation at any node plus the net injections through lines must equal the total power demanded (variable) at that node. Note that artificial demands have been introduced to take losses into account. The block of constraints (13) is equivalent to block (5).

It should be noted that the market-clearing price for each hour is not explicitly obtained from solving problem (6)–(13). In this paper, the market-clearing price is defined as the price of the last accepted production bid. According to the actual practice in many electricity markets, such as the one in mainland Spain, a uniform-price auction is considered.

D. Market Simulation

The model described in this paper considers the three typical participants in a pool-based electricity market; namely, generating companies (GENCOs), consuming companies (CONCOs), and the market operator (MO). This section describes the process used for the simulations. The main steps of that iterative process are described below.

- Step 0) An initial solution and initial price-quota curves for all GENCOs are obtained by clearing a market considering that all units offer all of their power blocks for all time periods at their corresponding marginal costs. This provides an initial solution.
- Step 1) Once the market is cleared, all necessary information is made available to the participants. The aggregated offer and the aggregated demand for every hour are made public. Injections at all nodes for every hour are also made public.
- Step 2) With the information obtained from step 1) and with the knowledge of its own previous offer to the market, every GENCO derives its price-quota curve for every hour. Assuming that all other companies do not change their offers, any given GENCO solves problem (1)–(5) described above. The solution obtained allows deriving the optimal offer of that GENCO for the next iteration.
- Step 3) Once all GENCOs have calculated and submitted their offers, the MO clears the market and calculates productions and market-clearing prices for every hour; this is achieved by solving problem (6)–(13) above. If the desired number of iterations have been reached, the simulation concludes; otherwise, the simulation continues in step 1).

Note that the above procedure 1–3 is repeated a sufficient number of times to identify the behavior patterns of the GENCOs. These patterns are crucial to identify Nash equilibria.

Also note that our model differs from Cournot equilibrium models in the sense that the aggregated demand curves in the previous iteration are used to construct the price-quota curves. The calculation of the amount of power offered to the market is achieved solving problem (1)–(5), using the price-quota curves. Our model also differs from supply function equilibrium models, since the decision variables are the quantities offered, not the bid curves. The purpose of our model is to reproduce as close as possible the behavior of profit-seeker market agents. In summary, every producer gathers all of its available information, builds its price-quota curve, solves its problem (1)–(5), and decides the quantity to bid. In turn, the market operator clears the market using a detailed market clearing procedure.

IV. CASE STUDY

In the previous section, both the simulator and an iterative simulation procedure have been presented. This section is devoted to describing Nash equilibria of the market from a game perspective. In this game, the producers maximize their profits from selling their production through a multiperiod pool-based auction.

TABLE I GENERATING UNITS DATA

Туре	А	В	С	D	Е	F	G			
P [MW]	12	76	100	155	197	350	400			
<u>P</u> [MW]	2.4	15.2	25	54.25	68.95	140	100			
C1 ^(*)	25.63	18.98	20.36	10.95	21.02	11.13	7.82			
$C_2^{(*)}$	26.01	19.81	21.92	11.32	22.24	11.79	7.92			
C3 ^(*)	29.38	23.01	23.72	11.79	23.22	12.25	8.14			
$C_4^{(*)}$	33.28	26.46	24.87	12.43	24.22	12.94	8.34			
RR[MW/h]	12	76	100	155	180	120	400			
SC(\$)	114.1	789.6	949.9	1263	1300	5920	NA ^(**)			
MUT (h)	4	8	8	8	12	24	NA ^(**)			
MDT (h)	2	4	8	8	10	48	NA ^(**)			
	^(*) Units: [\$/MWh]. ^(**) NA: Not Applicable.									

TABLE IIUNITS OWNED BY EACH COMPANY

	Α	В	С	D	Е	F	G	Total
E1	2	2	2	2	2	3	6	19
E2	1	3	3	1	3	1	1	13
E3	2	2	2	3	4	3	2	18

The model presented has been tested using an all-thermal power system of realistic size. The considered electricity market comprises three price-maker companies: E1, E2, and E3. The market time horizon is 24 h. Data for all units are based on the 1996 IEEE RTS [24], and are detailed in Table I.

In this table, type indicates the unit type (A, B, C, D, E, F, or G); \overline{P} and \underline{P} indicate, respectively, maximum and minimum power output; every C_b value provides the production cost of the block b of the unit (four-block piecewise convex cost curves are considered); RR gives both ramp-up and ramp-down maximum values; SC is the constant start-up cost; and MUT and MDT represent the minimum up and down times, respectively. A total number of 50 units are considered in the system. Data regarding the distribution of the units among the GENCOs are presented in Table II.

The units are distributed over a 73-nodes, 108-lines transmission network that is based on the network described in [24]. All of the units are placed evenly distributed over the network.

After running 80 iterations of the simulator with data from Tables I and II, reasonable strategies played by market participants are detected. There is no convergence, but the oscillations of the total production are either negligible, small, or bounded, as seen in Fig. 3. Similar oscillatory behavior, although using supply functions, has been reported in [25], where convergence is never attained.

In particular, each of the three companies–E1, E2, and E3–shows a distinctive behavior depending on the time of the day. The first eight hours of the day show a gradual increase in prices, and the remaining 16 h present two peaks: around 11 A.M. and around 10 P.M., respectively. Therefore, it seems logical to divide a company's strategy into two elements: the morning portion and the evening portion. During the first eight hours, two strategies are enough due to the stability of the prices, but for the rest of the day, any company needs at least three strategies to account for the variation in prices of the not-so-stable price curve (see Fig. 4).



Fig. 3. Evolution of the total production for three representative hours.



Fig. 4. Maximum and minimum prices over the last 20 iterations of the simulation (for a total of 80).

A company's strategy is calculated making use of the simulator output. Any company, E1 for instance, competes against the other two in an iterative simulation process, as explained in Section III. As a result, E1 can identify the overall production (megawatts) that maximizes its own profit. This is the base scenario for E1, meaning that E1 may change its output by increasing or decreasing the number of megawatts offered. Given the sum of E1's optimized production for the first eight hours provided by the simulator after 80 iterations (P_{1M}) , two different morning strategies are allowed to E1. The "low-production morning strategy" allows E1's overall morning output to be as much as 90% of P_{1M} . In the same way, a "high-production morning strategy" allows E1's output to be as much as 110% of P_{1M} . Similarly, three different possibilities are considered for the evening period: low-, medium-, and high-production strategies. Fig. 5 shows all possible morning and evening strategies of company E1. The strategies are modeled as constraints of the type \leq ; thus, the missing "medium-production morning strategy" can be considered included in the "high-production morning strategy" as a particular case. Hence, it is clear that nothing prevents the "medium-production morning strategy" from taking place. The +/-10% limits only apply to total production. The sum of E1's optimized production after 80 iterations for the last 16 h is calculated (P_{1E}) . Now the overall output constraints are set to 90, 100, and 110% of P_{1E} , respectively. All of these morning and evening constraints may be changed to a different set of values, and the resulting Nash equilibria would also be



Fig. 5. Morning and evening selected strategies of company E1.

different. However, we restrict our analysis to this case, which provides enough detail.

In conclusion, a company has to choose from six possible strategies, with two options in the morning and three in the evening. It makes a total of $2 \times 3 = 6$ strategies. Following the notation explained in Section II, the set of strategies of company E_i , S_i , contains the following elements:

$$S_{i} = \{s_{i1}, s_{i2}, s_{i3}, s_{i4}, s_{i5}, s_{i6}\}$$

= {(low,low), (low,mid), (low,high),
(high,low), (high,mid), (high,high)}. (14)

Thus, the resulting set of strategies for the three companies contains $6 \times 6 \times 6 = 216$ elements. Each one of these 216 elements in the set is the result of combining all possible morning and evening strategies for all companies. This implies running 216 times the market-clearing algorithm with six extra production constraints enforced. One constraint is taken from the morning period and one from the evening for each company.

Table III provides the results of solving the market-clearing problem for the aforementioned 216 scenarios, with production limits enforced. Each cell in the table represents the profit or payoff in thousands of dollars for companies E1, E2, and E3, respectively. For example, the values of the third row, second column of Table III(c) are: 942.1; 209.5; 301.9. These values are the payoffs when the strategy of E1 is fixed to $s_1 = (low,high)$ and the strategies of E2 and E3 are also fixed to $s_2 = (low,high)$ and $s_3 = (low,mid)$. Table III contains six subtables that represent all strategies of E2 and E3, if E1's strategy is fixed for each subtable.

An iterated deletion process to remove dominated strategies has been performed as follows. First, the strategies in Table III(a) are discarded because E1's (low, low) profit is always lower than E1's (low, high) profit, which happens in Table III(c). Next, the same argument can be used to discard Tables III(b), III(d), and III(e). At this point, only Tables III(c) and III(f) remain eligible to search for Nash equilibria. Note that these two tables represent E1's (low, high) and (high, high) strategies, respectively. E1, with more units that E2 and E3, is always better off playing the "high-production evening strategy;" this is a result of its market power.

 $s_2 = (high, high)$

$s_1 = (low, low)$	$s_3 = (low, low)$	$s_3 = (low, mid)$	$s_3 = (low, high)$	$s_3 = (high, low)$	$s_3 = (high, mid)$	$s_3 = (high, high)$
$s_2 = (low, low)$	264.7; 46.5; 84.2	265.5; 48.4; 85.1	264.7; 48.4; 89.3	269.5; 49.5; 86.8	270.3; 49.5; 88.3	269.5; 49.5; 91.8
$s_2 = (low, mid)$	482.2; 106.0; 143.8	462.9; 100.4; 136.5	330.8; 64.7; 107.7	335.1; 66.7; 103.0	619.6; 145.6; 179.5	335.9; 66.7; 109.2
$s_2 = (low, high)$	658.0; 158.6; 177.8	660.0; 159.6; 188.4	654.8; 158.0; 202.1	664.1; 160.6; 180.5	664.1; 160.6; 192.8	659.6; 159.1; 202.0
$s_2 = (high, low)$	276.2; 51.2; 90.5	276.2; 51.2; 91.2	276.2; 51.2; 93.0	269.0; 49.3; 86.5	269.0; 48.4; 88.3	269.0; 49.3; 91.5
$s_2 = (high, mid)$	436.3; 93.2; 130.2	436.5; 94.9; 131.7	665.5; 156.8; 216.4	372.4; 77.5; 110.5	520.4; 114.4; 208.7	472.6; 101.6; 164.4
$s_2 = (high, high)$	666.8; 161.5; 183.5	670.7; 162.3; 197.6	668.0; 161.5; 209.2	659.5; 159.7; 179.5	658.2; 158.9; 188.6	660.7; 159.7; 205.3
(1			B		(1 • 1 • 1)	(1 + 1 + 1 + 1)
$s_1 = (low, mid)$	$s_3 = (low, low)$	$s_3 = (low, mid)$	$s_3 = (low, high)$	$s_3 = (high, low)$	$s_3 = (high, mid)$	$s_3 = (high, high)$
$\frac{s_2 = (low, low)}{(l + i)}$	265.7; 47.4; 84.2	265.7; 48.4; 86.0	264.9; 48.4; 89.3	270.5; 49.5; 86.8	271.0; 49.6; 87.0	272.7; 50.0; 92.9
$\frac{s_2 = (low, mid)}{(l + 1)}$	338.1; 65.7; 99.8	338.1; 64.7; 104.9	506.2; 105.6; 167.2	448.6; 90.4; 141.5	701.6; 154.0; 206.8	446.7; 92.0; 152.8
$\frac{s_2 = (low, high)}{(link)}$	699.8; 158.8; 177.2	701.1; 158.8; 188.2	701.9; 156.9; 201.6	707.5; 160.6; 180.5	711.1; 161.2; 194.2	709.6; 159.3; 206.1
$\frac{s_2 = (high, low)}{(high, low)}$	277.1; 51.2; 90.5	277.1; 51.2; 93.7	277.9; 51.2; 94.4	269.9; 49.3; 86.5	269.9; 49.3; 88.3	271.6; 49.8; 91.0
$\frac{s_2 = (high, mid)}{(1 + 1 + 1)}$	434.0; 89.4; 123.7	663.9; 145.8; 195.2	514.5; 109.0; 166.2	342.4; 66.6; 102.1	448.1; 92.6; 129.3	573.5; 123.2; 179.3
$s_2 = (high, high)$	714.2; 162.3; 184.2	715.5; 162.3; 194.5	716.3; 162.3; 208.7	706.9; 160.5; 180.1	705.3; 159.7; 191.3	709.0; 160.5; 206.0
$s_1 = (low, high)$	$s_3 = (low, low)$	$s_3 = (low, mid)$	$s_3 = (low, high)$	$s_3 = (high, low)$	$s_3 = (high, mid)$	$s_3 = (high, high)$
$\frac{1}{s_2 = (low, low)}$	942.1.199.3.287.9	747 8. 152 9. 192 9	747 8. 153 9. 205 8	932.0.197.2.296.3	753 5: 155 0: 194 6	752.6:155.0:210.4
$\frac{s_2}{s_2 = (low mid)}$	748 7: 162 9: 180 9	928 2: 202 7: 298 5	749.9:160.8:206.2	763 2: 167 5: 185 5	940 4: 205 9: 308 3	753 1: 163 8: 208 6
$\frac{s_2}{s_2} = (low high)$	747.4, 162.5, 180.9	942 1, 200 5, 301 0	750.6: 164.1: 207.1	752 2: 162 6: 192 4	752.6: 164.6: 105.4	765 3: 166 6: 210.0
$\frac{s_2 = (low, high)}{s_2 = (high low)}$	747.4, 102.3, 180.8	942.1 ; 209.3 ; 301.9	750.0, 104.1, 207.1	732.2, 103.0, 183.4	752.0, 104.0, 195.4	763.3, 160.0, 210.0
$\frac{s_2 - (mgn, low)}{s_2 - (high, mid)}$	935.5; 204.1; 286.0	757.9; 156.7; 200.1	759.2; 156.6; 212.0	932.2; 199.3; 296.0	752.0; 154.8; 194.0	752.0; 153.9; 209.5
$\frac{s_2 = (nign, mia)}{(1 + 1 + 1)}$	927.0; 212.2; 284.2	759.2; 165.4; 202.2	807.6; 175.8; 238.1	756.0; 162.8; 184.0	752.0; 162.6; 193.7	931.0; 207.6; 320.9
$s_2 = (nign, nign)$	770.6; 169.5; 188.4	774.0; 170.1; 201.4	759.2; 166.2; 213.5	939.4; 211.2; 297.6	752.9; 163.5; 194.8	762.6; 167.2; 212.1
			D			
$s_1 = (high, low)$	$s_3 = (low, low)$	$s_3 = (low, mid)$	D $s_3 = (low, high)$	$s_3 = (high, low)$	$s_3 = (high, mid)$	$s_3 = (high, high)$
$s_1 = (high, low)$ $s_2 = (low, low)$	<i>s</i> ₃ = (<i>low</i> , <i>low</i>) 269.1; 48.4; 86.5	<i>s</i> ₃ = (<i>low</i> , <i>mid</i>) 269.1; 49.3; 87.1	D s ₃ = (<i>low</i> , <i>high</i>) 269.1; 49.3; 91.3	<i>s</i> ₃ = (<i>high</i> , <i>low</i>) 269.1; 49.3; 85.2	<i>s</i> ₃ = (<i>high</i> , <i>mid</i>) 269.1; 49.3; 88.0	<i>s</i> ₃ = (<i>high</i> , <i>high</i>) 270.9; 49.8; 91.6
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$	<i>s</i> ₃ = (<i>low</i> , <i>low</i>) 269.1; 48.4; 86.5 390.2; 80.9; 115.6	<i>s</i> ₃ = (<i>low</i> , <i>mid</i>) 269.1; 49.3; 87.1 608.4; 137.7; 188.3	D s ₃ = (<i>low</i> , <i>high</i>) 269.1; 49.3; 91.3 392.2; 81.1; 126.6	<i>s</i> ₃ = (<i>high</i> , <i>low</i>) 269.1; 49.3; 85.2 334.7; 66.6; 102.1	<i>s</i> ₃ = (<i>high</i> , <i>mid</i>) 269.1; 49.3; 88.0 522.6; 118.9; 156.8	<i>s</i> ₃ = (<i>high</i> , <i>high</i>) 270.9; 49.8; 91.6 335.5; 66.6; 110.9
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$	<i>s</i> ₃ = (<i>low</i> , <i>low</i>) 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5	<i>s</i> ₃ = (<i>low</i> , <i>mid</i>) 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8	D s ₃ = (low, high) 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7	<i>s</i> ₃ = (<i>high</i> , <i>low</i>) 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1	<i>s</i> ₃ = (<i>high</i> , <i>mid</i>) 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5	<i>s</i> ₃ = (<i>high</i> , <i>high</i>) 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$ $s_2 = (high, low)$	<i>s</i> ₃ = (<i>low</i> , <i>low</i>) 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5	<i>s</i> ₃ = (<i>low</i> , <i>mid</i>) 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2	$s_3 = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2	<i>s</i> ₃ = (<i>high</i> , <i>mid</i>) 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1	<i>s</i> ₃ = (<i>high</i> , <i>high</i>) 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$ $s_2 = (high, low)$ $s_2 = (high, mid)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7	$s_3 = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7	$s_3 = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$ $s_2 = (high, low)$ $s_2 = (high, mid)$ $s_2 = (high, mid)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9	$s_3 = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1	$s_3 = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$ $s_2 = (high, low)$ $s_2 = (high, mid)$ $s_2 = (high, mid)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5	$s_3 = (low,mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E	$s_3 = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1	$s_3 = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$ $s_2 = (high, low)$ $s_2 = (high, mid)$ $s_2 = (high, high)$ $s_1 = (high, mid)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_3 = (low, low)$	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low, mid)$	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_3 = (high, high)$
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$ $s_2 = (high, low)$ $s_2 = (high, mid)$ $s_2 = (high, mid)$ $s_1 = (high, mid)$ $s_2 = (low, low)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_3 = (low, low)$ 270.1; 49.3; 86.5	$s_3 = (low,mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low,mid)$ 270.1; 49.3; 88.4	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7	$s_{3} = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_{3} = (high, high)$ 270.1; 49.3; 91.3
$s_1 = (high, low)$ $s_2 = (low, low)$ $s_2 = (low, mid)$ $s_2 = (low, high)$ $s_2 = (high, low)$ $s_2 = (high, mid)$ $s_2 = (high, mid)$ $s_1 = (high, mid)$ $s_2 = (low, low)$ $s_2 = (low, mid)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_3 = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_3 = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$ $s_{1} = (high, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$	$s_{3} = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_{3} = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1 708.4; 160.5; 191.5	$s_{3} = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_{3} = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$ $s_{2} = (low, high)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (low, high)$	$s_{3} = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5	$s_{3} = (low,mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_{3} = (low,mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7 270.1; 49.3; 86.3	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1 708.4; 160.5; 191.5 270.1; 49.3; 88.3	$s_{3} = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_{3} = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (high, low)$ $s_{2} = (high, low)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_3 = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 469.2; 97.3; 130.4	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7 270.1; 49.3; 86.3 471.2; 97.5; 141.8	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3 562.3; 119.9; 210.3	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 449.1; 92.6; 124.4	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1 708.4; 160.5; 191.5 270.1; 49.3; 88.3 749.6; 165.9; 251.8	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_3 = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3 555.4; 117.5; 173.1
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (low, high)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$	$s_{3} = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_{3} = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 469.2; 97.3; 130.4 705.5; 159.7; 179.5	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 192.4	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3 562.3; 119.9; 210.3 706.3; 159.7; 204.0	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 449.1; 92.6; 124.4 705.5; 159.7; 179.5	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1 708.4; 160.5; 191.5 270.1; 49.3; 88.3 749.6; 165.9; 251.8 708.4; 160.5; 193.4	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_3 = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3 555.4; 117.5; 173.1 706.3; 159.7; 205.2
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, low)$ $s_{2} = (high, low)$ $s_{2} = (high, low)$ $s_{3} = (high, low)$ $s_{4} = (high, high)$	$s_3 = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_3 = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 469.2; 97.3; 130.4 705.5; 159.7; 179.5	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7 270.1; 49.3; 86.3 471.2; 97.5; 141.8 708.4; 160.5; 192.4 $s_3 = (low, mid)$	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3 562.3; 119.9; 210.3 706.3; 159.7; 204.0 F $s_3 = (low, high)$	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 449.1; 92.6; 124.4 705.5; 159.7; 179.5 $s_{3} = (high, low)$	$s_{3} = (high, mid)$ $269.1; 49.3; 88.0$ $522.6; 118.9; 156.8$ $663.6; 160.5; 194.5$ $269.9; 49.3; 87.1$ $562.3; 131.0; 165.7$ $663.6; 160.5; 190.4$ $s_{3} = (high, mid)$ $270.1; 49.3; 86.7$ $448.1; 93.4; 129.1$ $708.4; 160.5; 191.5$ $270.1; 49.3; 88.3$ $749.6; 165.9; 251.8$ $708.4; 160.5; 193.4$ $s_{3} = (high, mid)$	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_3 = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3 555.4; 117.5; 173.1 706.3; 159.7; 205.2 $s_3 = (high, high)$
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (high, mid)$ $s_{3} = (high, mid)$ $s_{4} = (high, mid)$ $s_{5} = (high, mid)$	$s_{3} = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_{3} = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 469.2; 97.3; 130.4 705.5; 159.7; 179.5 $s_{3} = (low, low)$ 751.0; 154.4; 182.7	$s_{3} = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_{3} = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7 270.1; 49.3; 86.3 471.2; 97.5; 141.8 708.4; 160.5; 192.4 $s_{3} = (low, mid)$ 752.2; 154.8; 105.2	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3 562.3; 119.9; 210.3 706.3; 159.7; 204.0 F $s_3 = (low, high)$ 950.4.107.8; 311.5	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 449.1; 92.6; 124.4 705.5; 159.7; 179.5 $s_{3} = (high, low)$ 752.2; 153.9; 183.0	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1 708.4; 160.5; 191.5 270.1; 49.3; 88.3 749.6; 165.9; 251.8 708.4; 160.5; 193.4 $s_{3} = (high, mid)$ 762.8: 157.1: 107.4	$s_{3} = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_{3} = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3 555.4; 117.5; 173.1 706.3; 159.7; 205.2 $s_{3} = (high, high)$ 941.8: 196.0; 316.2
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (high, high)$ $s_{3} = (low, low)$ $s_{4} = (low, mid)$ $s_{5} = (low, mid)$ $s_{5} = (low, mid)$	$s_{3} = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_{3} = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 469.2; 97.3; 130.4 705.5; 159.7; 179.5 $s_{3} = (low, low)$ 751.0; 154.4; 182.7 752.2; 164.8; 182.2	$s_{3} = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_{3} = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7 270.1; 49.3; 86.3 471.2; 97.5; 141.8 708.4; 160.5; 192.4 $s_{3} = (low, mid)$ 752.2; 154.8; 195.3 752.2; 163.5; 106.9	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3 562.3; 119.9; 210.3 706.3; 159.7; 204.0 F $s_3 = (low, high)$ 950.4; 197.8; 311.5 800.2; 172.2; 232.0	$s_{3} = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_{3} = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 449.1; 92.6; 124.4 705.5; 159.7; 179.5 $s_{3} = (high, low)$ 752.2; 153.9; 183.0 752.2; 163.8; 182.0	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1 708.4; 160.5; 191.5 270.1; 49.3; 88.3 749.6; 165.9; 251.8 708.4; 160.5; 193.4 $s_{3} = (high, mid)$ 762.8; 157.1; 197.4 753.1; 161.9; 105.5	$s_{3} = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_{3} = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3 555.4; 117.5; 173.1 706.3; 159.7; 205.2 $s_{3} = (high, high)$ 941.8; 196.0; 316.2 944 9: 205 9: 317 7
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (high, low)$ $s_{2} = (low, high)$ $s_{2} = (low, mid)$ $s_{3} = (low, low)$ $s_{4} = (low, low)$ $s_{5} = (low, mid)$	$s_{3} = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_{3} = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 469.2; 97.3; 130.4 705.5; 159.7; 179.5 $s_{3} = (low, low)$ 751.0; 154.4; 182.7 752.2; 164.8; 183.2 045.1; 206.0; 284.9	$s_{3} = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_{3} = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7 270.1; 49.3; 86.3 471.2; 97.5; 141.8 708.4; 160.5; 192.4 $s_{3} = (low, mid)$ 752.2; 154.8; 195.3 752.2; 163.5; 196.8	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3 562.3; 119.9; 210.3 706.3; 159.7; 204.0 F $s_3 = (low, high)$ 950.4; 197.8; 311.5 800.2; 172.2; 232.9	$s_3 = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_3 = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 449.1; 92.6; 124.4 705.5; 159.7; 179.5 $s_3 = (high, low)$ 752.2; 153.9; 183.0 752.2; 163.8; 182.9 P40 2: 207 5: 202 7	$s_{3} = (high, mid)$ $269.1; 49.3; 88.0$ $522.6; 118.9; 156.8$ $663.6; 160.5; 194.5$ $269.9; 49.3; 87.1$ $562.3; 131.0; 165.7$ $663.6; 160.5; 190.4$ $s_{3} = (high, mid)$ $270.1; 49.3; 86.7$ $448.1; 93.4; 129.1$ $708.4; 160.5; 191.5$ $270.1; 49.3; 88.3$ $749.6; 165.9; 251.8$ $708.4; 160.5; 193.4$ $s_{3} = (high, mid)$ $762.8; 157.1; 197.4$ $753.1; 161.9; 195.5$ $762.6; 167.9; 106.5$	$s_{3} = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_{3} = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3 555.4; 117.5; 173.1 706.3; 159.7; 205.2 $s_{3} = (high, high)$ 941.8; 196.0; 316.2 944.9; 205.8; 317.7 760.6, 166, 6; 200.0
$s_{1} = (high, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, high)$ $s_{2} = (high, low)$ $s_{2} = (high, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$ $s_{2} = (high, mid)$ $s_{2} = (low, mid)$ $s_{2} = (low, low)$ $s_{2} = (low, low)$ $s_{2} = (low, mid)$ $s_{2} = (low, mid)$ $s_{3} = (low, low)$ $s_{4} = (low, mid)$ $s_{5} = (low, low)$ $s_{5} = (low, mid)$ $s_{5} = (low, low)$ $s_{5} = (low, low)$ $s_{5} = (low, low)$	$s_{3} = (low, low)$ 269.1; 48.4; 86.5 390.2; 80.9; 115.6 661.0; 159.7; 179.5 269.1; 48.4; 86.5 645.1; 150.1; 189.1 661.1; 159.7; 179.5 $s_{3} = (low, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 469.2; 97.3; 130.4 705.5; 159.7; 179.5 $s_{3} = (low, low)$ 751.0; 154.4; 182.7 752.2; 164.8; 183.2 945.1; 206.0; 284.8 759.5; 159.2; 104.2	$s_3 = (low, mid)$ 269.1; 49.3; 87.1 608.4; 137.7; 188.3 663.6; 160.5; 191.8 269.9; 49.3; 86.6 485.2; 108.7; 145.2 663.6; 159.6; 193.3 $s_3 = (low, mid)$ 270.1; 49.3; 88.4 754.5; 163.8; 253.4 708.4; 160.5; 190.7 270.1; 49.3; 86.3 471.2; 97.5; 141.8 708.4; 160.5; 192.4 $s_3 = (low, mid)$ 752.2; 154.8; 195.3 752.2; 163.5; 196.8 750.9; 163.4; 178.2	D $s_3 = (low, high)$ 269.1; 49.3; 91.3 392.2; 81.1; 126.6 661.8; 158.8; 203.7 269.1; 49.3; 90.2 430.0; 92.6; 136.7 661.8; 159.7; 203.9 E $s_3 = (low, high)$ 270.1; 49.3; 91.6 490.1; 104.2; 150.1 706.3; 158.4; 203.9 270.1; 49.3; 83.3 562.3; 119.9; 210.3 706.3; 159.7; 204.0 F $s_3 = (low, high)$ 950.4; 197.8; 311.5 800.2; 172.2; 232.9 760.6; 166.6; 210.2	$s_3 = (high, low)$ 269.1; 49.3; 85.2 334.7; 66.6; 102.1 663.6; 160.5; 180.1 269.1; 48.4; 85.2 334.7; 65.7; 101.7 663.6; 160.5; 180.1 $s_3 = (high, low)$ 270.1; 49.3; 86.5 342.8; 66.6; 102.1 705.5; 159.7; 179.4 270.1; 49.3; 86.5 449.1; 92.6; 124.4 705.5; 159.7; 179.5 $s_3 = (high, low)$ 752.2; 153.9; 183.0 752.2; 163.8; 182.9 949.2; 207.5; 293.7 759.5, 156.2, 100.2	$s_{3} = (high, mid)$ 269.1; 49.3; 88.0 522.6; 118.9; 156.8 663.6; 160.5; 194.5 269.9; 49.3; 87.1 562.3; 131.0; 165.7 663.6; 160.5; 190.4 $s_{3} = (high, mid)$ 270.1; 49.3; 86.7 448.1; 93.4; 129.1 708.4; 160.5; 191.5 270.1; 49.3; 88.3 749.6; 165.9; 251.8 708.4; 160.5; 193.4 $s_{3} = (high, mid)$ 762.8; 157.1; 197.4 753.1; 161.9; 195.5 763.6; 167.8; 196.1	$s_3 = (high, high)$ 270.9; 49.8; 91.6 335.5; 66.6; 110.9 661.8; 159.7; 204.0 269.1; 49.3; 92.8 410.7; 88.0; 133.0 661.8; 159.7; 204.0 $s_3 = (high, high)$ 270.1; 49.3; 91.3 342.8; 65.8; 110.1 703.4; 158.9; 205.6 269.3; 49.3; 90.3 555.4; 117.5; 173.1 706.3; 159.7; 205.2 $s_3 = (high, high)$ 941.8; 196.0; 316.2 944.9; 205.8; 317.7 760.6; 166.6; 209.9
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769.1; 168.8; 186.3 751.8; 164.4; 195.0 760.6; 166.6; 210.1 753.1; 164.4; 183.0 763.6; 167.7; 196.3 756.4; 165.5; 210.0

TABLE III PROFITS AND NASH EQUILIBRIA FOR ALL COMPANIES

#	S	TRATEGIE	PROFIT				
#	E1	E2	E3	E1	E2	E3	Total
1	low, high	low, high	low, mid	942.1	209.5	301.9	1453.5
2	low, high	low, mid	high, mid	940.4	205.9	308.3	1454.6
3	low, high	high, high	high, low	939.4	211.2	297.6	1448.2
4	low, high	high, mid	high, high	931.0	207.6	320.9	1459.5
5	high, high	high, mid	low, high	948.3	209.4	310.9	1468.6
6	high, high	low, high	high, low	949.2	207.5	293.7	1450.4
7	high, high	low, mid	high, high	944.9	205.8	317.7	1468.4

TABLE IV Nash Equilibria

Considering Tables III(c) and III(f), only 72 different strategy vectors are possible Nash equilibria. Having in mind the definition of a Nash equilibrium presented in Section II and analyzing in detail all of the remaining possibilities, seven pure Nash equilibria are found. Table IV describes each of them. Note that pure Nash equilibria may not exist in other cases, but a mixed strategy equilibrium always exists for finite games [26]. Therefore, mixed equilibria are also possible and can be obtained by solving a system of equations that guarantees the same profit for every pure strategy selected by any player (condition of mixed strategies equilibrium). The system of equations is highly nonlinear. The computation involved and the analysis of its solution are outside the scope of this paper.

All equilibria are similar in terms of individual and overall profits, although the strategies are quite different. Note that E2 never plays its "low-production evening strategy," because E2 has many units, albeit not as many as E1. In addition, Nash equilibrium #2 is "dominated in profits" by equilibrium #5 (i.e., the three companies make more money in equilibrium #5 than in equilibrium #2). This makes equilibrium #2 unlikely. Finally, equilibrium #6 is the one with the second lowest total profit, although it is the most profitable for E1. Note that strategies that only differ in a $\pm 10\%$ margin have a big impact in the companies' profits. For example, E1's profits range between thousands of U.S.\$: 264.7 and 950.4.

The simulations presented were performed using CPLEX 7.5 under GAMS [27] on a DELL PowerEdge 2500 biprocessor (1.26 GHz) with 2 GB of RAM. The total required CPU time to solve the 216 scenarios was 503.6 min. This represents an average of 2.3 min per scenario.

V. CONCLUSION

A methodology to study Nash equilibria in auction-based multiperiod electricity markets has been presented in this paper. Nash equilibria are valuable instruments because they predict if a market can be stable in the long term. To obtain these equilibria, a step-by-step technique has been used. First, a very detailed market model has been developed. Second, an iterative simulation process has provided bidding criteria to the producers. And third, these bidding criteria have been used to build realistic strategies. In addition, unrealistic strategies have been removed by iterated deletion.

The proposed methodology can be of interest to regulators that study the behavior of oligopolistic markets and to bidders who want to know the best strategy to follow. One extra advantage comes from the ex-post analysis of the results, providing rules based on the Nash outcomes. A case study illustrating the technique has shown seven possible Nash equilibria. These equilibria are comparatively characterized. In the future, we will study the effect of different production constraints in the Nash equilibria results.

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