ALGEBRAS OF QUOTIENTS OF GRADED LIE ALGEBRAS

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Abstract. In this paper we explore graded algebras of quotients of Lie algebras with special emphasis on the 3-graded case and answer some natural questions concerning its relation to maximal Jordan systems of quotients.

Introduction

The study of algebras of quotients of Lie algebras was initiated by the second author in [23]. Inspired by the associative definition of ring of quotients given by Utumi in [25] and adapting some ideas from [16], a notion of general algebra of quotients of a Lie algebra was introduced and also, as a special concrete example, a maximal algebra of quotients for every semiprime Lie algebra was built.

The introductory paper [23] was followed by [4], where abstract properties of algebras of quotients were considered, and by [3], where the main objective was to compute maximal algebras of quotients of Lie algebras of the form $A^-/Z_A$ and $K/Z_K$, for $A$ a prime associative algebra, $K$ the Lie algebra of skew elements of a prime associative algebra with involution and $Z_A$ and $Z_K$ their respective centers.

In the present article we are interested in algebras of quotients of graded Lie algebras. One of the reasons to consider them is the relationship between 3-graded Lie algebras and Jordan pairs: recall that given a Jordan pair $V$, there exists a 3-graded Lie algebra, $TKK(V)$, such that $TKK(V)^- = V^-$ and $TKK(V)^+ = V^+$ (and other additional properties), where $TKK(V)$ stands for the Tits-Kantor-Koecher Lie algebra associated to $V$.

Apart from a preliminary section, we have divided the paper into three parts. We start by introducing in Section 2 graded algebras of quotients of Lie algebras. As in the non-graded case, graded algebras of quotients of graded Lie algebras inherit primeness, semiprimeness and strong nondegeneracy. We study the relationship between the graded and the non-graded notions of quotients, and give important examples of graded algebras of quotients of Lie algebras. Concretely, for $A = \bigoplus_{\sigma \in G} A_\sigma$ a graded semiprime associative algebra with an involution $*$ such that $A_\sigma^* = A_\sigma$, $K_A$ the Lie algebra of all the skew elements of $A$, and $Z_{K_A}$ the annihilator of $K_A$, we prove that if $Q$ is a $G$-graded $*$-subalgebra of $Q_s(A)$ (the Martindale symmetric algebra of quotients of $A$) containing $A$, then $K_Q/Z_{K_Q}$ is a $G$-graded Lie algebra of quotients of $K_A/Z_{K_A}$, and $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$ is a graded Lie algebra of quotients of $[K_A, K_A]/Z_{[K_A, K_A]}$.

As a corollary we obtain that $Q^-/Z_Q$ and $[Q^-, Q^-]/Z_{[Q, Q]}$ are graded algebras of quotients of $A^-/Z_A$ and $[A^-, A^-]/Z_{[A, A]}$, respectively.

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Section 3 deals with the particular case of 3-graded Lie algebras. Finally, in Section 4 we relate Jordan systems of quotients and Lie algebras of quotients, where by a Jordan system we understand a Jordan pair, a triple system or a Jordan algebra. We prove that the maximal system of quotients of a strongly nondegenerate Jordan system is the associated system to the maximal Lie algebra of quotients of its TKK (see Section 4 for the definitions).

1. Preliminaries

In this paper we will deal with Lie algebras over an arbitrary unital and commutative ring of scalars $\Phi$.

Recall that a $\Phi$-module $L$ together with a bilinear map $[,] : L \times L \to L$, denoted by $(x, y) \mapsto [x, y]$ (called the bracket of $x$ and $y$), is said to be a Lie algebra over $\Phi$ if the following axioms are satisfied:

(i) $[x, x] = 0$,

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

Given an abelian group $G$ (whose neutral element will be denoted by $e$), a Lie algebra $L$ is called $G$-graded if $L = \oplus_{\sigma \in G} L_{\sigma}$, where $L_{\sigma}$ is a $\Phi$-subspace of $L$ and $[L_{\sigma}, L_{\tau}] \subseteq L_{\sigma \tau}$ for all $\sigma, \tau \in G$. When the group is understood we will use the term “graded” instead of “$G$-graded”.

For any subset $X$ of $L$, its support is defined as $\text{Supp}(X) = \{ \sigma \in G \mid x_{\sigma} \neq 0 \text{ for some } x \in X \}$. The grading on $L$ is called finite if $\text{Supp}(L)$ is a finite set.

Every Lie algebra $L$ can be seen as a graded algebra over any group $G$ by considering the trivial grading, that is $L = \oplus_{\sigma \in G} L_{\sigma}$, where $L_e = L$ (being $e$ the neutral element of the group $G$) and $L_{\sigma} = \{0\}$ for any $\sigma \in G \setminus \{e\}$. Thanks to this fact, every statement for graded algebras can be read in terms of algebras.

For a graded Lie algebra $L = \oplus_{\sigma \in G} L_{\sigma}$, the set of homogeneous elements is $\bigcup_{\sigma \in G} L_{\sigma}$. The elements of $L_{\sigma}$ are said to be homogeneous of degree $\sigma$. Graded subalgebras are defined in the natural way.

An ideal $I$ of a graded Lie algebra $L = \oplus_{\sigma \in G} L_{\sigma}$ is called a graded ideal if whenever $y = \sum y_{\sigma} \in I$ we have $y_{\sigma} \in I$, for every $\sigma \in G$. It is straightforward to show that the sum, the intersection and the product of graded ideals are again graded ideals.

Let $X$ and $Y$ be two subsets of a Lie algebra $L$. The set

$$\text{Ann}_X(Y) := \{ x \in X \mid [x, y] = 0 \text{ for every } y \in Y \}$$

is called the annihilator of $Y$ in $X$, while we will refer to

$$\text{QAnn}_X(Y) := \{ x \in X \mid [x, [x, y]] = 0 \text{ for every } y \in Y \}$$

as the quadratic annihilator of $Y$ in $X$. It is easy to check, by using the Jacobi identity, that $\text{Ann}_L(X)$ is a (graded) ideal of $L$ when $X$ is also a (graded) ideal of $L$, although the quadratic annihilator of an ideal needs not be an ideal (see $[22$, Examples 1.1$])$. In the special situation that $X = L = Y$, $\text{Ann}_L(L)$ is called the center of $L$ and is denoted by $Z_L$; the elements of the center are called total zero divisors. If there is no risk of confusion, we will write $\text{Ann}(Y)$ for $\text{Ann}_L(Y)$.

We say that a (graded) Lie algebra $L$ is (graded) semiprime if for every nonzero (graded) ideal $I$ of $L$, $[I, I] \neq 0$. In the sequel we shall usually denote $[I, I]$ by $I^2$. Next, $L$ is said to be (graded) prime if for nonzero (graded) ideals $I$ and $J$ of $L$, $[I, J] \neq 0$. A (graded) ideal $I$ of
$L$ is said to be (graded) essential if its intersection with any nonzero (graded) ideal is again a nonzero (graded) ideal.

**Lemma 1.1.** Let $I$ be a graded ideal of a graded Lie algebra $L = \oplus_{\sigma \in G} L_{\sigma}$. Then:

(i) $\text{Ann}(I)$ is a graded Lie ideal of $L$. In particular, $Z_L$ is a graded ideal of $L$.

If moreover $L$ is graded semiprime, then:

(ii) $I \cap \text{Ann}(I) = 0$.

(iii) $I$ is a graded essential ideal of $L$ if and only if $\text{Ann}(I) = 0$.

**Proof.** (i). The only thing we are going to show is that every homogeneous component of any element $x \in \text{Ann}(I)$ is again in $\text{Ann}(I)$. Fix $20 \tau \in G$. Note that $[x_{\sigma}, I_{\tau}] = 0$ for every $\sigma \in G$ because otherwise there would exist $y_{\tau} \in I_{\tau}$ such that $[x_{\sigma}, y_{\tau}] \neq 0$ for some $\sigma \in G$; this would imply $0 = [x, I_{\tau}] \subseteq [x, I] = 0$, a contradiction. Hence $[x_{\sigma}, I] = \oplus_{\tau \in G} [x_{\sigma}, I_{\tau}] = 0$.

To obtain (ii) and (iii), follow the proofs of conditions (i) and (ii) in [23, Lemma 1.2]. \( \square \)

Note that if $L$ is (graded) semiprime, then $I^2$ is a (graded) essential ideal if $I$ is so. Further, the intersection of (graded) essential ideals is clearly again a (graded) essential ideal.

Given an element $x$ of a Lie algebra $L$, we may define a map $\text{ad} x: L \to L$ by $\text{ad} x(y) = [x, y]$. A (homogeneous) element $x$ of a (graded) Lie algebra $L$ is a (homogeneous) absolute zero divisor if $(\text{ad} x)^2 = 0$. The algebra $L$ is said to be (graded) strongly nondegenerate if it does not contain nonzero (homogeneous) absolute zero divisors. It is obvious from the definitions that (graded) strongly nondegenerate Lie algebras are (graded) semiprime, but the converse does not hold (see [23, Remark 1.1]).

## 2. Graded algebras of quotients of graded Lie algebras

Following the pattern of [23] it is possible to introduce algebras of quotients of graded Lie algebras, and to build a maximal graded algebra of quotients for every graded semiprime Lie algebra. We devote this section to this end and to relate both notions: the graded and the non-graded ones. Some interesting examples of graded algebras of quotients have also been found.

**Definitions 2.1.** Let $L = \oplus_{\sigma \in G} L_{\sigma}$ be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma \in G} Q_{\sigma}$. We say that $Q$ is a graded algebra of quotients of $L$ or also that $L$ is a graded subalgebra of quotients of $Q$ if the following equivalent conditions are satisfied:

(i) Given $0 \neq p_{\sigma} \in Q_{\sigma}$ and $q_{\tau} \in Q_{\tau}$, there exists $x_{\alpha} \in L_{\alpha}$ such that $[x_{\alpha}, p_{\sigma}] \neq 0$ and $[x_{\alpha}, L(q_{\tau})] \subseteq L$, where $L(q_{\tau})$ denotes the linear span in $Q$ of $q_{\tau}$ and the elements of the form $\text{ad}_{x_{1}} \cdots \text{ad}_{x_{n}} q_{\tau}$, with $n \in \mathbb{N}$ and $x_{1}, \ldots, x_{n} \in L$.

(ii) $Q$ is graded ideally absorbed into $L$, i.e., for every nonzero element $q_{\tau} \in Q_{\tau}$ there exists a nonzero graded ideal $I$ of $L$ with $\text{Ann}_{L}(I) = 0$ and such that $0 \neq [I, q_{\tau}] \subseteq L$.

If for any nonzero $p_{\sigma} \in Q_{\sigma}$ there exists $x_{\alpha} \in L_{\alpha}$ such that $0 \neq [x_{\alpha}, p_{\sigma}] \in L$, then we say that $Q$ is a graded weak algebra of quotients of $L$, and $L$ is called a graded weak subalgebra of quotients of $Q$.

**Remarks 2.2.** The definitions before are consistent with the non-graded ones (see [23, Definitions 2.1 and 2.5]) in the sense that if $Q$ is a (weak) algebra of quotients of a Lie algebra $L$,
then it is also a graded (weak) algebra of quotients of $L$ when considering the trivial gradings on $L$ and $Q$.

The necessary and sufficient condition for a graded Lie algebra to have a graded (weak) algebra of quotients is the absence of homogeneous total zero divisors different from zero, condition that turns out to be equivalent to have zero center.

**Remark 2.3.** Although every graded algebra of quotients is a graded weak algebra of quotients, the converse is not true, as shown in the following example.

Consider the $\mathbb{C}$-module $P$ of all polynomials $\sum_{r=0}^{m} \alpha_r x^r$, with $\alpha_i \in \mathbb{C}$ and $m \in \mathbb{N}$, with the natural $\mathbb{Z}$-grading. Denote by $\sigma : \mathbb{C} \to \mathbb{C}$ the complex conjugation. Then the following product makes $P$ into a $\mathbb{Z}$-graded Lie algebra:

$$
\left[ \sum_{r=0}^{m} \alpha_r x^r, \sum_{s=0}^{n} \beta_s x^s \right] = \sum_{r,s} (\alpha_r \beta^s_s - \beta_s \alpha^s_r) x^{r+s}.
$$

Let $Q$ be the $\mathbb{Z}$-graded Lie algebra $P/I$, where $I$ denotes the $\mathbb{Z}$-graded ideal of $P$ consisting of all polynomials whose first nonzero term has degree at least 4, and let $L$ be the following graded subalgebra of $Q$:

$$
L = \{ \overline{\alpha_0} + \overline{\alpha_2} x^2 + \overline{\alpha_3} x^3 \mid \alpha_0, \alpha_2, \alpha_3 \in \mathbb{C} \},
$$

where $\overline{\cdot}$ denotes the class of an element $y \in P$ in $P/I$. Then $Q$ is a graded weak algebra of quotients of $L$, but $Q$ is not a graded quotient algebra of $L$ since no $l \in L$ satisfies $[l, \overline{x}] \in L$ and $[l, \overline{x}^3] \neq 0$ (see [23, Remark 2.6]).

As it happened in the non-graded case (see [23, Proposition 2.7]), some properties of a graded Lie algebra $L$ are inherited by each of its graded weak algebras of quotients. We give here a different approach to show that graded strong nondegeneracy is inherited.

**Proposition 2.4.** Let $Q = \bigoplus_{\sigma \in G} Q_{\sigma}$ be a graded weak algebra of quotients of a graded subalgebra $L$ and suppose that $\Phi$ is 2 and 3-torsion free. Then $L$ graded strongly nondegenerate implies $Q$ graded strongly nondegenerate.

**Proof.** Suppose that there exists an element $0 \neq q_{\tau}$ in $Q_{\tau}$ such that $(ad q_{\tau})^2 = 0$. Since $Q$ is a graded weak algebra of quotients of $L$, $0 \neq y := [q_{\tau}, x_{\sigma}] \in L$ for some $x_{\sigma} \in L_{\sigma}$. As $q_{\tau}$ is in $\text{QAnn}_Q(Q) \subseteq \text{QAnn}_Q(L)$ we have, by [22, Theorem 2.1], that $[y, [y, u]] \in \text{QAnn}_L(L)$ for every $u \in L$ (observe that the map $u \mapsto ad u$ gives an isomorphism between $L$ and its image inside $A(Q)$, the Lie subalgebra of $\text{End}(Q)$ generated by the elements $ad x$ for $x$ in $Q$; this allows to apply the result in [22]). But $\text{QAnn}_L(L)$ is zero, because $L$ is graded strongly nondegenerate, therefore $[y, [y, u]] = 0$ for every $u \in L$. Again the same reasoning leads to $y = 0$, a contradiction. This shows the statement. \qed

Now we show the relationship between graded (weak) algebras of quotients and (weak) algebras of quotients, a useful tool that combined with other results will provide with examples of graded algebras of quotients.

**Lemma 2.5.** Let $L$ be a graded subalgebra of a graded Lie algebra $Q = \bigoplus_{\sigma \in G} Q_{\sigma}$. If $Q$ is a weak algebra of quotients of $L$ then $Q$ is also a graded weak algebra of quotients of $L$.

**Proof.** For $0 \neq q_{\tau} \in Q_{\tau}$, apply the hypothesis to find $x \in L$ such that $0 \neq [x, q_{\tau}] \in L$; in particular, $0 \neq [x_{\alpha}, q_{\tau}] \in L_{\alpha \tau}$ for some $\alpha \in G$. \qed
Recall that given a subalgebra $L$ of a Lie algebra $Q$ and an element $q \in Q$, the set
\[
(L : q) := \{ x \in L \mid [x, L(q)] \subseteq L \}
\]
is an ideal of $L$ (see [23, Lemma 2.10 (i)]). In case of being $L$ a graded subalgebra of a graded Lie algebra $Q$ and $q_r \in Q$, it can be proved $(L : q_r)$ is indeed a graded ideal of $L$.

**Lemma 2.6.** Let $Q = \oplus_{\sigma \in G} Q_{\sigma}$ be a graded algebra of quotients of a graded semiprime Lie algebra $L$. Then, given $0 \neq p_\sigma \in Q_{\sigma}$ and $q_{\tau_i} \in Q_{\tau_i}$, with $\tau_i \in G$ and $i = 1, \ldots, n$ (for any $n \in \mathbb{N}$), there exist $\alpha \in G$ and $x_\alpha \in L_\alpha$ such that $[x_\alpha, p_\sigma] \neq 0$ and $[x_\alpha, L(q_{\tau_i})] \subseteq L$ for every $i = 1, \ldots, n$.

**Proof.** Consider $0 \neq p_\sigma \in Q_{\sigma}$ and $q_{\tau_i} \in Q_{\tau_i}$, with $i = 1, \ldots, n$. By [23, Lemma 2.10 (i)], $(L : q_{\tau_i})$ is a graded essential ideal of $L$ for every $i$, hence $I = \bigcap_{i=1}^n (L : q_{\tau_i})$ is again a graded essential ideal of $L$. Condition (iii) in Lemma 1.1 implies $\text{Ann}_L(I) = 0$ and by [23, Lemma 2.11] we obtain $\text{Ann}_Q(I) = 0$. So, there exists $x \in I$ such that $[x, p_\sigma] \neq 0$, and if we decompose $x$ into its homogeneous components we find some $\alpha \in G$ satisfying $[x_\alpha, p_\sigma] \neq 0$. Now the proof is complete because $x_\alpha \in I$ as $I$ is a graded ideal and $x \in I$. \hfill \□

**Proposition 2.7.** Let $L$ be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma \in G} Q_{\sigma}$. Consider the following conditions:

(i) $Q$ is an algebra of quotients of $L$.

(ii) $Q$ is a graded algebra of quotients of $L$.

Then (i) implies (ii). Moreover, if $L$ is graded semiprime then (ii) implies (i).

**Proof.** (i) $\Rightarrow$ (ii). Given $0 \neq p_\sigma \in Q_{\sigma}$ and $q_r \in Q_r$, by the hypothesis there exists $x \in L$ satisfying $[x, p_\sigma] \neq 0$ and $[x, L(q_r)] \subseteq L$, that is, $x \in (L : q_r)$. This means by the considerations above that $x_\alpha \in (L : q_r)$.

(ii) $\Rightarrow$ (i). Suppose now that $Q$ is a graded algebra of quotients of $L$, with $L$ graded semiprime. Take $p, q$ in $Q$, with $p \neq 0$; let $\sigma \in G$ be such that $p_\sigma \neq 0$ and write $\tau_1, \tau_2, \ldots, \tau_n$ to denote the elements of $\text{Supp}(q)$. By Lemma 2.6 it is possible to find an element $x_\alpha \in L_\alpha$ satisfying $[x_\alpha, p_\sigma] \neq 0$ and $[x_\alpha, L(q_{\tau_i})] \subseteq L$ for every $i = 1, \ldots, n$, hence $[x_\alpha, p] \neq 0$ and $[x_\alpha, L(q)] \subseteq L$; this shows that $Q$ is an algebra of quotients of $L$. \hfill \□

We continue the section with some important examples of graded algebras of quotients of graded Lie algebras. For brevity, we will not include all the definitions involved in the first example (we refer the reader to [5, 5.4]). However, in Section 4 we will explain what the TKK-algebra of a Jordan pair is. Recall that any strongly prime hermitian Jordan pair $V$ is sandwiched as follows (see [5, 5.4]):

$$H(R, *) \trianglelefteq V \leq H(Q(R), *),$$

where $R$ is a $*$-prime associative pair with involution and $Q(R)$ is its associative Martindale pair of symmetric quotients.

**Example 2.8.** Let $R$ be a $*$-prime associative pair with involution, and $Q(R)$ its Martindale pair of symmetric quotients. Then $TKK(H(Q(R), *))$ is a 3-graded algebra of quotients of $TKK(H(R, *))$. 
Proof. From [6, Proposition 4.2 and Corollary 4.3], \( Q := TKK(H(Q(R), *)) \) is ideally absorbed into the strongly prime Lie algebra \( L := TKK(H(R, *)) \); use [23, Proposition 2.15] to obtain that \( Q \) is an algebra of quotients of \( L \) and Proposition 2.7 to reach the conclusion. \( \square \)

Now, we provide with examples of graded Lie algebras of quotients of graded Lie algebras arising from associative algebras graded by abelian groups.

Let \( A \) be an associative algebra with an involution \(*\); then the set of skew elements \( K_A = \{ x \in A : x^* = -x \} \) is a Lie subalgebra of \( A^- \). If \( A \) is \( G \)-graded and \( A^*_G = A_G \), for all \( \sigma \in G \), then \( K_A \) and \( K_A/Z_KA \) are \( G \)-graded Lie algebras too.

For a semiprime associative algebra, denote by \( Q_A(A) \) its Martindale symmetric ring of quotients (see, for example [2] for more information about this ring of quotients).

**Theorem 2.9.** Let \( A \) be a semiprime \( G \)-graded associative algebra with an involution \(*\) such that \( A^*_G = A_G \), for every \( \sigma \in G \), and let \( Q = \oplus_{\sigma \in G} Q_\sigma \) be a \( G \)-graded overalgebra of \( A \) contained in \( Q_A(A) \) and satisfying \( Q^*_G = Q_G \) for every \( \sigma \in G \). Then:

(i) \( KQ/Z_KQ \) is a graded algebra of quotients of \( K_A/Z_KA \).

(ii) \( [KQ, KQ]/Z_{[KQ, KQ]} \) is a graded algebra of quotients of \( [K_A, K_A]/Z_{[K_A, K_A]} \).

**Proof.** By [4, Theorem 1], \( KQ/Z_KQ \) and \( [KQ, KQ]/Z_{[KQ, KQ]} \) are algebras of quotients of \( K_A/Z_KA \) and \( [K_A, K_A]/Z_{[K_A, K_A]} \); respectively. Since \( A \) is semiprime, and so is \( Q \) (by [2, Lemma 2.1.9.(i)]), it follows from [15, Theorem 6.1] that \( K_A/Z_KA \) and \( KQ/Z_KQ \) are semiprime Lie algebras. In particular, they are graded semiprime, hence Proposition 2.7 applies to get the result. \( \square \)

Note that for an arbitrary graded associative algebra \( A \), the graded Lie algebra \( A^- \) is graded isomorphic to \( K_{A \oplus A^0} \) and hence \( A^-/Z_A \) is graded isomorphic to \( K_{A \oplus A^0}/Z_{K_{A \oplus A^0}} \), where \( A^0 \) denotes the opposite algebra of \( A \), and \( A \oplus A^0 \) is endowed with the exchange involution. This fact allows to obtain the following consequence of the theorem before.

**Corollary 2.10.** Let \( A \) be a semiprime graded associative algebra and \( Q \) be a graded subalgebra of \( Q_A(A) \) containing \( A \). Then

(i) \( Q^-/Z_Q \) is a graded algebra of quotients of \( A^-/Z_A \).

(ii) \( [Q, Q]/Z_{[Q, Q]} \) is a graded algebra of quotients of \( [A, A]/Z_{[A, A]} \).

Following the construction given in [23] of the maximal algebra of quotients of a semiprime Lie algebra, it is possible to build a maximal graded algebra of quotients for every graded semiprime Lie algebra. Taking into account that the elements of the maximal algebra of quotients of a semiprime Lie algebra arise from derivations defined on essential ideals, it seems natural to consider instead graded derivations defined on graded essential ideals. With this idea in mind, we proceed to introduce a new graded algebra of quotients.

Let \( I \) be an ideal of a Lie algebra \( L \). A linear map \( \delta : I \to L \) is said to be a **derivation of \( L \)** if

\[
\delta([x, y]) = [\delta x, y] + [x, \delta y]
\]

for every \( x, y \in I \), and \( \text{Der}(I, L) \) will stand for the set of all derivations from \( I \) into \( L \). Suppose now that the Lie algebra \( L \) is graded by a group \( G \), and that \( I \) is a graded ideal of \( L \). We say that a derivation \( \delta \) has **degree** \( \sigma \in G \) if it satisfies \( \delta(I_\tau) \subseteq L_{\tau\sigma} \) for every \( \tau \in G \). In this
case, \( \delta \) is called a \textit{graded derivation of degree} \( \sigma \). Denote by \( \text{Der}_{\text{gr}}(I, L)_{\sigma} \) the set of all graded derivations of degree \( \sigma \). Clearly, it becomes a \( \Phi \)-module by defining operations in the natural way and, consequently, \( \text{Der}_{\text{gr}}(I, L) := \bigoplus_{\sigma \in G} \text{Der}_{\text{gr}}(I, L)_{\sigma} \) is also a \( \Phi \)-module.

For any element \( x \) in a Lie algebra \( L \), the adjoint map \( \text{ad} \): \( I \to L \) defined by \( \text{ad} x(y) = [x, y] \) is a derivation of \( L \). If \( L \) is graded by a group \( G \) and \( x \) is a homogeneous element of degree \( \sigma \), then \( \text{ad} x \) is in fact a derivation of degree \( \sigma \). In general, for any \( x \) in the graded Lie algebra \( L \),

\[
\text{ad} x = \sum_{\sigma \in G} \text{ad} x_{\sigma} \in \bigoplus_{\sigma \in G} \text{Der}_{\text{gr}}(I, L)_{\sigma} = \text{Der}_{\text{gr}}(I, L).
\]

In order to ease the notation, denote by \( I_{\text{gr}-e}(L) \) the set of all graded essential ideals of a graded Lie algebra \( L \). If \( L \) is a \( G \)-graded semiprime Lie algebra, it can be shown (as in [23, Theorem 3.4]) that the direct limit

\[
Q_{\text{gr}-m}(L) := \lim_{\longleftarrow I \in I_{\text{gr}-e}(L)} \text{Der}_{\text{gr}}(I, L)
\]

of graded derivations of \( L \) defined on graded essential ideals of \( L \) is a graded algebra of quotients of \( L \) containing \( L \) as a graded subalgebra, via the following graded Lie monomorphism:

\[
\varphi : L \to Q_{\text{gr}-m}(L) \quad x \mapsto (\text{ad} x)_{L}
\]

where \( \delta \) stands for any arbitrary element of \( Q_{\text{gr}-m}(L) \).

Moreover, \( Q_{\text{gr}-m}(L) \) is maximal among the graded algebras of quotients of \( L \). It is called the \textit{maximal graded algebra of quotients} of \( L \). The following result characterizes it. We omit its proof because it is similar to that of [23, Theorem 3.8].

\textbf{Theorem 2.11.} Let \( L \) be a graded semiprime Lie algebra and consider a graded overalgebra \( S \) of \( L \). Then \( S \) is graded isomorphic to \( Q_{\text{gr}-m}(L) \), under an isomorphism which is the identity on \( L \), if and only if \( S \) satisfies the following properties:

(i) For any \( s_{\sigma} \in S_{\sigma} \) (\( \sigma \in G \)) there exists \( I \in I_{\text{gr}-e}(L) \) such that \([I, s_{\sigma}] \subseteq L \).

(ii) For \( s_{\sigma} \in S_{\sigma} \) (\( \sigma \in G \)) and \( I \in I_{\text{gr}-e}(L) \), \([I, s_{\sigma}] = 0 \) implies \( s_{\sigma} = 0 \).

(iii) For \( I \in I_{\text{gr}-e}(L) \) and \( \delta \in \text{PDer}_{\text{gr}}(I, L)_{\sigma} \) (\( \sigma \in G \)) there exists \( s_{\sigma} \in S_{\sigma} \) such that \( \delta(x) = [s_{\sigma}, x] \) for every \( x \in I \).

\textbf{Remark 2.12.} Note the conditions (i) and (ii) in the theorem above are equivalent to the following one:

(ii)' \( S \) is a graded algebra of quotients of \( L \).

\textbf{Remark 2.13.} The notion of maximal graded algebra of quotients extends that of maximal algebra of quotients given in [23] as the maximal graded algebra of quotients and the maximal algebra of quotients of a semiprime Lie algebra coincide when considering the trivial grading over such an algebra.

3. \textbf{Maximal graded algebras of quotients of 3-graded Lie algebras}

Let \( L \) be a \( \mathbb{Z} \)-graded Lie algebra with a finite grading. We may write \( L = \bigoplus_{k=-n}^{n} L_k \) and we will say that \( L \) has a \((2n+1)\)-\textit{grading}. In what follows, we will deal with 3-graded Lie algebras.
In this section we show that for a 3-graded semiprime Lie algebra $L$, the maximal algebra of quotients of $L$ is 3-graded too and coincides with the maximal graded algebra of quotients of $L$, as defined in Section 4.

**Lemma 3.1.** Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a 3-graded Lie algebra and $I$ an ideal of $L$. Denote by $\pi_i$ the canonical projection from $L$ into $L_i$ (with $i \in \{-1, 0, 1\}$) and consider $\hat{I} := J + \pi_{-1}(J) + \pi_1(J)$, where $J := ([I, I], [I, I])$. Then:

(i) $\hat{I}$ is a graded ideal of $L$ contained in $I$.

If moreover $L$ is semiprime, then:

(ii) Suppose that $I$ is a graded ideal. Then $I$ is an essential ideal of $L$ if and only if $\hat{I}$ is an essential ideal of $L$.

(iii) It is trivial that $I$ essential as an ideal implies $I$ essential as a graded ideal.

**Proof.** (i). Note that $\pi_0(J) \subseteq \hat{I}$ since $\pi_0 = \text{Id} - \pi_{-1} - \pi_1$. Show first that $\hat{I}$ is an ideal of $L$: take $x \in \hat{I}$ and $y \in L$ and write $x = u + z_{-1} + t_1$, where $u$ and the elements $z = z_{-1} + z_0 + z_1$ and $t = t_{-1} + t_0 + t_1$ are in $J$. We have

$$(\dagger) \quad [x, y] = [u, y] + [z_{-1}, y] + [t_1, y].$$

Now, since $u$ is in $J$, which is an ideal of $L$, we obtain $[u, y] \in J \subseteq \hat{I}$. On the other hand, writing $y = y_{-1} + y_0 + y_1$ we have $[z_{-1}, y] = [z_{-1}, y_0] + [z_{-1}, y_1]$; apply again that $J$ is an ideal to obtain $[z, y_1], [z, y_0] \in J$, which implies that the elements $[z, y_1]_0 = [z_{-1}, y_1]$ and $[z, y_0]_1 = [z_{-1}, y_0]$ are in $\hat{I}$. Hence, $[z_{-1}, y] \in \hat{I}$. Analogously, it can be shown $[t_1, y] \in \hat{I}$.

Put together $(\dagger)$ and this to obtain $[x, y] \in \hat{I}$, as desired.

We claim that $\hat{I}$ is in fact a graded ideal: consider $xx_{-1} + x_0 + x_1 \in \hat{I}$ and write, as above, $x = u + z_{-1} + t_1$, with $u, z = z_{-1} + z_0 + z_1$ and $t = t_{-1} + t_0 + t_1$ elements in $J$. Then $x_{-1} = u_{-1} + z_{-1}, x_0 = u_0$ and $x_1 = u_1 + t_1$. Thus, taking into account the definition of $\hat{I}$ we obtain that $x_i \in \hat{I}$ for $i \in \{-1, 0, 1\}$.

Finally, we prove that $\hat{I}$ is contained in $I$ by showing that $\pi_{-1}(J)$ and $\pi_1(J)$ are contained in $I$. Define $\delta := \pi_1 - \pi_{-1}$. Then $\delta^2 = \pi_{-1} + \pi_1$ implies $2\pi_1 = \delta^2 + \delta$ and $2\pi_{-1} = \delta^2 - \delta$. Hence, to prove that $\pi_{-1}(J)$ and $\pi_1(J)$ are contained in $I$, it is enough to check that $\delta^2(J)$ and $\delta(J)$ are contained in $I$. Take $x, y \in I$ and write $x = x_{-1} + x_0 + x_1$ and $y = y_{-1} + y_0 + y_1$ where $x_i, y_i \in L_i$ for $i \in \{-1, 0, 1\}$. A computation gives

$$[x, y]_{-1} = [x_{-1}, y_0] + [x_0, y_{-1}] = [x_{-1}, y] - [x_{-1}, y_1] + [x, y_{-1}] - [x_1, y_{-1}]$$

$$[x, y]_1 = [x_0, y_1] + [x_1, y_0] = [x_1, y] - [x_1, y_1] + [x, y_1] - [x_{-1}, y_1].$$

So $\delta([x, y]) = [x, y]_1 - [x, y]_{-1} = [x_1, y] + [x, y_1] - [x_{-1}, y] - [x, y_{-1}] \in I$, that is, $\delta([I, I]) \subseteq I$; it can be proved analogously $\delta(J) \subseteq [I, I] \subseteq I$, therefore $\delta^2(J) \subseteq \delta([I, I]) \subseteq I$.

(ii). Consider $I$ as an essential ideal of $L$. Note that the semiprimeness of $L$ implies that $J$ is also an essential ideal of $L$. Hence, $J \cap K \neq 0$ for any nonzero ideal $K$ of $L$ and so $\hat{I} \cap K \neq 0$. This shows that $\hat{I}$ is an essential ideal of $L$.

To prove the converse, suppose that $\hat{I}$ is an essential ideal of $L$. As $\hat{I} \subseteq I$ (by (i)), the ideal $I$ must be essential too.

(iii). It is trivial that $I$ essential as an ideal implies $I$ essential as a graded ideal.
Suppose now that $I$ is a graded essential ideal and let $U$ be a nonzero ideal of $L$. Being $L$ semiprime, $K := [U, U]$, $[U, U]$ is a nonzero ideal of $L$. Apply (i) to obtain that $\tilde{U} := K + \pi_1(K) + \pi_1(K)$ is a graded ideal of $L$ contained in $U$. As $I$ is a graded essential ideal, $I \cap \tilde{U} \neq 0$ and hence $I \cap U \neq 0$. □

**Theorem 3.2.** Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a semiprime 3-graded Lie algebra. Then:

(i) $Q_m(L)$ is graded and graded isomorphic to $Q_{gr-m}(L)$.

(ii) If $L$ is strongly nondegenerate and $\Phi$ is 2 and 3-torsion free, then $Q_m(L)$ is a 3-graded strongly nondegenerate Lie algebra.

**Proof.** (i). Observe first that $L$, viewed as a 3-graded Lie algebra, is graded semiprime (since it is semiprime), so it has sense to consider $Q_{gr-m}(L)$.

Define

$$\varphi : Q_m(L) \to Q_{gr-m}(L)$$

where for an essential ideal $I$ of $L$, $\tilde{I} \subseteq I$ is the graded essential ideal defined in Lemma 3.1.

The Lie algebra $Q_m(L)$ is 5-graded: let $I$ be an essential ideal of $L$ and $\tilde{I}$ as before. It is easy to check, by considering the canonical projections onto the subspaces $\tilde{I}, (i \in \{-1, 0, 1\})$, that $PDer(\tilde{I}, L)$ is just $\oplus_{i=-2}^2 PDer_{gr}(\tilde{I}, L)$, which coincides, by definition, with $PDer_{gr}(\tilde{I}, L)$. This shows our claim and that the map $\varphi$ is well-defined. Finally, keeping in mind Lemma 3.1 (iii) it is straightforward to verify that $\varphi$ is a graded Lie algebra isomorphism.

(ii). Apply (i) and [7, Proposition 1.7]. □

### 4. Jordan pairs of quotients and 3-graded Lie algebras of quotients

Our target in this section is to analyze the relationship between the notions of Jordan pair of quotients and of (graded) Lie algebra of quotients, via the Tits-Kantor-Koecher construction. In the particular case of maximal quotients, we will prove that under the suitable hypothesis, the maximal Lie algebra of quotients of the TKK-algebra of a Jordan pair $V$ is the TKK-algebra of the maximal Jordan pair of quotients of $V$.

A *Jordan pair over* $\Phi$ is a pair $V = (V^+, V^-)$ of $\Phi$-modules together with a pair $(Q^+, Q^-)$ of quadratics maps $Q^\sigma : V^\sigma \to \text{Hom}(V^{-\sigma}, V^\sigma)$ (for $\sigma = \pm$) with linearizations denoted by $Q_{\pm,x}^\sigma y = \{x, y, z\} = D_{\pm,x}^\sigma y, z$, where $Q_{\pm,x}^\sigma = Q_{\pm,x}^\sigma - Q_{\pm,x}^{-\sigma}$, satisfying the following identities in all the scalar extensions of $\Phi$:

(i) $D_{x,y}^\sigma Q_x^\sigma = Q_x^\sigma D_{y,x}^\sigma$

(ii) $D_{Q_x^\sigma y, x}^\sigma = D_{x, Q_y^\sigma x}^\sigma$

(iii) $Q_{Q_x^\sigma y, x}^\sigma = Q_x^\sigma Q_{Q_y^\sigma x}^\sigma$

for every $x \in V^\sigma$ and $y \in V^{-\sigma}$.

From now on, we shall deal with Jordan pairs $V = (V^+, V^-)$ over a ring of scalars $\Phi$ containing $\frac{1}{2}$. In order to ease the notation, Jordan products will be denoted by $Q_{xy}$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$. The reader is referred to [14] for basic results, notation and terminology on Jordan pairs. Nevertheless, we recall here some notions and basic properties.

Let $V = (V^+, V^-)$ be a Jordan pair. An element $x \in V^\sigma$ is called an *absolute zero divisor* if $Q_x = 0$, while the pair $V$ is said to be *strongly nondegenerate* (nondegenerate in
the terminology of [7]) if it has no nonzero absolute zero divisors. The pair \( V \) is semiprime if \( Q_{I^1} = 0 \) imply \( I = 0 \), being \( I \) an ideal of \( V \), and is called prime if \( Q_{I^1} = 0 \) imply \( I = 0 \) or \( J = 0 \), for \( I \) and \( J \) ideals of \( V \). A strongly prime pair is a prime and strongly nondegenerate pair.

For a subset \( X = (X^+, X^-) \) of \( V \), the annihilator of \( X \) in \( V \) is \( \text{Ann}_V(X) = (\text{Ann}_V(X)^+, \text{Ann}_V(X)^-) \), where, for \( \sigma = \pm \)

\[
\text{Ann}_V(X)^\sigma = \{ z \in V^\sigma \mid \{ z, X^{-\sigma}, V^\sigma \} = \{ z, V^{-\sigma}, X^\sigma \} = \{ V^{-\sigma}, z, X^{-\sigma} \} = 0 \}.
\]

One can check that \( \text{Ann}_V(I) \) is an ideal of \( V \) if \( I \) is so. Ideals of Lie algebras having zero annihilator are essentials and when the Lie algebra where they live is semiprime, the reverse holds, i.e., every essential ideal has zero annihilator (see [23, Lemma 1.2]). In the context of Jordan pairs, a similar result can be shown.

**Lemma 4.1.** Let \( I = (I^+, I^-) \) be an ideal of a semiprime Jordan pair \( V = (V^+, V^-) \). Then:

(i) \( I \cap \text{Ann}_V(I) = 0 \).

(ii) \( I \) is an essential ideal of \( V \) if and only if \( \text{Ann}_V(I) = 0 \).

**Proof.** (i). If we show that the ideal \( K := I \cap \text{Ann}_V(I) \) satisfies that \( Q_{K^1} K^\mp = 0 \), for \( K^\sigma = I^\sigma \cap \text{Ann}_V(I)^\sigma \), \( \sigma = \pm \), the result follows by the semiprimeness of \( V \). Given \( x \in K^\sigma \subseteq \text{Ann}_V(I)^\sigma \) for \( \sigma = \pm \) we have \( \{ x, K^{-\sigma}, V^\sigma \} = 0 \) since \( K^{-\sigma} \subseteq I^{-\sigma} \). So \( \{ K^\sigma, K^{-\sigma}, V^\sigma \} = 0 \) for \( \sigma = \pm \) and hence \( Q_{K^1} K^\mp = 0 \), as desired.

(ii). Consider an essential ideal \( I = (I^+, I^-) \) of \( V \); then \( I \cap \text{Ann}_V(I) = 0 \) by (i), and by the essentiality, \( \text{Ann}_V(I) = 0 \). Conversely, suppose that \( \text{Ann}_V(I) = 0 \) and consider an ideal \( K = (K^+, K^-) \) of \( V \) satisfying \( I \cap K = 0 \).

For \( x \in K^\sigma \), with \( \sigma = \pm \), and taking into account that \( I \) and \( K \) are ideals of \( V \), we obtain

\[
\{ x, I^{-\sigma}, V^\sigma \}, \{ x, V^{-\sigma}, I^\sigma \}, \{ V^{-\sigma}, x, I^{-\sigma} \} \subseteq I \cap K = 0,
\]

hence, \( K \subseteq \text{Ann}_V(I) = 0 \). This shows that \( I \) is an essential ideal of \( V \). \( \square \)

Let us recall the connection between Jordan 3-graded Lie algebras and Jordan pairs.

A 3-graded Lie algebra \( L = L_{-1} \oplus L_0 \oplus L_1 \) is called Jordan 3-graded if \( [L_1, L_{-1}] = L_0 \) and there exists a Jordan pair structure on \( (L_1, L_{-1}) \) whose Jordan product is related to the Lie product by \( \{ x, y, z \} = [[x, y], z] \), for any \( x, y, z \in L_\sigma, y \in L_{-\sigma}, \sigma = \pm \). In this case, \( V = (L_1, L_{-1}) \) is called the associated Jordan pair.

Since \( \frac{1}{2} \in \Phi \), the product on the associated Jordan pair is unique and given by \( Q_{x,y} = \frac{1}{2}\{ x, y, x \} = \frac{1}{2} [[x, y], x] \). Conversely, for any 3-graded Lie algebra, the formula above defines a pair structure on \( (L_1, L_{-1}) \) whenever \( \frac{1}{2} \in \Phi \) (see [21, 1.2]).

One important example of a Jordan 3-graded Lie algebra is the TKK-algebra of a Jordan pair. It is built as explained below.

Let \( V = (V^+, V^-) \) be a Jordan pair; a pair \( (\delta^+, \delta^-) \in \text{End}_\Phi(V^+) \times \text{End}_\Phi(V^-) \) is a derivation of \( V \) if it satisfies

\[
\delta^\sigma(\{ x, y, z \}) = \{ \delta^\sigma(x), y, z \} + \{ x, \delta^-\sigma(y), z \} + \{ x, y, \delta^\sigma(z) \}
\]

for any \( x, z \in V^\sigma \) and \( y \in V^{-\sigma}, \sigma = \pm \). For \( x, y \in V \) the map \( \delta(x, y) := (D_{x,y}^+, -D_{y,x}^-) \) is a derivation of \( V \) (by the identity (JP12) in [14]) called inner derivation. Denote by \( \text{I} \text{D} \text{e}r(V) \) the \( \Phi \)-module spanned by all inner derivations of \( V \) and define on the \( \Phi \)-module \( \text{TKK}(V) := V^+ \oplus \text{I} \text{D} \text{e}r(V) \oplus V^- \) the following product:
Let $\mathfrak{L}$. Lemma 4.3.

The algebras of quotients of their respective TKK-algebras.

Check that the following provides TKK($V$ is called the Tits-Kantor-Koecher algebra of $V$).

As this construction has its origin in the fundamental papers [9, 10, 11] by Kantor, in [12, 13] by Koecher and in [24] by Tits, TKK($V$) is called the Tits-Kantor-Koecher algebra of $V$ or the TKK-algebra for short. It is easy to check that the following provides TKK($V$) with a 3-grading:

$$
TKK(V)_1 = V^+, \ TKK(V)_0 = \text{IDer}(V), \ TKK(V)_{-1} = V^-.
$$

Moreover, TKK($V$) is a Jordan 3-graded Lie algebra with $V$ as associated Jordan pair.

If $L$ is a Jordan 3-graded Lie algebra with associated Jordan pair $V$, then the TKK-algebra associated to $V$ is not in general isomorphic to $L$. Rather, we have:

**Lemma 4.2.** ([20, 2.8]). Let $L$ be a Jordan 3-graded Lie algebra with associated Jordan pair $V$. Then TKK($V$) $\cong L/C_V$, where $C_V = \{x \in L_0 \mid [x, L_1] = 0 = [x, L_{-1}]\} = Z(L) \cap L_0$.

Now, we are going to show the equivalence between Jordan pairs of quotients and Lie algebras of quotients of their respective TKK-algebras.

**Lemma 4.3.** Let $V$ be a semiprime Jordan pair, and $I = (I^+, I^-)$ an ideal of $V$. Define by $\operatorname{Id}_{\text{TKK}(V)}(I) = I^+ \oplus ([I^+, V^-] + [V^+, I^-]) \oplus I_-$ the graded ideal of TKK($V$) generated by $I$.

Then $\text{Ann}_{\text{TKK}(V)}(\operatorname{Id}_{\text{TKK}(V)}(I)) = 0$ if and only if $\text{Ann}_V(I) = 0$.

**Proof.** See [7, Lemma 2.9].

**Definition 4.4.** (See [7, 2.5]). Let $V$ be a semiprime Jordan pair contained in a Jordan pair $W$. It is said that $W$ is a pair of $\mathfrak{M}$-quotients of $V$ if for every $0 \neq q \in W^\sigma$ (with $\sigma = \pm$) there exists an ideal $I$ of $V$ with $\text{Ann}_V(I) = 0$ such that $\{q, I^-\sigma, V^\sigma\} \subseteq V^\sigma$ and $\{I^-\sigma, q, V^-\sigma\} \subseteq V^-\sigma$, with either $\{q, I^-\sigma, V^\sigma\} + \{q, V^-\sigma, I^\sigma\} \neq 0$ or $\{I^-\sigma, q, V^-\sigma\} \neq 0$.

**Theorem 4.5.** Let $V$ be a semiprime subpair of a Jordan pair $W$. Then the following conditions are equivalent:

(i) $W$ is a pair of $\mathfrak{M}$-quotients of $V$.
(ii) TKK($W$) is an algebra of quotients of TKK($V$).

**Proof.** (i) $\Rightarrow$ (ii) is [7, Theorem 2.10].

(ii) $\Rightarrow$ (i). Take $0 \neq q^\sigma \in W^\sigma$ ($\sigma = \pm$) and apply Propositions 2.7 to find a 3-graded ideal $I$ of TKK($V$) with $\text{Ann}_{\text{TKK}(V)}(I) = 0$ and such that $0 \neq [I, q^\sigma] \subseteq \text{TKK}(V)$. We claim that $I_V := I_1 \oplus ([I_1, V^-] + [V^+, I_{-1}]) \oplus I_{-1}$ is an essential 3-graded ideal of TKK($V$), where $I = I_1 \oplus I_0 \oplus I_{-1}$. Let $K = K_1 \oplus K_0 \oplus K_{-1}$ be a nonzero 3-graded ideal of TKK($V$); the semiprimeness of $V$ implies that either $I_1 \cap K_1 \neq 0$ or $I_{-1} \cap K_{-1} \neq 0$ (see the proof of [8, Proposition 2.6]) and therefore $I_V \cap K \neq 0$. By Lemma and 1.1 (iii), $\text{Ann}_{\text{TKK}(V)}(I_V) = 0$, and by Lemma 4.3, $\text{Ann}_V((I_1, I_{-1})) = 0$. 

Denote $I_1$ and $I_{-1}$ by $I^+$ and $I^-$, respectively. Then, for $\sigma = \pm$ we have:
\[
\{q^\sigma, I^{-\sigma}, V^\sigma\} \subseteq [[q^\sigma, I^{-\sigma}], V^\sigma] \subseteq V^\sigma
\]
\[
\{I^{-\sigma}, q^\sigma, V^{-\sigma}\} \subseteq [[I^{-\sigma}, q^\sigma], V^{-\sigma}] \subseteq V^{-\sigma}
\]
\[
\{q^\sigma, V^{-\sigma}, I^\sigma\} \subseteq [[q^\sigma, V^{-\sigma}], I^\sigma] \subseteq [[V^{-\sigma}, I^\sigma], q^\sigma] \subseteq V^\sigma
\]
To complete the proof we have to check that either $\{q^\sigma, I^{-\sigma}, V^\sigma\} + \{q^\sigma, V^{-\sigma}, I^\sigma\} \neq 0$ or $\{I^{-\sigma}, q^\sigma, V^{-\sigma}\} \neq 0$. We have just showed that $\text{Ann}_{\text{TKK}(V)}(I_V) = 0$; using [23, Lemma 2.11] we obtain that $\text{Ann}_{\text{TKK}(W)}(I_V) = 0$ and hence $0 \neq [I_V, q^\sigma] \subseteq [I, q^\sigma] \subseteq \text{TKK}(V)$ which implies that either $[(I_V)_0, q^\sigma] \neq 0$ or $[I^{-\sigma}, q^\sigma] \neq 0$. In the first case, we have:
\[
0 \neq [(I_V)_0, q^\sigma] = [[[I^\sigma, V^{-\sigma}], q^\sigma] + [V^\sigma, I^{-\sigma}, q^\sigma], q^\sigma] \subseteq \{I^\sigma, V^{-\sigma}, q^\sigma\} + \{V^\sigma, I^{-\sigma}, q^\sigma\}
\]
In the second case, apply that the representation of $\text{I} \text{Der}(V)$ on $V$ is faithful to obtain
\[
0 \neq [[I^{-\sigma}, q^\sigma], V^{-\sigma}] \cup [I^{-\sigma}, q^\sigma, V^{-\sigma}] \quad \text{or}
\]
\[
0 \neq [[I^{-\sigma}, q^\sigma], V^\sigma] \subseteq [[V^\sigma, I^{-\sigma}, q^\sigma], q^\sigma] = \{V^\sigma, I^{-\sigma}, q^\sigma\} \{q^\sigma, I^{-\sigma}, V^\sigma\}
\]
\[\Box\]

We continue the section by examining the relationship between maximal Jordan pairs of $\mathfrak{m}$-quotients (see [7, 3.1 and Theorem 3.2] for definition) and maximal algebras of quotients of Jordan 3-graded Lie algebras.

**Lemma 4.6.** Let $V = (V^+, V^-)$ be a strongly nondegenerate Jordan pair. If $I$ is an essential ideal of $\text{TKK}(V)$, then there exists an essential ideal $I$ of $V$ such that $\text{Id}_{\text{TKK}(V)}(I)$ is contained in $I$.

**Proof.** Consider an essential ideal $I$ of $\text{TKK}(V)$, which is a strongly nondegenerate Lie algebra (by [6, Proposition 2.6]); in particular, it is semiprime. Therefore, we may apply Lemma 3.1 (i) and (ii) to find an essential graded ideal $\hat{I}_1 = \hat{I}_0 \oplus \hat{I}_1$ of $\text{TKK}(V)$ contained in $I$. It can be shown, as in the proof of Theorem 4.5, that $\hat{I}_1 = ([\hat{I}_1, V^-] + [V^+, \hat{I}_1]) \oplus \hat{I}_1 \subseteq I$ is an essential ideal of $\text{TKK}(V)$ and, by means of Lemma 4.3, $\hat{I} := (\hat{I}_1, \hat{I}_1)$ is an essential ideal of $V$.

For a strongly nondegenerate Jordan pair $V$, denote its maximal Jordan pair of $\mathfrak{m}$-quotients by $Q_m(V)$ (see [7] for its construction).

**Theorem 4.7.** Assume that $\frac{1}{b} \in \Phi$.

(i) Let $V$ be a strongly nondegenerate Jordan pair. Then
\[
Q_m(V) = \left( (Q_m(\text{TKK}(V)))_1, (Q_m(\text{TKK}(V)))_{-1} \right)
\]
is the maximal Jordan pair of $\mathfrak{m}$-quotients of $V$.

(ii) If $L = L_{-1} \oplus L_0 \oplus L_1$ is a strongly non-degenerate Jordan 3-graded Lie algebra satisfying that $Q_m(L)$ is Jordan 3-graded, then
\[
Q_m(L) \cong Q_m(\text{TKK}(V)) \cong \text{TKK}(Q_m(V)),
\]
where $V = (L_1, L_{-1})$ is the associated Jordan pair of $L$. 

Proof. (i). The Lie algebras $Q_{TKK}(TKK(V))$ and $Q_m(TKK(V))$ are isomorphic by Lemmas 4.3 and 4.6 (see [7] for the definition of $Q_{TKK}(TKK(V))$). On the other hand, Theorem 3.2 (i) implies that they are isomorphic to $Q_{gr-m}(TKK(V))$. (Note that $TKK(V)$ is a strongly nondegenerate Lie algebra by [7, Proposition 2.6] so, it has sense to consider its maximal graded algebra of quotients). Now, the result follows by [7, Theorem 3.2].

(ii). The Lie algebra $L$ has zero center because it is strongly nondegenerate, hence $L \cong TKK(V)$ (use Lemma 4.2) and, obviously, $Q_m(L) \cong Q_m(TKK(V))$. This one is a strongly non-degenerate Lie algebra (by 2.4) and has a 3-grading (by (i)) with associated Jordan pair $Q_m(V)$. The hypothesis on $Q_m(L)$ allows us to use again Lemma 4.2 obtaining $Q_m(L) \cong TKK(Q_m(V))$. □

The following is an example of a strongly non-degenerate Jordan 3-graded Lie algebra $L$ such that its maximal (graded) algebra of quotients $Q_m(L)$ is not Jordan 3-graded. If we denote by $V$ the associated Jordan pair of $L$, we obtain that $TKK(Q_m(V))$ is not (graded) isomorphic to $Q_m(L)$ (since $TKK(Q_m(V))$ is Jordan 3-graded) which thereby means that the condition on $L$ in Theorem 4.7 (ii) is necessary.

Example 4.8. Denote by $M_\infty(\mathbb{R}) = \bigcup_{n=1}^\infty M_n(\mathbb{R})$ the algebra of infinite matrices with a finite number of nonzero entries and consider

$$L := sl_\infty(\mathbb{R}) = \{ x \in M_\infty(\mathbb{R}) \mid tr(x) = 0 \},$$

which is a simple Lie algebra of countable dimension (see [1, Theorem 1.4]).

Denote by $e_{ij}$ the matrix whose entries are all zero except for the one in row $i$ and column $j$ and consider the orthogonal idempotents $e := e_{11}$ and $f := \text{diag}(0,1,1,\ldots)$ (note that $f \notin M_\infty(\mathbb{R})$); we can see $L$ as a 3-graded Lie algebra by doing $L = L_{-1} \oplus L_0 \oplus L_1$, where $L_{-1} = eL_f$, $L_0 = \{ exe + fxf \mid x \in L \}$ and $L_1 = fLe$.

Let $exe + fxf$ be an element of $L_0$ with $x = (x_{ij}) \in M_n(\mathbb{R})$ for some $n \in \mathbb{N}$. Taking into account that $tr(x) = 0$, we obtain:

$$exe + fxf = \sum_{i=2}^n \sum_{j=2}^n [-e_{1j}, x_{ij}e_{11}] \in [L_{-1}, L_1],$$

This shows that $L_0 = [L_{-1}, L_1]$, i.e., $L$ is Jordan 3-graded.

In what follows, we will prove that $\text{Der}(L)$ is not Jordan 3-graded. The simplicity of $L$ implies that $Q_m(L) \cong \text{Der}(L)$; on the other hand, the strongly nondegeneracy of $L$ allows us to apply [7, Proposition 1.7] obtaining that $\text{Der}(L)$ is 3-graded. Now, take, $\delta := \text{ad} e$; one can easily check that:

$$\delta(L_{-1}) \subseteq L_{-1}, \delta(L_0) = 0 \text{ and } \delta(L_1) \subseteq L_1,$$

which means that $\delta \in \text{Der}(L)_0$. But note that $\delta \notin [\text{Der}(L)_{-1}, \text{Der}(L)_1]$ since the elements of $[\text{Der}(L)_{-1}, \text{Der}(L)_1]$ have zero trace on every finite dimensional subspace of $L$ while the trace of $\delta$ is always nonzero. Therefore, $[\text{Der}(L)_{-1}, \text{Der}(L)_1] \not\subseteq \text{Der}(L)_0$, i.e., $\text{Der}(L)$ is not Jordan 3-graded.

Remark 4.9. Note that there exist non-trivial Jordan 3-graded Lie algebras such that their maximal (graded) algebra of quotients are also Jordan 3-graded Lie algebras. For example:

Let $F$ be a field and consider the Lie algebra

$$L := sl_2(F) = \{ x \in M_2(F) \mid tr(x) = 0 \}.$$
We have that $L$ is a Jordan 3-graded Lie algebra with the grading $L = L_{-1} \oplus L_0 \oplus L_1$, where

$L_{-1} = Fe_{21}$, $L_0 = F(e_{11} - e_{22})$ and $L_1 = Fe_{12}$.

Moreover, $L$ is a finite dimensional semisimple Lie algebra and applying [23, Lemma 3.9] we obtain that $L \cong Q_m(L)$.

We finish the paper with analogues to Theorem 4.7 but for Jordan triple systems and Jordan algebras. In order to not enlarge the paper we refer the reader to [14, 19, 18] for basic definitions and results on Jordan triple systems and algebras.

**Theorem 4.10.** Let $T$ be a strongly nondegenerate Jordan triple system over a ring of scalars $\Phi$ containing $\frac{1}{6}$. Then the maximal Jordan triple system of $\mathfrak{m}$-quotients of $T$ is the first component of the maximal algebra of quotients of the TKK-algebra of the double Jordan pair $V(T) = (T, T)$ associated to $T$, i.e.,

$$Q_m(T) = (Q_m(TKK(V(T))))_1.$$

**Proof.** The Jordan pair $V(T) = (T, T)$ is strongly nondegenerate since $T$ is so. By Theorem 4.7 (i) we have

$$Q_m(V(T)) = \left( (Q_m(TKK(V(T))))_1, (Q_m(TKK(V(T))))_{-1} \right).$$

The conclusion follows now from the versions of Lemmas 4.3 and 4.6 for Jordan triple systems and from [7, 4.5 and Theorem 4.6].

**Theorem 4.11.** Let $J$ be a strongly nondegenerate Jordan algebra over a ring of scalars $\Phi$ containing $\frac{1}{6}$. Then

$$Q_m(J) = Q_m(J_T) = (Q_m(TKK(V(J_T))))_1,$$

is the maximal Jordan algebra of quotients of $J$, where $J_T$ denotes the Jordan triple system associated to $J$ and $V(J_T) = (J_T, J_T)$ is the double Jordan pair associated to $J_T$.

**Proof.** Note that $J_T$ is a strongly nondegenerate Jordan triple system by the strong nondegeneracy of the Jordan algebra $J$. From [7, 5.4 and Theorem 5.5] it follows that the maximal Jordan algebra of quotients $Q_m(J)$ is $Q_m(J_T)$. Finally apply Theorem 4.10 to reach the conclusion.

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