Contents

Contents **1**

1 Definitions, examples. Finite dimensional Leavitt path algebras **3**
- 1.1 The IBN property and the type of a ring. 3
- 1.2 Path algebras and Leavitt path algebras 5
- 1.3 Examples of Leavitt path algebras 7
- 1.4 Finite dimensional Leavitt path algebras 8

2 Uniqueness theorems. Simple Leavitt path algebras **13**
- 2.1 Semiprimeness in path algebras and in Leavitt path algebras . 13
- 2.2 Uniqueness theorems 17
- 2.3 Simple Leavitt path algebras 18
- 2.4 Purely infinite Leavitt path algebras 25
- 2.5 The dichotomy principle for simple Leavitt path algebras . . . 29

3 Graph C^*-algebras **31**
- 3.1 Basics on C^*-algebras 31
- 3.2 Generators and relations 34
- 3.3 The C^*-algebra of a graph E 36
- 3.4 First applications of G.I.U.T. 40
- 3.5 Gauge invariant ideals 41

4 Links between $C^*(E)$ and $L_{\mathbb{C}}(E)$. **45**
- 4.1 Leavitt path algebras versus graph C^*-algebras 45
- 4.2 Tomforde's Graded Invariant Uniqueness Theorem for LPAs . 46
- 4.3 Comparison of properties 50
  - 4.3.1 Simple & Purely infinite simple algebras 51
  - 4.3.2 Real rank zero versus Exchange property 52
  - 4.3.3 K-Theory 54
  - 4.3.4 Stable rank 57

Chapter 1

Definitions, examples. Finite dimensional Leavitt path algebras

Introduction

Leavitt path algebras are a specific type of path K-algebras associated to a graph E, modulo some relations. Its appearance, for row-finite graphs, took place in [2] and [12]. They can be considered, on the one hand, natural generalizations of Leavitt algebras $L(1, n)$ of type $(1, n)$, introduced and investigated by Leavitt in [41] in order to give examples of algebras not satisfying the IBN property.

On the other hand, they are the algebraic version of Cuntz-Krieger graph C*-algebras, a class of algebras intensively investigated by analysts for more than two decades. For a complete explanation of the history of Leavitt path algebras see [20].

While the analytic aspects of Leavitt path algebras will be analyzed in subsequent chapters, in this one we will relate them to the work by Leavitt and will give some important and interesting examples.

1.1 The IBN property and the type of a ring.

Let R be a unital ring. We say that R satisfies the *invariant basis number (IBN) property* if any two bases (i.e., linearly independent spanning sets) for a free left R-module have the same number of elements. In words, the IBN property says that if m and n are integers with the property that the free left modules ${}_RR^m$ and ${}_RR^n$ are isomorphic, then $m = n$.

Noetherian rings and commutative rings are included among the many classes of rings having this property . But the IBN property does not hold for all rings, as the following example shows.

Example 1.1.1 For a field K, let $V = K^{(\mathbb{N})}$, whis is a countably infinite dimensional vector space over K, and let $R = \mathrm{End}_K(V)$. It is not difficult to see that $R \cong \mathrm{RFM}_{\mathbb{N}}(K)$, the *countable row-finite matrices over* K (use the standard basis for V, view the elements of V as row-vectors, and apply transformations on the right hand side). Then ${}_RR^m \cong {}_RR^n$ for all $m, n \in \mathbb{N}$:

The first step is to show ${}_RR^1 \cong {}_RR^2$; such an isomorphism is given by the map which associates $X \in R$ with the pair of matrices (X_1, X_2), where X_1 (resp. X_2) is built from the odd-numbered (resp. even numbered) columns of X. But then ${}_RR^1 \cong {}_RR^2$ gives ${}_RR^1 \oplus {}_RR^1 \cong {}_RR^2 \oplus {}_RR^1$, so ${}_RR^2 \cong {}_RR^3$, and the result follows by continuing in this way.

It is easy to determine, algebraically, whether or not for a ring R we have ${}_RR^1 \cong {}_RR^n$ for some $n > 1$. Note that such an isomorphism exists if and only if there is a set of $2n$ elements in R which produce the appropriate isomorphisms as matrix multiplications by an n-row vector and an n-column vector with entries in R. Specifically, it is easy to show that ${}_RR^1 \cong {}_RR^n$ for some $n > 1$ if and only if there exist elements $x_1, ..., x_n, y_1, ..., y_n \in R$ for which $x_i y_j = \delta_{ij} 1_R$ for all i, j, and $\sum_{i=1}^n y_i x_i = 1_R$.

Suppose that a unital ring R does not have IBN. Let $m \in \mathbb{N}$ be minimal with ${}_RR^m \cong {}_RR^n$ for some $n > m$ and find the minimal such n for m. Then it is said that R has *module type* (m, n). (Warning: some authors call the module type of such a ring $(m, n-m)$.) For example, $\mathrm{RFM}_{\mathbb{N}}(K)$ has module type (1,2).

In his paper, Leavitt proved that for each pair of positive integers $n > m$ and any field K there exists a K-algebra of module type (m, n). To do this, observe that, as it has been used before, isomorphisms between free modules can be realized as matrix multiplications by matrices having coefficients in R. So we need only construct algebras which contain elements which behave "correctly". Do this as a quotient of a free associative K-algebra in the appropriate number of variables satisfying the appropriate relations.

For example, to get an algebra of type (1,3) we need an algebra containing elements $x_1, x_2, x_3, y_1, y_2, y_3$ for which $x_i y_j = \delta_{ij} 1_R$ for all i, j, and $\sum_{i=1}^3 y_i x_i = 1_R$. Consider the polynomial algebra over a field K in 6 non-commuting variables. Then factor by the ideal generated by the appropriate relations. It is not difficult to show that this quotient is not zero if $m \geq 2$, but this is much more difficult to show (directly) if $m = 1$.

The quotient algebra described above is denoted $L_K(m,n)$, and called the *Leavitt K-algebra of type* (m,n).

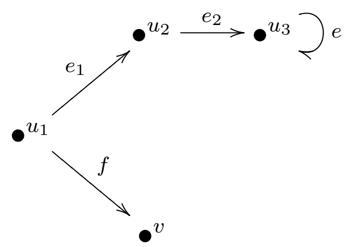
1.2 Path algebras and Leavitt path algebras

Definitions 1.2.1 A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. If E^0 is finite then, by the row-finite hypothesis, E^1 must necessarily be finite as well; in this case we simply say that E is *finite*. A vertex which emits no edges is called a *sink*. A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source* of μ, $r(\mu) := r(e_n)$ is the *range* of μ, and n is the *length* of μ, i.e, $l(\mu) = n$. We denote by μ^0 the set of its vertices, that is: $\mu^0 = \{s(e_1), r(e_i) : i = 1, \dots, n\}$.

Although much work has been done on arbitrary graphs, we will be concerned only with row-finite graphs.

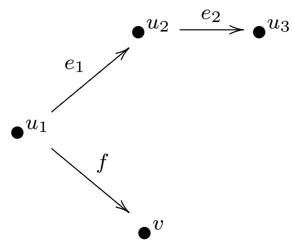
Example 1.2.2 Consider the following graph:



Then $E^0 = \{u_1, u_2, u_3, v\}$, $E^1 = \{e_1, e_2, f\}$, $r(e_1) = u_1$, $s(e_1) = u_2$, etc. The vertex v is a sink. Some paths are, for example, v, f, e_1e_2, e, ee, e_2eee, etc. For $\mu = e_1e_2e^3$, $\mu^0 = \{u_1, u_2, u_3\}$.

Definition 1.2.3 Now let K be a field and let KE denote the K-vector space which has as a basis the set of paths. It is possible to define an algebra structure on KE as follows: for any two paths $\mu = e_1 \dots e_m, \nu = f_1 \dots f_n$, we define $\mu\nu$ as zero if $r(\mu) \neq s(\nu)$ and as $e_1 \dots e_m f_1 \dots f_n$ otherwise. This K-algebra is called the *path algebra* of E over K.

Example 1.2.4 Consider a field K and the following graph E:



Then the path algebra KE has, as a vector space over K, dimension 8, while the path K-algebra asociated to the graph in example 1.2.2 is infinite dimensional.

There are several ways of defining Leavitt path algebras.

Definition 1.2.5 Given a graph E we define the *extended graph of* E as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as

$$r'|_{E^1} = r,\ s'|_{E^1} = s,\ r'(e_i^*) = s(e_i) \text{ and } s'(e_i^*) = r(e_i).$$

Definition 1.2.6 Let K be a field and E be a row-finite graph. The *Leavitt path algebra of E with coefficients in K* is defined as the path algebra over the extended graph $\widehat{E}$, with relations:

(CK1) $e_i^* e_j = \delta_{ij} r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$.

(CK2) $v_i = \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^*$ for every $v_i \in E^0$ which is not a sink.

This algebra is denoted by $L_K(E)$ (or by $L(E)$ if there is no risk of confusion with the field K).

The conditions (CK1) and (CK2) are called the *Cuntz-Krieger relations.* In particular condition (CK2) is the *Cuntz-Krieger relation at v_i*. If v_i is a sink, we do not have a (CK2) relation at v_i. Note that the condition of row-finiteness is needed in order to define the equation (CK2).

There exists a natural inclusion of the path algebra KE into the Leavitt path algebra $L_K(E)$ sending vertices to vertices and edges to edges. We will use this monomorphism without any explicit mention to it.

Another way of introducing Leavitt path algebras is as follows.

Definition 1.2.7 For a field K and a row-finite graph E, the Leavitt path K-algebra $L_K(E)$ is defined as the universal K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* : e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e\in E^1 | s(e)=v\}} ee^*$ for every $v \in E^0$ that emits edges.

Universal means that if A is a K-algebra containing a set of pairwise orthogonal idempotents $\{a_v : v \in E^0\}$ and a set of elements $\{b_e, b_{e^*} : e \in E^1\}$ satisfying the relations (1)-(4), then there exists an algebra homomorphism $\Phi : L_K(E) \to A$ satisfying $\Phi(v) = a_v$ for all $v \in E^0$, $\Phi(e) = b_e$ and $\Phi(e^*) = b_{e^*}$ for all $e \in E^1$.

The uniqueness of the Leavitt path algebra associated to a graph E and to a field K follows from the universal property.

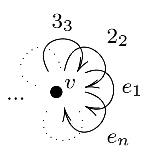
Again the identities (3) and (4) are called the *Cuntz-Krieger relations.*

The elements of E^1 are called *(real) edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

1.3 Examples of Leavitt path algebras

Many well-known examples of K- algebras can be seen as Leavitt path K-algebras over concrete graphs.

Example 1.3.1 (Leavitt algebras of type $(1, n)$, $n > 1$). The Leavitt path K-algebra of the following graph



is the Leavitt algebra of type $(1, n)$, for $n > 1$.

Example 1.3.2 (Matrix algebras). Consider the graph

$$\bullet^{u_1} \xrightarrow{e_1} \bullet^{u_2} \cdots\cdots \bullet^{u_{n-1}} \xrightarrow{e_{n-1}} \bullet^{u_n}$$

Then $M_{n+1}(K) \cong L_K(E)$, via the map $u_i \mapsto e_{ii}, e_i \mapsto e_{i+1i}$, and $e_i^* \mapsto e_{ii+1}$, where e_{ij} denotes the matrix unit in $M_n(K)$ with all entries equal zero except that in row i and column j.

Example 1.3.3 ((The Laurent polynomial ring). Consider the graph

$$\bullet^{u} \circlearrowright e$$

Then the map $\varphi : L_K(E) \to K[x, x^{-1}]$ given on generators (as a K- vector space) by $\varphi(e^n) = x^n$ and $\varphi((e^*)^n) = x^{-n}$ produces an isomorphism between $L_K(E)$ and $K[x, x^{-1}]$.

A path μ is called a *cycle* if $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$. A graph E without cycles is said to be *acyclic*.

1.4 Finite dimensional Leavitt path algebras

We finish this chapter by showing that the Leavitt path K- algebra of every finite and acyclic graph is a direct sum of matrices of finite size over K. In particular, they are semisimple and artinian. The converse is also true, that is, every semisimple and artinian Leavitt path algebra is associated to a finite and acyclic graph.

First we need to state that Leavitt path algebras are $\mathbb{Z}$-graded algebras, and that every set of paths is linearly independent.

Lemma 1.4.1 *Every monomial in $L_K(E)$ is of the following form:*

(i) $k_i v_i$ *with* $k_i \in K$ *and* $v_i \in E^0$*, or*

(ii) $k e_{i_1} \dots e_{i_\sigma} e^*_{j_1} \dots e^*_{j_\tau}$ *where* $k \in K$; $\sigma,\ \tau \geq 0,\ \sigma + \tau > 0,\ e_{i_s} \in E^1$ *and* $e^*_{j_t} \in (E^1)^*$ *for* $0 \leq s \leq \sigma, 0 \leq t \leq \tau$.

Proof. Follow the proof of [47, Corollary 1.15], a straightforward induction argument on the length of the monomial $kx_1 \dots x_n$ with $x_i \in E^0 \cup E^1 \cup (E^1)^*$. □

Although not every Leavitt path algebra is unital (this happens, for example, when the number of vertices is infinite), they are “nearly unital”, concretely, they are algebras with *local units*, i.e., for E a graph, K a field and $L_K(E)$ the associated Leavitt path algebra, there exists a set of idempotentes $\{u_n\}_{n \in \mathbb{N}}$ in $L_K(E)$ satisfying the following properties:

– $u_n \in u_{n+1} L_K(E) u_{n+1}$,

– for every finite subset $X \subseteq L_K(E)$ there exists $m \in \mathbb{N}$ such that $X \subseteq u_m L_K(E) u_m$.

This statement is proved in the following result.

Lemma 1.4.2 *If E^0 is finite then $L(E)$ is a unital K-algebra. If E^0 is infinite, then $L(E)$ is an algebra with local units (specifically, the set generated by finite sums of distinct elements of E^0).*

Proof. First assume that E^0 is finite: We will show that $\sum_{i=1}^n v_i$ is the unit element of the algebra. First we compute $(\sum_{i=1}^n v_i)v_j = \sum_{i=1}^n \delta_{ij}v_j = v_j$. Now if we take $e_j \in E^1$ we may use the equations (2) in the definition of path algebra together with the previous computation to get $(\sum_{i=1}^n v_i)e_j = (\sum_{i=1}^n v_i)r(e_j)e_j = r(e_j)e_j = e_j$. In a similar manner we see that $(\sum_{i=1}^n v_i)e_j^* = e_j^*$. Since $L(E)$ is generated by $E^0 \cup E^1 \cup (E^1)^*$, then it is clear that $(\sum_{i=1}^n v_i)\alpha = \alpha$ for every $\alpha \in L(E)$, and analogously $\alpha(\sum_{i=1}^n v_i) = \alpha$ for every $\alpha \in L(E)$. Now suppose that E^0 is infinite. Consider a finite subset $\{a_i\}_{i=1}^t$ of $L(E)$ and use 1.4.1 to write $a_i = \sum_{s=1}^{n_i} k_s^i v_{j_s{}^i} + \sum_{l=1}^{m_i} k_l'^i p_l^i$ where $k_s^i, k_l'^i \in K-\{0\}$, and p_l^i are monomials of type (b). Then with the same ideas as above it is not difficult to prove that $\alpha = \sum_{i=1}^t(\sum_{s=1}^{n_i} v_{j_s{}^i} + \sum_{l=1}^{m_i} r(p_l^i) + \sum_{l=1}^{m_i} s(p_l^i))$ is a finite sum of vertices such that $\alpha a_i = a_i\alpha = a_i$ for every i. $\square$

Now we see that every Leavitt path algebra is a $\mathbb{Z}$-graded algebra and describe the grading.

Lemma 1.4.3 *$L(E)$ is a $\mathbb{Z}$-graded algebra, with grading induced by*

$$deg(v_i) = 0 \text{ for all } v_i \in E^0; deg(e_i) = 1 \text{ and } deg(e_i^*) = -1 \text{ for all } e_i \in E^1.$$

That is, $L(E) = \bigoplus_{n\in\mathbb{Z}} L(E)_n$, where $L(E)_0 = KE^0 + A_0$, $L(E)_n = A_n$ for $n \neq 0$ where

$$A_n = \sum\{ke_{i_1}\dots e_{i_\sigma}e_{j_1}^*\dots e_{j_\tau}^* : \sigma+\tau > 0,\ e_{i_s} \in E^1,\ e_{i_t} \in (E^1)^*,\ k \in K,\ \sigma-\tau = n\}.$$

Proof. The fact that $L(E) = \sum_{n\in\mathbb{Z}} L(E)_n$ follows from 1.4.1. The grading on $L(E)$ follows directly from the fact that $A(\widehat{E})$ is $\mathbb{Z}$-graded, and that the relations (CK1) and (CK2) are homogeneous in this grading. $\square$

Note that the natural monomorphism from the path algebra KE into the Leavitt path algebra $L_K(E)$ is graded, hence KE is a $\mathbb{Z}$-graded subalgebra of $L_K(E)$.

The following result appears in [52].

Lemma 1.4.4 *Let E be a graph and K a field. Any set of different paths is K-linearly independent in $L_K(E)$.*

Proof. Let $\mu_1, \ldots, \mu_n$ be different paths. Write $\sum_i k_i \mu_i = 0$, for $k_i \in K$. Applying that $L_K(E)$ is $\mathbb{Z}$-graded we may suppose that all the paths have the same length. Since $\mu_j^* \mu_i = \delta_{ij} r(\mu_j)$ then $0 = \sum_i k_i \mu_j^* \mu_i = k_j r(\mu_j)$; this implies $k_j = 0$. □

The results that follow appear in [6].

Lemma 1.4.5 *Let E be a finite and acyclic graph and $v \in E^0$ a sink. Then $I_v := \sum\{k\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = v = r(\beta),\ k \in K\}$ is an ideal of $L(E)$, and $I_v \cong \mathbb{M}_{n(v)}(K)$.*

Proof. Consider $\alpha\beta^* \in I_v$ and a nonzero monomial $e_{i_1} \ldots e_{i_n} e_{j_1}^* \ldots e_{j_m}^* = \gamma\delta^* \in L(E)$. If $\gamma\delta^*\alpha\beta^* \neq 0$ we have two possibilities: Either $\alpha = \delta p$ or $\delta = \alpha q$ for some paths $p, q \in E^*$.

In the latter case $deg(q) \geq 1$ cannot happen, since v is a sink.

Therefore we are in the first case (possibly with $deg(p) = 0$), and then

$$\gamma\delta^*\alpha\beta^* = (\gamma p)\beta^* \in I_v$$

because $r(\gamma p) = r(p) = v$. This shows that I_v is a left ideal. Similarly we can show that I_v is a right ideal as well.

Let $n = n(v)$ (which is clearly finite because the graph is acyclic, finite and row-finite), and rename $\{\alpha \in E^* : r(\alpha) = v\}$ as $\{p_1, \ldots, p_n\}$ so that

$$I_v := \sum\{kp_ip_j^* : i, j = 1, \ldots, n; k \in K\}.$$

Take $j \neq t$. If $(p_ip_j^*)(p_tp_l^*) \neq 0$, then as above, $p_t = p_jq$ with $deg(q) > 0$ (since $j \neq t$), which contradicts that v is a sink.

Thus, $(p_ip_j^*)(p_tp_l^*) = 0$ for $j \neq t$. It is clear that

$$(p_ip_j^*)(p_jp_l^*) = p_ivp_l^* = p_ip_l^*.$$

We have shown that $\{p_ip_j^* : i, j = 1, \ldots, n\}$ is a set of matrix units for I_v, and the result now follows. □

Proposition 1.4.6 *Let E be a finite and acyclic graph. Let $\{v_1, \ldots, v_t\}$ be the sinks. Then*

$$L(E) \cong \bigoplus_{i=1}^{t} \mathbb{M}_{n(v_i)}(K).$$

Proof. We will show that $L(E) \cong \bigoplus_{i=1}^{t} I_{v_i}$, where I_{v_i} are the sets defined in Lemma 1.4.5.

Consider $0 \neq \alpha\beta^*$ with $\alpha, \beta \in E^*$. If $r(\alpha) = v_i$ for some i, then $\alpha\beta^* \in I_{v_i}$. If $r(\alpha) \neq v_i$ for every i, then $r(\alpha)$ is not a sink, and the relation (4) in the definition of $L_K(E)$ applies to yield:

$$\alpha\beta^* = \alpha \left(\sum_{\substack{e \in E^1 \\ s(e)=r(\alpha)}} ee^* \right) \beta^* = \sum_{\substack{e \in E^1 \\ s(e)=r(\alpha)}} \alpha e(\beta e)^*.$$

Now since the graph is finite and there are no cycles, for every summand in the expression above, either the summand is already in some I_{v_i}, or we can repeat the process (expanding as many times as necessary) until reaching sinks. In this way $\alpha\beta^*$ can be written as a sum of terms of the form $\alpha\gamma(\beta\gamma)^*$ with $r(\alpha\gamma) = v_i$ for some i. Thus $L(E) = \sum_{i=1}^{t} I_{v_i}$.

Consider now $i \neq j$, $\alpha\beta^* \in I_{v_i}$ and $\gamma\delta^* \in I_{v_j}$. Since v_i and v_j are sinks, we know as in Lemma 1.4.5 that there are no paths of the form $\beta\gamma'$ or $\gamma\beta'$, and hence $(\alpha\beta^*)(\gamma\delta^*) = 0$. This shows that $I_{v_i}I_{v_j} = 0$, which together with the facts that $L(E)$ is unital and $L(E) = \sum_{i=1}^{t} I_{v_i}$, implies that the sum is direct. Finally, Lemma 1.4.5 gives the result. □

We now get as corollaries to Proposition 1.4.6 the two results mentioned.

Theorem 1.4.7 *The Leavitt path algebra $L_K(E)$ is a finite dimensional K-algebra if and only if E is a finite and acyclic graph.*

Proof. If E is finite and acyclic, then Proposition 1.4.6 immediately yields that $L_K(E)$ is finite dimensional.

Suppose on the other hand that E is not finite; in other words, the set E^0 of vertices is infinite. But then $\{v \mid v \in E^0\}$ is a linearly independent set in $L_K(E)$. Furthermore, if E is not acyclic, then there is a vertex v and a closed path μ based at v. But then $\{\mu^n \mid n \geq 1\}$ is a linearly independent set in $L_K(E)$. □

Combining Proposition 1.4.6 with Theorem 1.4.7 immediately yields

Corollary 1.4.8 *The only finite dimensional K-algebras which arise as $L_K(E)$ for a graph E are of the form $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$.*

Chapter 2

Uniqueness theorems. Simple Leavitt path algebras

Introduction

We start this chapter by studying semiprimeness of path algebras and of Leavitt path algebras, property that differs in each context. Our following goal will be to describe graded ideals in Leavitt path algebras.

2.1 Semiprimeness in path algebras and in Leavitt path algebras

We first study the semiprimeness of the path algebra associated to a graph E. Recall that an algebra A is said to be *semiprime* if it has no nonzero ideals of zero square, equivalently, if $aAa = 0$ for $a \in A$ implies $a = 0$ (an algebra A that satisfies this last condition is called in the literature *nondegenerate*).

Proposition 2.1.1 *([52, Proposition 2.1]). For a graph E and a field K the path algebra KE is semiprime if and only if for every path μ there exists a path μ' such that $s(\mu') = r(\mu)$ and $r(\mu') = s(\mu)$.*

Proof. Suppose first that KE is semiprime. Given a path μ, since $\mu(KE)\mu \neq 0$, there exists a path $\nu \in KE$ such that $\mu\nu\mu \neq 0$. This means that $s(\nu) = r(\mu)$ and $r(\nu) = s(\mu)$.

Now, let us prove the converse. Note that by [44, Proposition II.1.4 (1)], a $\mathbb{Z}$-graded algebra is semiprime if and only if it is graded semiprime. Hence, and taking into account that being graded semiprime and graded nondegenerate are equivalent, it suffices to show that if x is any nonzero homogeneous element of KE, then $x(KE)x \neq 0$. Write $x = \sum_{i=1}^{n} k_i\alpha_i$, with $0 \neq k_i \in K$ and $\alpha_1, \dots, \alpha_n$ different paths of the same degree (i.e. of the same length). Denote the source and range of α_1 by u_1 and v_1, respectively. Then, by (3), $\alpha_1{}^*x = k_1\alpha_1{}^*\alpha_1 = k_1v_1$. By the hypothesis, there exists a path $\alpha_1{}'$ such that $s(\alpha_1{}') = v_1$ and $r(\alpha_1{}') = u_1$. Observe that $\alpha_1{}'x \neq 0$; otherwise $0 = (\alpha_1{}')^*\alpha_1{}'x = u_1x$, a contradiction since a set of different paths is always linearly independent over K (Lemma 1.4.4) and $\alpha_1 = u_1\alpha_1 \neq 0$. Therefore $0 \neq k_1\alpha_1{}'x = k_1v_1\alpha_1{}'x = \alpha_1{}^*x\alpha_1{}'x \in \alpha_1{}^*x(KE)x$. □

Lemma 2.1.2 *([18, Lemma 1.5]). Let E be an arbitrary graph. Let v be a vertex in E^0 such that there exists a cycle without exits c based at v. Then:*

$$vL_K(E)v = \left\{ \sum_{i=-m}^{n} k_ic^i \mid k_i \in K;\ m, n \in \mathbb{N} \right\} \cong K[x, x^{-1}],$$

where $\cong$ denotes a graded isomorphism of K-algebras, and considering (by abuse of notation) $c^0 = w$ and $c^{-t} = (c^*)^t$, for any $t \geq 1$.

Proof. First, it is easy to see that if $c = e_1 \dots e_n$ is a cycle without exits based at v and $u \in T(v)$, then $s(f) = s(g) = u$, for $f, g \in E^1$, implies $f = g$. Moreover, if $r(h) = r(j) = w \in T(v)$, with $h, j \in E^1$, and $s(h),\ s(j) \in T(v)$ then $h = j$. We have also that if $\mu \in E^*$ and $s(\mu) = u \in T(v)$ then there exists $k \in \mathbb{N}^*,\ 1 \leq k \leq n$ verifying $\mu = e_k\mu'$ and $s(e_k) = u$.

Let $x \in vL_K(E)v$ be given by $x = \sum_{i=1}^{p} k_i\alpha_i\beta_i^* + \delta v$, with $s(\alpha_i) = r(\beta_i^*) = s(\beta_i) = v$ and $\alpha_i,\ \beta_i \in E^*$. Consider $A = \{\alpha \in E^* \colon s(\alpha) = v\}$; we prove now that if $\alpha \in A$, $deg(\alpha) = mn + q,\ m,\ q \in \mathbb{N}$ with $0 \leq q < n$, then $\alpha = c^m e_1 \dots e_q$. We proceed by induction on $deg(\alpha)$. If $deg(\alpha) = 1$ and $s(\alpha) = s(e_1)$ then $\alpha = e_1$. Suppose now that the result holds for any $\beta \in A$ with $deg(\beta) \leq sn + t$ and consider any $\alpha \in A$, with $deg(\alpha) = sn + t + 1$. We can write $\alpha = \alpha' f$ with $\alpha' \in A,\ f \in E^1$ and $deg(\alpha') = sn + t$, so by the induction hypothesis $\alpha' = c^s e_1 \dots e_t$. Since $s(f) = r(e_t) = s(e_{t+1})$ implies $f = e_{t+1}$, then $\alpha = \alpha' f = c^s e_1 \dots e_{t+1}$.

We shall show that the elements $\alpha_i\beta_i^*$ are in the desired form, i.e., c^d with $d \in \mathbb{Z}$. Indeed, if $deg(\alpha_i) = deg(\beta_i)$ and $\alpha_i\beta_i^* \neq 0$, we have $\alpha_i\beta_i^* = c^p e_1 \dots e_k e_k^* \dots e_1^* c^{-p} = v$ by (4). On the other hand $deg(\alpha_i) > deg(\beta_i)$ and $\alpha_i\beta_i^* \neq 0$ imply $\alpha_i\beta_i^* = c^{d+q} e_1 \dots e_k e_k^* \dots e_1^* c^{-q} = c^d,\ d \in \mathbb{N}^*$. In a similar way, from $deg(\alpha_i) < deg(\beta_i)$ and $\alpha_i\beta_i^* \neq 0$ it follows that $\alpha_i\beta_i^* = c^q e_1 \dots e_k e_k^* \dots e_1^* c^{-q-d} = c^{-d},\ d \in \mathbb{N}^*$. Define $\varphi \colon K[x, x^{-1}] \to L_K(E)$ by $\varphi(1) = v,\ \varphi(x) = c$ and $\varphi(x^{-1}) = c^*$. It is a straightforward routine to check that φ is a graded monomorphism with image $vL_K(E)v$, so that $vL_K(E)v$ is graded isomorphic to $K[x, x^{-1}]$ as a graded K-algebra. □

For a not necessarily associative K-algebra A, and fixed $x, y \in A$, the left and right *multiplication operators* $L_x, R_y \colon A \to A$ are defined by $L_x(y) := xy$ and $R_y(x) := xy$. Denoting by $\mathrm{End}_K(A)$ the K-algebra of K-linear maps $f \colon A \to A$, the *multiplication algebra* of A (denoted $\mathcal{M}(A)$) is the subalgebra of $\mathrm{End}_K(A)$ generated by the unit and all left and right multiplication operators $L_a, R_a : A \to A$. There is a natural action of $\mathcal{M}(A)$ on A such that A is an $\mathcal{M}(A)$-module whose submodules are just the ideals of A. This is given by $\mathcal{M}(A) \times A \longrightarrow A$, where $f \cdot a := f(a)$ for any $(f, a) \in \mathcal{M}(A) \times A$. Given $x, y \in A$ we shall say that x *is linked to* y if there is some $f \in \mathcal{M}(A)$ such that $y = f(x)$. This fact will be denoted by $x \vdash y$.

The result that follows states that any nonzero element in a Leavitt path algebra is linked to either a vertex or to a nonzero polynomial in a cycle with no exits. So it gives a full account of the action of $\mathcal{M}(L_K(E))$ on $L_K(E)$. This result is very powerful as the main ingredient to show that the socle of a Leavitt path algebra of a row-finite graph is the ideal generated by the line points. Other interesting results are also obtained as a consequence of it.

Proposition 2.1.3 *([17, Proposition 3.1]). Let E be an arbitrary graph. Then, for every nonzero element $x \in L_K(E)$, there exist $\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s \in E^0 \cup E^1 \cup (E^1)^*$ such that:*

1. *$\mu_1 \ldots \mu_r x \nu_1 \ldots \nu_s$ is a nonzero element in Kv, for some $v \in E^0$, or*
2. *there exist a vertex $w \in E^0$ and a cycle without exits c based at w such that $\mu_1 \ldots \mu_r x \nu_1 \ldots \nu_s$ is a nonzero element in $wL_K(E)w$.*

Both cases are not mutually exclusive.

Proof. Show first that for a nonzero element $x \in L(E)$, there exists a path $\mu \in L(E)$ such that $x\mu$ is nonzero and in only real edges.

Consider a vertex $v \in E^0$ such that $xv \neq 0$. Write $xv = \sum_{i=1}^m \beta_i e_i^* + \beta$, with $e_i \in E^1$, $e_i \neq e_j$ for $i \neq j$ and $\beta_i, \beta \in L(E)$, β in only real edges and such that this is a minimal representation of xv in ghost edges.

If $xve_i = 0$ for every $i \in \{1, \ldots, m\}$, then $0 = xve_i = \beta_i + \beta e_i$, hence $\beta_i = -\beta e_i$, and $xv = \sum_{i=1}^m -\beta e_i e_i^* + \beta = \beta(\sum_{i=1}^m -e_i e_i^* + v) \neq 0$. This implies that $\sum_{i=1}^m -e_i e_i^* + v \neq 0$ and since $s(e_i) = v$ for every i, this means that there exists $f \in E^1$, $f \neq e_i$ for every i, with $s(f) = v$. In this case, $xvf = \beta f \neq 0$ (because β is in only real edges), with βf in only real edges, which would conclude our discussion.

If $xve_i \neq 0$ for some i, say for $i = 1$, then $0 \neq xve_1 = \beta_1 + \beta e_1$, with $\beta_1 + \beta e_1$ having strictly less degree in ghost edges than x.

Repeating this argument, in a finite number of steps we prove our first statement.

Now, assume $x = xv$ for some $v \in E^0$ and x in only real edges. Let $0 \neq x = \sum_{i=1}^{r} k_i\alpha_i$ be a linear combination of different paths α_i with $k_i \neq 0$ for any i. We prove by induction on r that after multiplication on the left and/or the right we get a vertex or a polynomial in a cycle with no exit. For $r = 1$ if α_1 has degree 0 then it is a vertex and we have finished. Otherwise we have $x = k_1\alpha_1 = k_1 f_1 \cdots f_n$ so that $k_1^{-1} f_n^* \cdots f_1^* x = v$ where $v = r(f_n) \in E^0$.

Suppose now that the property is true for any nonzero element which is a sum of less than r paths in the conditions above. Let $0 \neq x = \sum_{i=1}^{r} k_i\alpha_i$ such that $\deg(\alpha_i) \leq \deg(\alpha_{i+1})$ for any i. If for some i we have $\deg(\alpha_i) = \deg(\alpha_{i+1})$ then, since $\alpha_i \neq \alpha_{i+1}$, there is some path μ such that $\alpha_i = \mu f \nu$ and $\alpha_{i+1} = \mu f' \nu'$ where $f, f' \in E^1$ are different and ν, ν' are paths. Thus $0 \neq f^*\mu^* x$ and we can apply the induction hypothesis to this element. So we can go on supposing that $\deg(\alpha_i) < \deg(\alpha_{i+1})$ for each i.

We have $0 \neq \alpha_1^* x = k_1 v + \sum_i k_i\beta_i$, where $v = r(\alpha_1)$ and $\beta_i = \alpha_1^*\alpha_i$. If some β_i is null then apply the induction hypothesis to $\alpha_1^* x$ and we are done. Otherwise if some β_i does not start (or finish) in v we apply the induction hypothesis to $v\alpha_1^* x \neq 0$ (or $\alpha_1^* xv \neq 0$). Thus we have

$$0 \neq z := \alpha_1^* x = k_1 v + \sum_{i=1}^{r} k_i\beta_i,$$

where $0 < \deg(\beta_1) < \cdots < \deg(\beta_r)$ and all the paths β_i start and finish in v.

Now, if there is a path τ such that $\tau^*\beta_i = 0$ for some β_i but not for all of them, then we apply our inductive hypothesis to $0 \neq \tau^* z \tau$. Otherwise for any path τ such that $\tau^*\beta_j = 0$ for some β_j, we have $\tau^*\beta_i = 0$ for all β_i. Thus $\beta_{i+1} = \beta_i r_i$ for some path r_i and z can be written as

$$z = k_1 v + k_2\gamma_1 + k_3\gamma_1\gamma_2 + \cdots + k_r\gamma_1\cdots\gamma_{r-1},$$

where each path γ_i starts and finishes in v. If the paths γ_i are not identical we have $\gamma_1 \neq \gamma_i$ for some i, then $0 \neq \gamma_i^* z \gamma_i = k_1 v$ proving our thesis. If the paths are identical then z is a polynomial in the cycle $c = \gamma_1$ with independent term $k_1 v$, that is, an element in $vL(E)v$.

If the cycle has an exit, it can be proved that there is a path η such that $\eta^* c = 0$, in the following way: Suppose that there is a vertex $w \in T(v)$, and two edges e, f, with $e \neq f$, $s(e) = s(f) = w$, and such that $c = aweb = aeb$, for a and b paths in $L(E)$. Then $\eta = af$ gives $\eta^* c = f^* a^* aeb = f^* eb = 0$. Therefore, $\eta^* z \eta$ is a nonzero scalar multiple of a vertex.

Moreover, if c is a cycle without exits, by Lemma 2.1.2,

$$vL(E)v = \left\{ \sum_{i=-m}^{n} l_i c^i, \text{ with } l_i \in K \text{ and } m, n \in \mathbb{N} \right\},$$

where we understand $c^{-m} = (c^*)^m$ for $m \in \mathbb{N}$ and $c^0 = v$.

Finally, consider the graph E consisting of one vertex and one loop based at the vertex to see that both cases can happen at the same time. This completes the proof. $\square$

Corollary 2.1.4 *For any nonzero $x \in Ł$ we have $x \vdash v$ for some $v \in E^0$ or $x \vdash p(c, c^*)$ where c is a cycle with no exits and p a nonzero polynomial in c and c^*.*

Proof. Use Lemma 2.1.2 together with Proposition 2.1.3. $\square$

Proposition 2.1.5 *Let E be an arbitrary graph. Then $L_K(E)$ is semiprime.*

Proof. Take a nonzero ideal I such that $I^2 = 0$. If I contains a vertex we are done. On the contrary there is a nonzero element $p(c, c^*) \in I$ by Corollary 2.1.4. If we consider the (nonzero) coefficient of maximum degree in c and write $p(c, c^*)^2 = 0$ we immediately see that this scalar must be zero, a contradiction. $\square$

2.2 Uniqueness theorems

An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. A graph is said to satisfy *Condition* (L) if every cycle in the graph has an exit.

For any K-algebra A the $\mathcal{M}(A)$-submodules of A are just the ideals of A and the cyclic $\mathcal{M}(A)$-submodules of A are the ideals generated by one element (*principal ideals* in the sequel), so Corollary 2.1.4 states that the nonzero principal ideals of any Leavitt path algebra contain either vertices or nonzero elements of the form $p(c, c^*)$. Therefore, for graphs in which every cycle has an exit, each nonzero ideal contains a vertex.

The following result is a consequence of Proposition 2.1.3.

Corollary 2.2.1 *Let E be an arbitrary graph.*

(i) Every $\mathbb{Z}$-graded nonzero ideal of $L_K(E)$ contains a vertex.

(ii) Suppose that E satisfies Condition (L). *Then every nonzero ideal of $L_K(E)$ contains a vertex.*

Proof. The second assertion has been proved above. So assume that I is a graded ideal of Ł which contains no vertices. Let $0 \neq x \in I$ and use Corollary 2.1.4 to find elements $y, z \in L_K(E)$ such that $yxz = \sum_{i=-m}^{n} k_i c^i \neq 0$. But

I being a graded ideal implies that every summand is in I. In particular, for $t \in \{-m, \ldots, n\}$ such that $k_t c^t \neq 0$ we have $0 \neq (k_t)^{-1} c^{-t} k_t c^t = w \in I$, which is absurd. □

Theorem 2.2.2 *Let E be an arbitrary graph, and let $L_K(E)$ be the associated Leavitt path algebra.*

(1) Graded Uniqueness Theorem.

If A is a $\mathbb{Z}$-graded ring and $\pi : L_K(E) \to A$ is a graded ring homomorphism with $\pi(v) \neq 0$ for every vertex $v \in E^0$, then π is injective.

(2) Cuntz-Krieger Uniqueness Theorem.

Suppose that E satisfies Condition (L)*. If $\pi : L_K(E) \to A$ is a ring homomorphism with $\pi(v) \neq 0$, for every vertex $v \in E^0$, then π is injective.*

Proof. In both cases, the kernel of the ring homomorphism π is an algebra ideal (a graded ideal in the first one). By Corollary 2.2.1, $\mathrm{Ker}(\pi)$ must be zero because otherwise it would contain a vertex (apply (i) in the corollary to (1) and (ii) to the other case), which is not possible by the hypotheses. □

2.3 Simple Leavitt path algebras

In this section we use Proposition 2.1.3 to proof a characterization of simple Leavitt path algebras (see [2, Theorem 3.11] and [18, Corollary 3.8]).

Recall that an algebra A is said to be *simple* if $A^2 \neq 0$ and it has no nonzero proper ideals. If the algebra is graded by a group G, write $A = \sum_{\sigma \in G} A_\sigma$, it is called *graded simple* if $A^2 \neq 0$ and it has no nonzero proper graded ideals (an ideal I of A is *graded* if whenever $y = \sum_\sigma (y_\sigma)$, every $y_\sigma \in I$). In general, and in the particular case of Leavitt path algebras, simplicity and graded-simplicity are not equivalent, as we shall see.

For $n \geq 2$ we write E^n to denote the set of paths of length n, and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths. We define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0. For a graph E, the empty set and E^0 are hereditary and saturated subsets of E^0.

The set $T(v) = \{w \in E^0 \mid v \geq w\}$ is the *tree* of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x\in X} T(x)$. The *hereditary saturated closure* of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. It is shown in [15] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$\Lambda_0(X) = T(X)$, and

$\Lambda_n(X) = \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

One way of constructing graded ideals in a Leavitt path algebra is the following:

Lemma 2.3.1 *Let H be a hereditary subset of E^0, for a graph E. Then*

$$I(H) = \left\{\sum k\alpha\beta^*, \text{ with } k \in K, \alpha, \beta \text{ paths such that } r(\alpha) = r(\beta) \in H\right\}.$$

In fact, $I(H)$ is a graded ideal.

Proof. Denote by J the following set:

$$J = \left\{\sum k\alpha\beta^* \mid k \in K, \alpha, \beta \text{ are paths and } r(\alpha) = r(\beta) \in H\right\}.$$

The containment $J \subseteq I(H)$ is clear. For the converse, consider μ, ν, α, β paths in $L(E)$, and $u \in H$ such that $\mu\nu^* u\alpha\beta^* \neq 0$. By [53, Lemma 3.1], $\mu\nu^* u\alpha\beta^*$ is $\mu\alpha'\beta^*$ if $\alpha = \nu\alpha'$ or $\mu\nu'^*\beta^*$ if $\nu = \alpha\nu'$. Note that $\alpha = \nu\alpha'$, $u = s(\alpha)$ and H hereditary imply $r(\alpha') \in H$, hence $\mu\alpha'\beta^* \in J$. In the second case, $\nu = \alpha\nu'$, $u = s(\alpha)$ and H hereditary imply $s(\nu'^*) = r(\nu') \in H$, hence $\mu r(\nu')\nu'^*\beta^* \in J$, therefore $I(H) \subseteq J$.

The last statement follows immediately by the form the elements of $I(H)$ have.

$\square$

Moreover, it was proved in [2, Lemma 3.9]

Lemma 2.3.2 *For every ideal I of a Leavitt path algebra $L_K(E)$, $I \cap E^0$ is a hereditary and saturated subset of E^0.*

In fact, all graded ideals of a Leavitt path algebra come from hereditary and saturated subsets of vertices.

Remark 2.3.3 An ideal J of $L(E)$ is graded if and only if it is generated by idempotents; in fact, $J = I(H)$, where $H = J \cap E^0 \in \mathcal{H}_E$. (See the proofs of [15, Proposition 4.2 and Theorem 4.3].)

Now, the question that arise is if every graded ideal of a Leavitt path algebra is again a Leavitt path algebra (note that this question has sense only for graded ideals as every Leavitt path algebra is graded). Other natural question is if the quotient of a Leavitt path algebra by an ideal is a Leavitt path algebra too.

In both cases, as we shall see, the answer is yes.

For a graph E and a hereditary subset H of E^0, we denote by E/H the *quotient graph*

$$(E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and by E_H the *restriction graph*

$$(H, \{e \in E^1 \mid s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Observe that while $L(E_H)$ can be seen as a subalgebra of $L(E)$, the same cannot be said about $L(E/H)$.

Lemma 2.3.4 *([19, Lemma 2.3]) Let E be a graph and consider a proper $H \in \mathcal{H}_E$. Define $\Psi : L(E) \to L(E/H)$ by setting $\Psi(v) = \chi_{(E/H)^0}(v)v$, $\Psi(e) = \chi_{(E/H)^1}(e)e$ and $\Psi(e^*) = \chi_{((E/H)^1)^*}(e^*)e^*$ for every vertex v and every edge e, where $\chi_{(E/H)^0} : E^0 \to K$ and $\chi_{(E/H)^1} : E^1 \to K$ denote the characteristic functions. Then:*

1. *The map Ψ extends to a K-algebra epimorphism of $\mathbb{Z}$-graded algebras with $Ker(\Psi) = I(H)$ and therefore $L(E)/I(H) \cong L(E/H)$.*
2. *If X is hereditary in E, then $\Psi(X) \cap (E/H)^0$ is hereditary in E/H.*
3. *For $X \supseteq H$, $X \in \mathcal{H}_E$ if and only if $\Psi(X) \cap (E/H)^0 \in \mathcal{H}_{(E/H)}$.*
4. *For every $X \supseteq H$, $\overline{\Psi(X) \cap (E/H)^0} = \Psi(\overline{X}) \cap (E/H)^0$.*

Proof. (1) It was shown in [2, Proof of Theorem 3.11] that Ψ extends to a K-algebra morphism. By definition, Ψ is $\mathbb{Z}$-graded and onto. Moreover, $I(H) \subseteq \mathrm{Ker}(\Psi)$.

Since Ψ is a graded morphism, $\mathrm{Ker}(\Psi) \in \mathcal{L}_{gr}(L(E))$. By [15, Theorem 4.3], there exists $X \in \mathcal{H}_E$ such that $\mathrm{Ker}(\Psi) = I(X)$. By Lemma **??**, $H = I(H) \cap E^0 \subseteq I(X) \cap E^0 = X$. Hence, $I(H) \neq \mathrm{Ker}(\Psi)$ if and only if there exists $v \in X \setminus H$. But then $\Psi(v) = v \neq 0$ and $v \in \mathrm{Ker}(\Psi)$, which is impossible.

(2) It is clear by the definition of Ψ.

(3) Since Ψ is a graded epimorphism, there is a bijection between graded ideals of $L(E/H)$ and graded ideals of $L(E)$ containing $I(H)$. Thus, the result holds by [15, Theorem 4.3].

(4) It is immediate by part (3). $\square$

Lemma 2.3.5 *Let E be a graph. For every hereditary and saturated subset H of E, the ideal $I(H)$ is isomorphic to $L(_HE)$.*

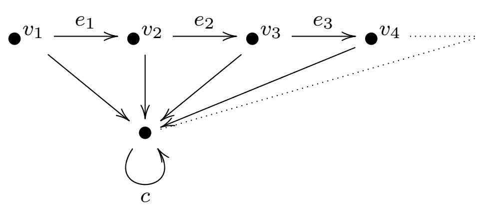
Corollary 2.3.6 *Every graded ideal of a Leavitt path algebra is again a Leavitt path algebra.*

One interesting property for Leavitt path algebras is that cycles without exists behaves in a similar way to sinks, so, roughly speaking, for a graph having no cycles with exits, and such that every vertex connects to the cycle (the so called C_n-comet), the corresponding Leavitt path algebra is a direct sum of matrices over something appropriate.

The notion of C_n-comet was introduced in [7] to describe the locally finite Leavitt path algebras. The role of the cycle C_n within a C_n-comet is similar to that played by sinks in more general graphs. If a graph E is a C_n-comet, then its associated Leavitt path algebra is isomorphic to $\mathbb{M}_n(K[x, x^{-1}])$. Since C_n-comets have a finite number of vertices, it is natural to generalize this concept to the case of an infinite (numerable) set of vertices.

Definition 2.3.7 We say that a graph E is a *comet* if it has exactly one cycle c, $T(v) \cap c^0 \neq \emptyset$ for every vertex $v \in E^0$, and every infinite path ends in the cycle c.

Remark 2.3.8 The following is not an example of a C_n-comet:



and the reason is that the infinite path $\gamma = e_1 e_2 e_3 \ldots$ does not end either in a sink or in a cycle.

Proposition 2.3.9 *Let E be a graph which is a comet. Then the Leavitt path algebra $L(E)$ is isomorphic to $\mathbb{M}_n(K[x, x^{-1}])$, where $n \in \mathbb{N}$ if E is finite, or $n = \infty$ otherwise.*

Proof. We can adapt [7, Theorem 3.3] to our situation. Concretely, let c be the cycle in E, v a vertex at which the cycle c is based and consider $\{p_i\}$ the (perhaps infinite) family of all paths in E which end in v but do not contain the cycle c. Let $n \in \mathbb{N} \cup \{\infty\}$ be the number of all such paths. Denote by N the set $\{1, \dots, n\}$ when n is finite and $N = \mathbb{N}$ when $n = \infty$. Consider the family $\mathcal{B} := \{p_i c^k p_j^*\}_{i,j \in N, k \in \mathbb{N}}$ of monomials in $L(E)$ where we understand $c^0 = v$ and $c^n = (c^*)^{-n}$ for negative n.

As in [7, Theorem 3.3], we can show that $\mathcal{B}$ is a linearly independent set. We will prove that $\mathcal{B}$ generates $L(E)$ as a K-vector space. First, note that since E is a comet, then $T(v)$ is a finite set for every $v \in c^0$. Not only is this true for any vertex on the cycle c but also for any vertex in E as follows: Suppose on the contrary that there exists $w \in E$ with $|T(w)| = \infty$. In particular, w does not lie on the cycle. As E is row-finite, we are able to find and edge e_1 in E with $s(e_1) = w$ and $v_1 := r(e_1)$ such that $|T(v_1)| = \infty$. Again v_1 does not lie on the cycle. Repeating this process, we find an infinite path such that none of its vertices lie on c, which contradicts the fact that every infinite path in E ends in the cycle c.

Take an arbitrary element $\sum_i k_i \alpha_i \beta_i^*$ of $L(E)$, where α_i, β_i are paths in E and $k_i \in K$. Consider the set $\{r(\alpha_i)\}$. Some of these vertices could lie on the cycle c, in which case we leave the corresponding monomial as is. For those monomials $\alpha_k \beta_k^*$ whose $\{r(\alpha_k)\}$ is not on c, we proceed as in [6, Proof of Proposition 3.5] by using relation (4) to expand it as

$$\alpha_k \beta_k^* = \sum_{\{e \in E^1 : s(e) = r(\alpha_k)\}} \alpha_k e e^* \beta_k^* = \sum_{\{e \in E^1 : s(e) = r(\alpha_k)\}} (\alpha_k e)(\beta_k e)^*.$$

As we have just proved that the tree of any vertex is finite, so will be this process of expanding these monomials until reaching vertices of c.

Consider now a monomial $\alpha_k \beta_k^*$ with $r(\alpha_k) \in c^0$. Let t be the subpath of c with $s(t) = r(\alpha_k)$ and $r(t) = v$. Since c does not have exits then $\alpha_k \beta_k^* = \alpha_k t t^* \beta_k^* = (\alpha_k t)(\beta_k t)^* = \alpha \beta^*$, where α and β are paths in E that end in v. Finally, since E is a comet, we can always factor some powers of c out of α and β so that there exist integers m, n such that $\alpha = p_i c^m$ and $\beta = p_j c^n$ for some paths p_i, p_j which do not contain the path c. Hence, we obtain that $\alpha_k \beta_k^* = p_i c^{m-n} p_j^* \in \mathcal{B}$. This proves that $\mathcal{B}$ is a K-generator of $L(E)$.

Now, by defining $\phi : L(E) \to \mathbb{M}_n(K[x, x^{-1}])$ on the basis by setting $\phi(p_i c^k p_j^*) = x^k e_{ij}$ for e_{ij} the (i,j)-matrix unit, then again one easily checks that ϕ is a K-algebra isomorphism. □

For a graph E, denote by $P_c(E)$ the set of vertices in the cycles without exits of E.

Proposition 2.3.10 *Let E be a graph. Then:*

(i) $I(P_c(E)) = \bigoplus_{j\in\Upsilon} I(P_{c_j}(E))$, *where* Υ *is a countable set and* $\{c_j\}_{j\in\Upsilon}$ *is the set of all different cycles without exits of* E *(and by abuse of notation we identify two cycles that have the same vertices).*

(ii) $P_c(E)$ *is hereditary and if* H *denotes the saturated closure of* $P_c(E)$, *we have that*

$$I(P_c(E)) = I(H) \cong L({}_HE) \cong \bigoplus_{i\in\Upsilon_1} \mathbb{M}_{n_i}(K[x,x^{-1}]) \oplus \bigoplus_{j\in\Upsilon_2} \mathbb{M}_{m_j}(K[x,x^{-1}]),$$

where Υ_1 *and* Υ_2 *are countable sets,* $n_i \in \mathbb{N}$ *and* $m_j = \infty$.

Proof. We will use Lemma 2.3.1 implicitly. This can be done because $P_c(E)$ is, clearly, a hereditary set.

(i). To shorten the notation, write: $\mathcal{J} = I(P_c(E))$ and $\mathcal{J}_j = I(P_{c_j}(E))$.

Consider monomials $\gamma\delta^*$ with $r(\delta) \in (c_j)^0$ and $\sigma\tau^* \in \mathcal{J}$. Since the cycles c_j have no exits, they are disjoint and then, similar arguments to that of the previous paragraph show that $\gamma\delta^*\sigma\tau^*, \sigma\tau^*\gamma\delta^* \in \mathcal{J}_{c_j}$. Moreover, these arguments also yield that if $\sigma\tau^* \in \mathcal{J}_{c_k}$ with $j \neq k$, then $\gamma\delta^*\sigma\tau^* = \sigma\tau^*\gamma\delta^* = 0$. Thus, $\{\mathcal{J}_{c_j}\}$ is indeed a family of orthogonal ideals of $\mathcal{J}$.

To show that $\mathcal{J} = \sum_j \mathcal{J}_j$ apply Lemma 2.3.1 to $H = \cup_j c_j^0$, which is a hereditary set since the considered cycles have no exits.

(ii). $I(P_c(E)) = I(H)$ follows by [19, Lemma 2.1] and $I(H) \cong L({}_HE)$ by [16, Lemma 1.2] The same results applied to c_j instead of c imply $I(P_{c_j}(E)) = I(H_j) \cong L({}_{H_j}E)$, for H_j the saturated closure of P_{c_j}. By the definition of H_j, and since c_j has no exits, every vertex in H_j connects to c_j. The same can be said about ${}_{H_j}E$, where c_j can be seen as its only cycle. Now suppose that γ is an infinite path in ${}_{H_j}E$. Again, by the way this graph is constructed, there must exist a finite path p and an infinite path β such that $\gamma = p\beta$, with β being completely contained in E_{H_j}. Suppose that β does not end in the cycle c_j. This, together with the fact that c_j does not have exits, yield that $\beta^0 \cap c_j^0 = \emptyset$. On the other hand, because $\beta^0 \subseteq H_j$ we can consider m to be the minimum n such that $\Lambda_n(c_j^0) \cap \beta^0 \neq \emptyset$. Now, $\beta^0 \cap c_j^0 = \emptyset$ implies that $m > 0$ so that there exists $w \in \{v \in (E_{H_j})^0 \mid \emptyset \neq r(s^{-1}(v)) \subseteq \Lambda_{m-1}(c_j^0)\} \cap \beta^0$. As β is infinite, there is an edge e in β such that $s(e) = w$ and $r(e) \in \beta^0$. This contradicts the minimality of m. Therefore β ends in the cycle c, and consequently γ. Hence, ${}_{H_j}E$ is a comet. Apply Proposition 2.3.9 and (i) to obtain the result. $\square$

Theorem 2.3.11 *Let* E *be an arbitrary graph. Then* $L_K(E)$ *is simple if and only if* E *satisfies Condition* (L) *and the only hereditary and saturated subsets of* E^0 *are the trivial ones.*

Proof. Suppose first that $L_K(E)$ is simple. If there exists some cycle without exits in E, the simplicity of $L_K(E)$ and Proposition 2.3.10 imply that the Leavitt path algebra coincides with the ideal generated by this cycle, which is isomorphic to a matrix ring over the Laurent polynomial ring. But this rings are not simple, so this cannot happen, that is, every cycle in E must have an exit.

Now, if H were a hereditary and saturated subset of E^0, then the ideal it generates would be a proper nonzero ideal of $L_K(E)$, contradicting the hypothesis of simplicity.

For the converse take into account that Condition (L) implies that any nonzero element in $L_K(E)$ is linked to a vertex (see Proposition 2.2.1). Thus, there is a vertex in any nonzero ideal I of $L_K(E)$. But on the other hand $\emptyset \neq I \cap E^0$ is hereditary and saturated ([4, Lemma 2.3]), therefore it coincides with E^0 and so $I = L_K(E)$. $\square$

Recall that a *matricial algebra* is a finite direct product of full matrix algebras over K, while a *locally matricial algebra* is a direct limit of matricial algebras. At this point we have shown that finite and acyclic graphs produce matricial algebras. Our following target will be to show that acyclic Leavitt path algebras are locally matricial. The following results can be found in [3].

Lemma 2.3.12 *Let E be a finite acyclic graph. Then $L(E)$ is finite dimensional.*

Proof. Since the graph is row-finite, the given condition on E is equivalent to the condition that E^* is finite. The result now follows from the previous observation that $L(E)$ is spanned as a K-vector space by $\{pq^* : p, q \text{ are paths in } E\}$. $\square$

This lemma is just what we need to the following

Proposition 2.3.13 *Let E be a graph. Then E is acyclic if and only if $L_K(E)$ is a union of a chain of finite dimensional subalgebras. Concretely, it is a locally matricial K-algebra.*

Proof. Assume first that E is acyclic. If E is finite, then Lemma 3 gives the result. So now suppose E is infinite, and rename the vertices of E^0 as a sequence $\{v_i\}_{i=1}^{\infty}$. We now define a sequence $\{F_i\}_{i=1}^{\infty}$ of subgraphs of E. Let $F_i = (F_i^0, F_i^1, r, s)$ where $F_i^0 := \{v_1, \ldots, v_i\} \cup r(s^{-1}(\{v_1, \ldots, v_i\})$, $F_i^1 := s^{-1}(\{v_1, \ldots, v_i\})$, and r, s are induced from E. In particular, $F_i \subseteq F_{i+1}$ for all i. For any $i > 0$, $L(F_i)$ is a subalgebra of $L(E)$ as follows. First note that we can construct $\phi : L(F_i) \to L(E)$ a K-algebra homomorphism because the Cuntz-Krieger relations in $L(F_i)$ are consistent with those in $L(E)$, in the following way: Consider v a sink in F_i (which need not be a sink in E), then we do not have CK2 at v in $L(F_i)$. If v is not a sink in F_i, then there exists $e \in F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ such that $s(e) = v$. But $s(e) \in \{v_1, \dots, v_i\}$ and therefore $v = v_j$ for some j, and then $F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ ensures that all the edges coming to v are in F_i, so CK2 at v is the same in $L(F_i)$ as in $L(E)$. The other relations offer no difficulty. Now, with a similar construction and argument to that used in [2, Proof of Theorem 3.11] we find $\psi : L(E) \to L(F_i)$ a K-algebra homomorphism such that $\psi\phi = Id|_{L(F_i)}$, so that ϕ is a monomorphism, which we view as the inclusion map. By construction, each vertex in E^0 is in F_i for some i; furthermore, the edge e has $e \in F_j^1$, where $s(e) = v_j$. Thus we conclude that $L(E) = \cup_{i=1}^{\infty} L(F_i)$. (We note here that the embedding of graphs $j : F_i \hookrightarrow E$ is a complete graph homomorphism in the sense of [15], so that the conclusion $L(E) = \cup_{i=1}^{\infty} L(F_i)$ can also be achieved by invoking [15, Lemma 2.1].)

Since E is acyclic, so is each F_i. Moreover, each F_i is finite since, by the row-finiteness of E, in each step we add only finitely many vertices. Thus, by Lemma 2.3.12, $L(F_i)$ is finite dimensional, so that $L(E)$ is indeed a union of a chain of finite dimensional subalgebras.

For the converse, let $p \in E^*$ be a cycle in E. Then $\{p^m\}_{m=1}^{\infty}$ is a linearly independent infinite set, so that p is not contained in any finite dimensional subalgebra of $L(E)$. $\square$

A different proof derives from Corollary 1.4.8 and Lemma 4.3.3.

2.4 Purely infinite Leavitt path algebras

The concept of purely infinite simple C*-algebra was introduced by Cuntz in 1981 [28] and implied a significant advance in the development of the theory of C*-algebras. It was in 2002 that Ara, Goodearl and Pardo gave the definition of purely infinite (unital) simple ring (see [14]). Both definitions agree when considering C*-algebras. An idempotent e in a ring R is called *infinite* if eR is isomorphic as a right R-module to a proper direct summand of itself. The ring R is called *purely infinite* in case every nonzero right ideal of R contains an infinite idempotent.

The following characterization of purely infinite simple rings can be found in [3].

Proposition 2.4.1 *For a ring R with local units, the following are equivalent conditions:*

(i) R is purely infinite simple.

(ii) R is simple, and for each nonzero finitely generated projective right A-module P, every nonzero submodule C of P contains a direct summand T of P for which T is directly infinite. (In particular, the property 'purely infinite simple' is a Morita invariant of the ring.)

(iii) wRw is purely infinite simple for every nonzero idempotent $w \in R$.

(iv) R is simple, and there exists a nonzero idempotent w in R for which wRw is purely infinite simple.

(v) R is not a division ring, and A has the property that for every pair of nonzero elements α, β in R there exist elements a, b in R such that $a\alpha b = \beta$.

As in the case of simplicity for Leavitt path algebras, being purely infinite and simple can be characterized in terms of the graph.

Most of the results of this section belong to the paper [3], although some proofs differ from the original ones.

Lemma 2.4.2 *Suppose A is a union of finite dimensional subalgebras. Then A is not purely infinite. In fact, A contains no infinite idempotents.*

Proof. It suffices to show the second statement. So just suppose $e = e^2 \in A$ is infinite. Then eA contains a proper direct summand isomorphic to eA, which in turn, by definition and a standard argument, is equivalent to the existence of elements $g, h, x, y \in A$ such that $g^2 = g, h^2 = h, gh = hg = 0, e = g + h, h \neq 0, x \in eAg, y \in gAe$ with $xy = e$ and $yx = g$. But by hypothesis the five elements e, g, h, x, y are contained in a finite dimensional subalgebra B of A, which would yield that B contains an infinite idempotent, and thus contains a non-artinian right ideal, which is impossible. □

A *closed simple path based at v_{i_0}* is a path $\mu = \mu_1 \dots \mu_n$, with $\mu_j \in E^1$, $n \geq 1$ such that $s(\mu_j) \neq v_{i_0}$ for every $j > 1$ and $s(\mu) = r(\mu) = v_{i_0}$. Denote by $CSP(v_{i_0})$ the set of all such paths. We note that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at v_{i_0} is a cycle. We define the following subsets of E^0:

$$V_0 = \{v \in E^0 : CSP(v) = \emptyset\}$$

$$V_1 = \{v \in E^0 : |CSP(v)| = 1\}$$

$$V_2 = E^0 - (V_0 \cup V_1)$$

Proposition 2.4.3 *Let E be a graph. Suppose that $w \in E^0$ has the property that, for every $v \in E^0$, $w \geq v$ implies $v \in V_0$. Then the corner algebra $wL(E)w$ is not purely infinite.*

Proof. Consider the graph $H = (H^0, H^1, r, s)$ defined by $H^0 := \{v : w \geq v\}$, $H^1 := s^{-1}(H^0)$, and r, s induced by E. The only nontrivial part of showing that H is a well defined graph is verifying that $r(s^{-1}(H^0)) \subseteq H^0$. Take $z \in H^0$ and $e \in E^1$ such that $s(e) = z$. But we have $w \geq z$ and thus $w \geq r(e)$ as well, that is, $r(e) \in H^0$.

Using that H is acyclic, along with the same argument as given in Proposition 2.3.13, we have that $L(H)$ is a subalgebra of $L(E)$. Thus Proposition 2.3.13 applies, which yields that $L(H)$ is the union of finite dimensional subalgebras, and therefore contains no infinite idempotents by Lemma 2.4.2. As $wL(H)w$ is a subalgebra of $L(H)$, it too contains no infinite idempotents, and thus is not purely infinite.

We claim that $wL(H)w = wL(E)w$. To see this, given $\alpha = \sum p_i q_i^* \in L(E)$, then $w\alpha w = \sum p_{i_j} q_{i_j}^*$ with $s(p_{i_j}) = w = s(q_{i_j})$ and therefore $p_{i_j}, q_{i_j} \in L(H)$. Thus $wL(E)w$ is not purely infinite as desired. □

Lemma 2.4.4 *Let E be a graph. If $L(E)$ is simple, then $V_1 = \emptyset$.*

Proof. For any subset $X \subseteq E^0$ we define the following subsets. $H(X)$ is the set of all vertices that can be obtained by one application of the hereditary condition at any of the vertices of X; that is, $H(X) := r(s^{-1}(X))$. Similarly, $S(X)$ is the set of all vertices obtained by applying the saturated condition among elements of X, that is, $S(X) := \{v \in E^0 : \emptyset \neq \{r(e) : s(e) = v\} \subseteq X\}$. We now define $G_0 := X$, and for $n \geq 0$ we define inductively $G_{n+1} := H(G_n) \cup S(G_n) \cup G_n$. It is not difficult to show that the smallest hereditary and saturated subset of E^0 containing X is the set $G(X) := \bigcup_{n \geq 0} G_n$.

Suppose now that $v \in V_1$, so that $CSP(v) = \{p\}$. In this case p is clearly a cycle. By Theorem 2.3.11 we can find an edge e which is an exit for p. Let A be the set of all vertices in the cycle. Since p is the only cycle based at v, and e is an exit for p, we conclude that $r(e) \notin A$. Consider then the set $X = \{r(e)\}$, and construct $G(X)$ as described above. Then $G(X)$ is nonempty and, by construction, hereditary and saturated.

Now Theorem 2.3.11 implies that $G(X) = E^0$, so we can find $n = min\{m : A \cap G_m \neq \emptyset\}$. Take $w \in A \cap G_n$. We are going to show that $w \geq r(e)$. First, since $r(e) \notin A$, then $n > 0$ and therefore $w \in H(G_{n-1}) \cup S(G_{n-1}) \cup G_{n-1}$. Here, $w \in G_{n-1}$ cannot happen by the minimality of n. If $w \in S(G_{n-1})$ then $\emptyset \neq \{r(e) : s(e) = w\} \subseteq G_{n-1}$. Since w is in the cycle p, there exists $f \in E^1$ such that $r(f) \in A$ and $s(f) = w$. In that case $r(f) \in A \cup G_{n-1}$ again contradicts the minimality of n. So the only possibility is $w \in H(G_{n-1})$, which means that there exists $e_{i_1} \in E^1$ such that $r(e_{i_1}) = w$ and $s(e_{i_1}) \in G_{n-1}$.

We now repeat the process with the vertex $w' = s(e_{i_1})$. If $w' \in G_{n-2}$ then we would have $w \in G_{n-1}$, again contradicting the minimality of n. If

$w' \in S(G_{n-2})$ then, as above, $\{r(e) : s(e) = w'\} \subseteq G_{n-2}$, so in particular would give $w = r(e_{i_1}) \in G_{n-2}$, which is absurd. So therefore $w' \in H(G_{n-2})$ and we can find $e_{i_2} \in E^1$ such that $r(e_{i_2}) = w'$ and $s(e_{i_2}) \in G_{n-2}$.

After n steps we will have found a path $q = e_{i_n} \dots e_{i_1}$ with $r(q) = w$ and $s(q) = r(e)$. In particular we have $w \geq s(e)$, and therefore there exists a cycle based at w containing the edge e. Since e is not in p we get $|CSP(w)| \geq 2$. Since w is a vertex contained in the cycle p, we then get $|CSP(v)| \geq 2$, contrary to the definition of the set V_1. □

Theorem 2.4.5 *Let E be a graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

(ii) Every cycle in E has an exit.

(iii) Every vertex connects to a cycle.

Proof. First, assume (i), (ii) and (iii) hold. By Theorem 2.3.11 we have that $L(E)$ is simple. By Proposition 2.4.1 it suffices to show that $L(E)$ is not a division ring, and that for every pair of elements α, β in $L(E)$ there exist elements a, b in $L(E)$ such that $a\alpha b = \beta$. Conditions (ii) and (iii) easily imply that $|E^1| > 1$, so that $L(E)$ has zero divisors, and thus is not a division ring. We now apply Proposition 2.1.3 to find $\overline{a}, \overline{b} \in L(E)$ such that $\overline{a}\alpha\overline{b} = w \in E^0$. By condition (iii), w connects to a vertex $v \notin V_0$. Either $w = v$ or there exists a path p such that $r(p) = v$ and $s(p) = w$. By choosing $a' = b' = v$ in the former case, and $a' = p^*, b' = p$ in the latter, we have produced elements $a', b' \in L(E)$ such that $a'wb' = v$.

An application of Lemma 2.4.4 yields that $v \in V_2$, so there exist $p, q \in CSP(v)$ with $p \neq q$. For any $m > 0$ let c_m denote the closed path $p^{m-1}q$. Using [2, Lemma 2.2], it is not difficult to show that $c_m^* c_n = \delta_{mn} v$ for every $m, n > 0$.

Now consider any vertex $v_l \in E^0$. Since $L(E)$ is simple, there exist $\{a_i, b_i \in L(E) \mid 1 \leq i \leq t\}$ such that $v_l = \sum_{i=1}^t a_i v b_i$. But by defining $a_l = \sum_{i=1}^t a_i c_i^*$ and $b_l = \sum_{j=1}^t c_j b_j$, we get

$$a_l v b_l = \left(\sum_{i=1}^t a_i c_i^*\right) v \left(\sum_{j=1}^t c_j b_j\right) = \sum_{i=1}^t a_i c_i^* v c_i b_i = v_l.$$

Now let s be a left local unit for β (i.e., $s\beta = \beta$), and write $s = \sum_{v_l \in S} v_l$ for some finite subset of vertices S. By letting $\widetilde{a} = \sum_{v_l \in S} a_l c_l^*$ and $\widetilde{b} = \sum_{v_l \in S} c_l b_l$, we get

$$\widetilde{a} v \widetilde{b} = \sum_{v_l \in S} a_l c_l^* v c_l b_l = \sum_{v_l \in S} v_l = s.$$

Finally, letting $a = \widetilde{a}a'\overline{a}$ and $b = \overline{b}b'\widetilde{b}\beta$, we have that $a\alpha b = \beta$ as desired.

For the converse, suppose that $L(E)$ is purely infinite simple. By Theorem 2.3.11 we have (i) and (ii). If (iii) does not hold, then there exists a vertex $w \in E^0$ such that $w \leq v$ implies $v \in V_0$. Applying Proposition 2.4.3 we get that $wL(E)w$ is not purely infinite. But then Proposition 2.4.1 implies that $L(E)$ is not purely infinite, contrary to hypothesis. □

2.5 The dichotomy principle for simple Leavitt path algebras

We are now in a position to show that every simple Leavitt path algebra is locally matricial or purely infinite.

We denote by E^∞ the set of infinite paths $\gamma = (\gamma_n)_{n=1}^\infty$ of the graph E and by $E^{\leq\infty}$ the set E^∞ together with the set of finite paths in E whose end vertex is a sink. We say that a vertex v in a graph E is *cofinal* if for every $\gamma \in E^{\leq\infty}$ there is a vertex w in the path γ such that $v \geq w$. We say that a graph E is *cofinal* if so are all the vertices of E.

Lemma 2.5.1 *([19, Lemma 2.7]). A graph E is cofinal if and only if $\mathcal{H} = \{\emptyset, E^0\}$.*

Proof. Suppose E to be cofinal. Let $H \in \mathcal{H}$ with $\emptyset \neq H \neq E^0$. Fix $v \in E^0 \setminus H$ and build a path $\gamma \in E^{\leq\infty}$ such that $\gamma^0 \cap H = \emptyset$: If v is a sink, take $\gamma = v$. If not, then $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \not\subseteq H$; otherwise, H saturated implies $v \in H$, which is impossible. Hence, there exists $e_1 \in s^{-1}(v)$ such that $r(e_1) \notin H$. Let $\gamma_1 = e_1$ and repeat this process with $r(e_1) \notin H$. By recurrence either we reach a sink or we have an infinite path γ whose vertices are not in H, as desired. Now consider $w \in H$. By the hypothesis, there exists $z \in \gamma$ such that $w \geq z$, and by hereditariness of H we get $z \in H$, contradicting the definition of γ.

Conversely, suppose that $\mathcal{H} = \{\emptyset, E^0\}$. Take $v \in E^0$ and $\gamma \in E^{\leq\infty}$, with $v \notin \gamma^0$ (the case $v \in \gamma^0$ is obvious). By hypothesis the hereditary saturated subset generated by v is E^0, i.e., $E^0 = \bigcup_{n\geq 0} \Lambda_n(v)$. Consider m, the minimum n such that $\Lambda_n(v) \cap \gamma^0 \neq \emptyset$, and let $w \in \Lambda_m(v) \cap \gamma^0$. If $m > 0$, then by minimality of m it must be $s^{-1}(w) \neq \emptyset$ and $r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$. The first condition implies that w is not a sink and since $\gamma = (\gamma_n) \in E^{\leq\infty}$, there exists $i \geq 1$ such that $s(\gamma_i) = w$ and $r(\gamma_i) = w' \in \gamma^0$, the latter meaning that $w' \in r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$, contradicting the minimality of m. Therefore $m = 0$ and then $w \in \Lambda_0(v) = T(v)$, as we needed. □

Theorem 2.5.2 *Every simple Leavitt path algebra is locally matricial or purely infinite.*

Proof. Let E be a graph and K a field, and suppose that $L_K(E)$ is a simple algebra. If E has no cycles, then $L_K(E)$ is a locally matricial agebra, by Proposition 2.3.13.

Now, suposse that E has cycles. As it is simple, by Theorem 2.3.11 and Lemma 2.5.1, every vertex connects to a cycle, hence, by Theorem 2.4.5, the Leavitt path algebra $L_K(E)$ is purely infinite and simple. □

Chapter 3

Graph C^*-algebras

Introduction

C^*-algebras (originally W^*-algebras) appear around 1950 as a restriction of the properties defining von Neumann algebras. Roughly speaking, C^*-algebras restrict the scope of von Neumann algebras to the context of Functional Analysis. Even if they are "algebraic" objects with "analytic" structure, most of the work C^*-algebras is done from the analytic point of view.

We will try to connect, at least in the context of graph algebras, these objects with their purely algebraic nature.

3.1 Basics on C^*-algebras

For operator algebraists, a $*$-*algebra* is an associative algebra A over the complex numbers $\mathbb{C}$ with an *involution*: a map $a \mapsto a^*$ from A to A such that $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$, $(a^*)^* = a$ and $(ab)^* = b^* a^*$. A $*$-algebra may or may not have an identity element 1, but if so, 1^* is also an identity, and hence $1^* = 1$.

Definition 3.1.1 A C^*-*algebra* is a $*$-algebra A with a norm $\|a\| : A \to [0, \infty)$ which satisfies the usual axioms for a norm on a vector space, which satisfies

$$\|ab\| \leq \|a\| \, \|b\| \quad \text{and} \quad \|a\|^2 = \|a^* a\| \text{ (the } C^*\text{-}\textit{identity}\text{)}, \tag{3.1}$$

and for which the normed space $(A, \|\cdot\|)$ is complete in the sense that Cauchy sequences converge. It follows from 3.1 that the norm also satisfies $\|a^*\| = \|a\|$, and, if A has an identity 1, that $\|1\| = 1$.

It is important to notice that the C^*-identity forces any $*$-homomorphism between C^*-algebras to be continuous in the norm-topology, because they are contractive maps (see below). Also it is important the fact that a Banach algebra has (if such exists) a unique norm under which it becomes a C^*-algebra.

Example 3.1.2

1. Let X be a compact Hausdorff space (or a compact metric space if you prefer). Then the set
$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous on } X\}$$
is a C^*-algebra with the algebra operations defined pointwise, with $f^*(x) := \overline{f(x)}$, and with $\|f\| := \sup\{|f(x)| : x \in X\}$. This C^*-algebra is commutative (that is, $fg = gf$) with identity given by the function 1 with constant value $1 \in \mathbb{C}$.

2. Let H be a Hilbert space: an inner-product space over $\mathbb{C}$ which is complete in the norm $\|h\| := (h \,|\, h)^{1/2}$ defined by the inner product. A linear transformation $T : H \to H$ is *bounded* if it maps bounded sets to bounded sets (and then for no good reason we call it a bounded linear *operator* on H); a basic result says that T is bounded if and only if it is continuous. The set $B(H)$ of bounded linear operators on H is a C^*-algebra with addition and scalar multiplication given pointwise, with multiplication given by composition, with the *operator norm* defined by
$$\|T\|_{\mathrm{op}} = \sup\{\|Th\| : \|h\| \leq 1\},$$
and with the adjoint T^* of T given by the unique bounded operator satisfying
$$(T^*h \,|\, k) = (h \,|\, Tk) \quad \text{for all } h, k \in H$$
(it is a fundamental lemma that for each T there is exactly one such operator T^*).

   When $H = \mathbb{C}^n$, every linear transformation T is bounded, and passing from T to its matrix with respect to the usual basis for $\mathbb{C}^n$ identifies $B(H)$ with $M_n(\mathbb{C})$. This identification carries composition into matrix multiplication and adjoints into conjugate transposes, so $M_n(\mathbb{C})$ is a C^*-algebra in the operator norm.

Notice that, according to results of Gelfand and Naimark: (a) If A is a commutative C^*-algebra with identity, then there is a compact Hausdorff space X such that A is isomorphic to $C(X)$; (b) Every C^*-algebra A is isomorphic to a closed $*$-subalgebra (or C^*-*subalgebra*) of $B(H)$.

The existence of an involution provides special sets of elements, which plays a central role in the understanding of the structure of C^*-algebras. If A denotes a C^*-algebra, an element $a \in A$ is:

1. Self-adjoint, if $a = a^*$.
2. Positive, if $a = xx^*$ for some $x \in A$.
3. Projection, if $a = a^* = a^2$.
4. Unitary, if $aa^* = a^*a = 1$.
5. Isometry, if $a^*a = 1$.
6. Partial isometry, if $a^*a = p$ projection, called the source projection, while aa^* is called the range projection.

Also, given any element a of a C^*-algebra A, we define the Spectrum of a

$$\mathrm{Spec}(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin A^{-1}\}.$$

The spectrum of an element is a compact Hausdorff space, and in fact the sub-C^*-algebra of A generated by a, denoted $C^*(a)$, is $*$-isomorphic (so isometric) to the commutative C^*-algebra $C(\mathrm{Spec(a)})$. This produces one of the most powerful tools in C^*-algebra Theory: the functional calculus. Functional calculus is specially useful when a is a positive element, since then $\mathrm{Spec(a)} \subset \mathbb{R}^+$, so that the element is represented as the real function $f(x) = x$, and any function on this "picture" (e.g. $f(x) = \sqrt{x}$) corresponds to an element in A. This fact provides lots of information about the algebra. You can see several results in this line in [43] or [55], among others. For example

Lemma 3.1.3 ([55, Lemma 5.1.6]) *Let A be a C^*-algebra, and let $a \in A$ be a positive element with $\|a\| \leq 1$. If $\|a - a^2\| < \varepsilon \leq 1/4$, then there exists a projection $p \in A$ such that $\|p - a\| < 2\varepsilon \leq 1/2$.*

Proof. For every $t \in C(\mathrm{Spec}(a))$, t is a real number such that $0 \leq t - t^2 < \varepsilon \leq 1/4$. Then, there is a gap in $\mathrm{Spec}(a)$: $t \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1]$, where $\delta := \frac{1}{2}\sqrt{1 - 4\varepsilon}$. Thus, the function

$$f(t) := \begin{cases} 0 & \text{if } t < 1/2 \\ 1 & \text{if } t > 1/2 \end{cases}$$

is continuous over $\mathrm{Spec}(a)$. Then, $p = f(a) \in A$, and it is a projection because $f = \overline{f} = f^2$. Moreover,

$$\sup\{|f(t) - t| \mid t \in \mathrm{Spec}(a)\} = \frac{1}{2} - \delta,$$

so that $\|p - a\| = \|f(a) - \mathrm{id}(a)\| = \frac{1}{2} - \delta < 2\varepsilon$ for $\varepsilon < 1/4$. □

As a consequence, any idempotent in A can be approached by a projection near enough to guarantee that they are Murray-von Neumann equivalent, so they represent the same element in K-theoretic sense (see e.g. [24]).

When we say that I is an ideal in a C^*-algebra, we mean that I is norm-closed and 2-sided. It then follows that I is also closed under the adjoint operation [43, Theorem 3.1.3], so the quotient A/I is a $*$-algebra.

Theorem 3.1.4 *If I is an ideal in a C^*-algebra A, then the quotient A/I is a C^*-algebra in the quotient norm*

$$\|a\|_I := \inf\{\|a + i\| : i \in I\}.$$

For proofs of this theorem, see [43, Theorem 3.1.4]. The proofs are not as routine as one might think: it takes considerable ingenuity and some substantial general theory to prove that the quotient norm satisfies the C^*-identity.

There are other results that will be important in some places in this course (even if we use them implicitly):

1. Suppose that A and B are C^*-algebras and $\phi : A \to B$ is a homomorphism. Then ϕ is norm-decreasing: $\|\phi(a)\| \leq \|a\|$ for every $a \in A$. If ϕ is injective, then ϕ is norm-preserving: $\|\phi(a)\| = \|a\|$ for every $a \in A$ [20, Theorem 1.1.6].

2. The range of every homomorphism $\phi : A \to B$ between C^*-algebras is a C^*-subalgebra of B [20, Corollary 1.1.9].

3. Suppose that A is a C^*-algebra and $\{A_n : n \in \mathbb{N}\}$ are C^*-subalgebras of A such that $A_n \subset A_{n+1}$ and $A = \overline{\bigcup_{n=1}^{\infty} A_n}$. If a homomorphism $\phi : A \to B$ is injective on each A_n, then ϕ is injective on A [20, Corollary 1.1.12].

3.2 Generators and relations

As seen in Chapter 1, a graph algebra (i.e. a Leavitt path algebra) is a K-algebra given by <u>generators</u> and <u>relations</u> codified by a direct graph.

So, in order to construct a graph C^*-algebra over a graph E, we first need to understand the notion of C^*-algebras given by generators and relations. Our source will be [42].

From the algebraic point of view, for a given set of generators $\mathcal{G}$ and relations $\mathcal{R}$ on the alphabet $\mathcal{G}$, the K-algebra with presentation

$$K\langle \mathcal{G} \mid \mathcal{R} \rangle$$

is the quotient of the free K-algebra $F_K(\mathcal{G})$ by the two-sided ideal $I(\mathcal{R})$ of $F_K(\mathcal{G})$ generated by the relations $\mathcal{R}$. This construction always has sense in the context, even if $I(\mathcal{R}) \cap \mathcal{G} \neq \emptyset$ (so that some generators collapses), or $I(\mathcal{R}) = F_K(\mathcal{G})$ (so that $K\langle \mathcal{G} \mid \mathcal{R} \rangle = 0!$), with no restrictions on the sets of generators and relations.

From the C^*-algebraic point of view, though, the key point [42, §1.1] is that free C^*-algebras over sets of generators **do not exist in the above sense**. The reason is that, if $\mathcal{G}$ is a countable set and H is a separable (i.e, with a countable basis) complex Hilbert space, it should exists an injective $*$-morphism

$$\mathcal{F}^*_{\mathbb{C}}(\mathcal{G}) \to B(H)$$

so that any relation of $\mathcal{G}$ will generate a closed two-sided ideal. Now, pick $\alpha \in \mathbb{R}^{++}$ with $\alpha > 2$. Then, if $\mathcal{G} = \{x\}$ and we consider the relation $xx^* - \alpha x^* x$, then for any sequence $\{a_n\}_{n\geq 1} \subset \ell_2(\mathbb{C})$ will produce an element $\sum\limits_{i\geq 1} a_n xx^*$ with the property

$$\sum_{i\geq 1}(2^n a_n)x^*x < \sum_{i\geq 1} a_n x^* x = \sum_{i\geq 1} a_n xx^*$$

which clearly is unbounded as operator on H!

So, we need to consider <u>bounded</u> sets of generators, and to define $\mathcal{F}^*_{\mathbb{C}}(\mathcal{G} \mid \mathcal{R})$ using a universal property: it is C^*-algebra A with <u>a representation</u>

$$i : \mathcal{G} \rightarrowtail A$$

such that for any representation $\langle b_g \rangle$ of $\mathcal{R}$ into a C^*-algebra B there exists a unique extension $*$-homomorphism from A to B sending $g \in \mathcal{G}$ to b_g.

The standard way is to fix a such presentation in $B(H)$. Thus, the general problem is:

1. To find such a map

$$\pi : \mathcal{F}^*_{\mathbb{C}}(\mathcal{G} \mid \mathcal{R}) \to B(H).$$

2. To check the injectivity of π.

This links the question about the existence of this universal algebra to the existence of a Uniqueness Theorem which fix when a morphism with source $\mathcal{F}^*_{\mathbb{C}}(\mathcal{G} \mid \mathcal{R})$ is injective. This is the reason of the interest of graph C^*-algebraists in getting these Uniqueness Theorems.

3.3 The C^*-algebra of a graph E

Because of the remarks in Section 3.2, it is clear that we cannot consider the existence of the graph C^*-algebra $C^*(E)$ associated to a graph E (as an algebra given by generators and relations) as an automatic fact. Since we want that $C^*(E)$ enjoys the universal property associated to the presentation encoded by the graph E (in the terms of Chapter 1), we are in fact looking for

$$C^*(E) = \mathcal{F}^*_{\mathbb{C}}(\mathcal{G} \mid \mathcal{R}),$$

where $\mathcal{G} = \{p_v \mid v \in E^0\} \cup \{s_e \mid e \in E^1\}$, while $\mathcal{R}$ is the set of Cuntz-Krieger relations:

1. $p_v = p_v^* = p_v^2$ for every $v \in E^0$;
2. $p_v p_w = \delta_{v,w} p_v$ for every $v, w \in E^0$;
3. $s_e^* s_e = p_{s(e)}$ for every $e \in E^1$ ($p_{s(e)}$ is the source projection of the partial isometry s_e);
4. $s_e s_e^* \leq p_{r(e)}$ for every $e \in E^1$;
5. $p_v = \sum_{r(e)=v} s_e s_e^*$ for every $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$.

Notice that $(3-5)$ are the reverse on $(CK1-2)$ in the definition of a Leavitt path algebra. In fact, if we consider the *transpose graph* E^t (same set of vertices and edges, but reverse sense for the edges), then $(CK1-2)$ of $L_{\mathbb{C}}(E)$ coincide with $(3-5)$ of $C^*(E^t)$. The reason of C^*-algebraists' choice relies in the final comment of (3).

To proceed with the construction of $C^*(E)$, we need to fix the map

$$\pi : \mathcal{F}^*_{\mathbb{C}}(\mathcal{G} \mid \mathcal{R}) \to B(H),$$

so that we first need a faithful representation of the Cuntz-Krieger E-family as bounded operators on a separable Hilbert space. For, let H be an (infinite-dimensional) separative complex Hilbert space, and let $\mathcal{H} = \oplus_{v \in E^0} H_v$, where

$H_v = H$ for every $v \in E^0$. Now, for each $v \in E^0$, let $P_v : \mathcal{H} \to H_v$ be the natural projection map. Also, for each $v \in E^0$ decompose H_v as a direct sum $H_v = \oplus_{\{e\in E^1 | r(e)=v\}} H_{v,e}$ of infinite-dimensional subspaces, and take S_e to be a unitary isomorphism $S_e : H_{s(e)} \to H_{r(e),e}$, viewed as a partial isometry on $\mathcal{H}$ with initial space $H_{s(e)}$. Clearly, the set $\mathcal{G} = \{P_v \mid v \in E^0\} \cup \{S_e \mid e \in E^1\}$ satisfies the above Cuntz-Krieger relations, and every element in $\mathcal{G}$ is a nonzero element of $B(\mathcal{H})$. So, the $*$-subalgebra of $B(\mathcal{H})$ generated by $\{P_v \mid v \in E^0\} \cup \{S_e \mid e \in E^1\}$ give us a faithful representation of a C^*-algebra whose presentation corresponds to a Cuntz-Krieger E-family $\{S, P\}$. We will denote such algebras by $C^*(S, P)$.

Now, we are looking amongst the C^*-algebras $C^*(S, P)$ the one satisfying the desired universal property (if such an algebra exists!). We start by showing the existence of such an object, and later we proceed to show the uniqueness of it.

The key results for existence are

Proposition 3.3.1 [47, Proposition 1.12] *Let E be a row-finite graph, and let $C^*(S, P)$. Then:*

1. *The projections $\{S_e S_e^* \mid e \in E^1\}$ are mutually orthogonal.*
2. $S_e^* S_f \neq 0 \Rightarrow e = f$.
3. $S_e S_f \neq 0 \Rightarrow s(e) = r(f)$.
4. $S_e S_f^* \neq 0 \Rightarrow s(e) = s(f)$.

Corollary 3.3.2 [47, Corollary 1.14] *Let E be a row-finite graph, and let $\{S, P\}$ be a Cuntz-Krieger E-family in a C^*-algebra B. Let $\mu, \nu \in E^*$. Then:*

1. *If $|\mu| = |\nu|$ and $\mu \neq \nu$, then $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$.*
2. $S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^* \\ S_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise.} \end{cases}$
3. *If $S_\mu S_\nu \neq 0$, then $\mu\nu$ is a path in E and $S_\mu S_\nu = S_{\mu\nu}$.*
4. *If $S_\mu S_\nu^* \neq 0$, then $s(\mu) = s(\nu)$.*

Corollary 3.3.3 [47, Corollary 1.15] *Let E be a row-finite graph, and let $\{S, P\}$ be a Cuntz-Krieger E-family in a C^*-algebra B. For $\mu, \nu, \alpha, \beta \in E^*$, we have*

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'} S_\beta^* & \text{if } \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

It particular, it follows that every nonzero finite product of the partial isometries S_e *and* S_f^* *has the form* $S_\mu S_\nu^*$ *for some* $\mu, \nu \in E^*$ *with* $s(\mu) = s(\nu)$.

As a consequence we get

Corollary 3.3.4 [47, Corollary 1.16] *If* E *is a row-finite graph, and* $\{S, P\}$ *is a Cuntz-Krieger* E*-family, then*

$$C^*(S, P) = \overline{span}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}.$$

In terms of the Leavitt path algebra $L_{\mathbb{C}}(E)$, Corollary 3.3.4 means that $C^*(S, P) = \overline{\varphi_{S,P}(L_{\mathbb{C}}(E))}^{\|\cdot\|}$, where

$$\begin{array}{rccc} \varphi_{S,P}: & L_{\mathbb{C}}(E)) & \longrightarrow & C^*(S, P) \\ & v & \mapsto & P_v \\ & e & \mapsto & S_e \end{array} \tag{3.3}$$

is the map induced by the universal property of $L_{\mathbb{C}}(E)$. In fact, the keystone to guarantee the existence of $C^*(E)$ relies in the above fact, expressed in a formal way as follows: consider the formal symbols $d_{\mu,\nu}$ giving a basis of a complex vector space V, and complex coefficients $z_{\mu,\nu}$ so that only finitely many of them in each sum are nonzero. Then we have the following result.

Proposition 3.3.5 [47, Proposition 1.20] *Let* E *be a row-finite graph. Then the vector space* V *of formal linear combinations*

$$V = \left\{ \sum_{\mu,\nu} z_{\mu,\nu} d_{\mu,\nu} \mid \mu, \nu \in E^*, s(\mu) = s(\nu) \right\}$$

is a $*$*-algebra with* $d_{\mu,\nu}* = d_{\nu,\mu}$ *and*

$$(d_\mu d_\nu^*)(d_\alpha d_\beta^*) = \begin{cases} d_{\mu\alpha'} d_\beta^* & \text{if } \alpha = \nu\alpha' \\ d_\mu d_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

The reason is that Proposition 3.3.5 encodes the universal property of $L_{\mathbb{C}}(E)$, so that for a given Cuntz-Krieger E-family $\{S, P\}$ we have

$$\varphi_{S,P} : L_{\mathbb{C}}(E) \to C^*(S, P).$$

Since the C^*-identity imply that projections and partial isometries have norm 1, we get

$$\|\varphi_{S,P}(\sum \alpha_{\mu,\nu} \mu\nu^*)\| \leq \sum \|\alpha_{\mu,\nu}\| \cdot \|S_\mu S_\nu^*\| \leq \sum |\alpha_{\mu,\nu}|,$$

so that

$$\|a\|_1 = \sup\{\|\varphi_{S,P}(a)\| \mid \{S,P\} \text{ Cuntz-Krieger } E-\text{family}\}$$

is finite for every $a \in L_{\mathbb{C}}(E)$. Hence, $\|\cdot\|_1$ is an $*$-algebra seminorm satisfying the C^*-identity. By [53, Theorem 7.3] (we will come back to this result with more detail in next chapter), $\|\cdot\|_1$ is in fact a norm. Thus,

$$C^*(E) := \overline{L_{\mathbb{C}}(E)}^{\|\cdot\|_1}$$

turns out to be a C^*-algebra of $C^*(S,P)$ type related to the graph E. Moreover, since for any Cuntz-Krieger E-family $\{S,P\}$ the map $\varphi_{S,P}$ is $\|\cdot\|_1$-continuous, the map 3.3 extends uniquely to a $*$-morphism

$$\begin{array}{rccc} \Phi_{S,P}: & C^*(E) & \longrightarrow & C^*(S,P) \\ & p_v & \mapsto & P_v \\ & s_e & \mapsto & S_e \end{array} \tag{3.5}$$

whence we conclude

Theorem 3.3.6 [47, Proposition 1.21 & Corollary 1.22] *Let E be a row-finite graph. Then:*

1. *$C^*(E)$ is a C^*-algebra generated by a Cuntz-Krieger E-family $\{s,p\}$ such that for any Cuntz-Krieger E-family $\{T,Q\}$ in a C^*-algebra B, there is an injective homomorphism $\pi_{T,Q} : C^*(E) \to B$ satisfying $\pi_{T,Q}(s_e) = T_e$ for every $e \in E^1$ and $\pi_{T,Q}(p_v) = Q_v$ for every $v \in E^0$.*

2. *Suppose that C is a C^*-algebra generated by a Cuntz-Krieger E family $\{w,r\}$ such that for any Cuntz-Krieger E-family $\{T,Q\}$ in a C^*-algebra B, there is an injective homomorphism $\rho_{T,Q} : C \to B$ satisfying $\rho_{T,Q}(w_e) = T_e$ for every $e \in E^1$ and $\rho_{T,Q}(r_v) = Q_v$ for every $v \in E^0$. Then there is an isomorphism $\phi : C^*(E) \to C$ such that $\phi(s_e) = w_e$ for every $e \in E^1$ and $\phi(p_v) = r_v$ for every $v \in E^0$.*

The first part of Theorem 3.3.6 guarantees the existence of the graph C^*-algebra $C^*(E)$, and the second part the uniqueness (because is the universal property of the algebra). In fact, we can relax the hypotheses of the second part, getting the injectivity of the map from milder assumptions which are easier to be checked. The basis of that relies of the existence of a continuous action of the circle $\mathbb{T}$ on $C^*(E)$; is the standard Gauge action.

Proposition 3.3.7 [47, Proposition 2.1] *Let E be a row-finite graph. Then there is an action γ of $\mathbb{T}$ on $C^*(E)$ such that, for each $z \in \mathbb{T}$, $\gamma_z(s_e) = zs_e$ for every $e \in E^1$ and $\gamma_z(p_v) = p_v$ for every $v \in E^0$.*

By using this Gauge action, we are able to proof the celebrated Uniqueness Theorems for graph C^*-algebras, which turns out to be the most powerful tools in the development of this theory.

Theorem 3.3.8 [47, Theorem 2.2] *Let E be a row-finite graph, and let $\{T, Q\}$ be a Cuntz-Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for every $v \in E^0$. If there is a continuous action $\beta : \mathbb{T} \to Aut(B)$ such that $\beta_z(T_e) = zT_e$ for every $e \in E^1$ and $\beta_z(Q_v) = Q_v$ for every $v \in E^0$, then the homomorphism $\pi_{T,Q} : C^*(E) \to B$ is an isomorphism onto $C^*(T, Q)$.*

Theorem 3.3.9 [47, Theorem 2.4] *Let E be a row-finite graph satisfying Condition (L)[1], and let $\{T, Q\}$ be a Cuntz-Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for every $v \in E^0$. Then the homomorphism $\pi_{T,Q} : C^*(E) \to B$ is an isomorphism onto $C^*(T, Q)$.*

and as a consequence we get

Theorem 3.3.10 [47, Corollary 2.5] *Let E be a row-finite graph satisfying Condition (L), and let $\{S, P\}$ and $\{T, Q\}$ be two Cuntz-Krieger E-families on a separable Hilbert space H such that $P_v \neq 0$ and $Q_v \neq 0$ for every $v \in E^0$. Then there exists an isomorphism $\phi : C^*(S, P) \to C^*(T, Q)$ such that $\phi(S_e) = T_e$ for every $e \in E^1$ and $\phi(P_v) = Q_v$ for every $v \in E^0$.*

So, under Condition (L), any two Cuntz-Krieger E-families generate isomorphic C^*-algebras, and isomorphic to $C^*(E)$, whence in this case we need not to care about the choice of the representation.

3.4 First applications of G.I.U.T.

Let us to expend a few time in looking to other interesting and useful applications of these uniqueness theorems.

Corollary 3.4.1 [47, Corollary 2.6] *Suppose that E is a row-finite graph with no sources, and define the dual graph $\widehat{E}$ by $\widehat{E}^0 = E^1$, $\widehat{E}^1 = E^2$, $r_{\widehat{E}}(ef) = e$ and $s_{\widehat{E}}(ef) = f$. Then, $\widehat{E}$ is row-finite and $C^*(\widehat{E}) \cong C^*(E)$.*

There are general versions of this results in the C^*-algebra context [22, Theorem 3.2] and in the Leavitt path algebra context [1, Theorem 2.8].

Also, we can use these theorems to erase sources and sink from the graph (which existence is critical for some results), without lose of the "stable"

[1] Notice that, in this contex, Condition (L) means that every cycles has an entry.

properties of the algebra. For, we add a head to each source and a tail to each sink (i.e., and infinite line subgraph starting in each sink/ending in each source). The underlying idea is to state injective morphisms between non-unital graph algebras, which is strongly useful to state Morita equivalences between algebras.

Corollary 3.4.2 [47, Corollary 2.11] *Suppose that E is a row-finite graph, and F is the graph obtained by adding a head to every source of E and a tail to every sink of E. Denote by $\{s,p\}$ and $\{t,q\}$ the canonical Cuntz-Krieger families generating $C^*(E)$ and $C^*(F)$ respectively, and let $q_E \in \mathcal{M}(C^*(F))$ be the projection*

$$q_E = \sum_{v \in E^0} p_v.$$

Then, $q_E C^*(F) q_E$ is a full corner in $C^*(F)$, and there is an isomorphism

$$\phi : C^*(E) \to q_E C^*(F) q_E$$

such that $\phi(s_e) = t_e$ for every $e \in E^1$ and $\phi(p_v) = q_v$ for every $v \in E^0$.

Also, by using the link between AF-algebras and directed graphs (the Bratteli diagrams) (see e.g. [30]), is it possible to show that any AF-algebra is strongly Morita equivalent to a graph C^*-algebra, as follows.

Proposition 3.4.3 [47, Proposition 2.12] *Let A be a unital AF-algebra, and let $(E, \{V_n\}, v_0)$ be a Bratteli diagram for A. Then, $A \cong p_{v_0} C^*(E) p_{v_0}$, and $p_{v_0} C^*(E) p_{v_0}$ is a full corner of $C^*(E)$.*

Other examples of application can be found in [47, Chapter 2].

3.5 Gauge invariant ideals

A fundamental consequence of the uniqueness theorems is the description of the gauge-invariant ideals of a graph C^*-algebra. This will allows, with a slight extra effort, to characterize simple/purely infinite simple graph C^*-algebras (we delay this discussion to the next chapter).

We will describe such ideals in a "constructive" way. The starting point is try to decide which kind of closed two-sided ideals in $C^*(E)$ give quotients that turns out to be graph C^*-algebras too.

Let $I \lhd C^*(E)$ be a closed two-sided ideal, and consider the subset of E^0 given by the rule $H_I = \{v \in E^0 \mid p_v \in I\}$. Now consider the quotient map $q_I : C^*(E) \to C^*(E)/I$. Notice that $q_I(p_v) \neq 0$ for every $v \notin H_I$, and moreover, if $s(e) \notin H_I$, then $q_I(p_{s(e)}) \neq 0$, whence $0 \neq q_I(s_e)q_I(s_e)^* \leq q_I(p_{r(e)})$, and thus $r(e) \notin H_I$ too. So,

$$E \setminus H_I = (E^0 \setminus H_I, s^{-1}(E^0 \setminus H_I), s, r)$$

is a graph, and $\{q_I(s_e), q_I(p_v) \mid s(e), v \notin H_I\}$ is a Cuntz-Krieger $E \setminus H_I$-family. Hence, if E satisfies Condition (L), Theorem 3.3.10 implies that

$$C^*(E \setminus H_I) \cong C^*(E)/I.$$

Thus, in order to understand two-sided ideals, we need to boil up the abstract characterization of the sets of the form H_I.

Definition 3.5.1 We define a relation $\geq$ on E^0 by setting $w \geq v$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$.[2] A subset H of E^0 is called hereditary if $v \leq w$ and $v \in H$ imply $w \in H$. A hereditary set is saturated if every vertex which feeds into H and only into H is again in H, that is, if $r^{-1}(v) \neq \emptyset$ and $s(r^{-1}(v)) \subseteq H$ imply $v \in H$.

Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0. Then we have the following result

Lemma 3.5.2 [47, Lemma 4.5] *Suppose that I a nonzero ideal in $C^*(E)$. Then $H_I \in \mathcal{H}_E$.*

We need to recall a definition

Definition 3.5.3 A graph E satisfies Condition (K) if for each vertex v on a closed simple path there exists at least two distinct closed simple paths α, β based at v.

Lemma 3.5.4 [47, Lemma 4.7] *A graph E satisfies Condition (K) if and only if for every $H \in \mathcal{H}_E$ the graph $E \setminus H$ satisfies Condition (L).*

Thus, we get the main result in this section. First we need a definition

Definition 3.5.5 Let E be a row-finite graph, and let $H \in \mathcal{H}_E$. We define a graph $E_H = (H, r^{-1}(H), s, r)$. We will denote by p_H the projection $\sum_{v \in H} p_v \in \mathcal{M}(C^*(E))$.

[2] Recall that C^*-algebraists read the information of the graph reversing the sense of the arrows.

Theorem 3.5.6 [47, Theorem 4.9] *Let E be a row-finite graph satisfying Condition (K). Then:*

1. *For each $H \in \mathcal{H}_E$, let I_H be the ideal generated by $\{p_v \mid v \in H\}$. Then $H \mapsto I_H$ is a bijection between $\mathcal{H}_E$ and the set $\mathcal{L}(C^*(E))$ of two-sided ideals of $C^*(E)$, with inverse given by $I \mapsto H_I$.*

2. *For every $H \in \mathcal{H}_E$, $C^*(E \setminus H) \cong C^*(E)/I_H$ and $C^*(E_H)$ is isomorphic to the full corner $p_H I_H p_H$.*

Moreover, the map defined in Theorem 3.5.6(1) is a lattice isomorphism (see e.g. [20, Theorem 2.1.6]).

Remark 3.5.7 If E do not satisfy Condition (K), then Theorem 3.5.6(1) works by replacing $\mathcal{L}(C^*(E))$ by the lattice $\mathcal{L}_\gamma(C^*(E))$ of gauge-invariant ideals of $C^*(E)$, simply using Theorem **??** instead of Theorem 3.3.9.

Let us sketch the proof of Theorem 3.5.6, because it illustrates the usage of the uniqueness theorem in this context.

We are assuming that E satisfies Condition (K), and that $I \lhd C^*(E)$, $H := H_I \in \mathcal{H}_E$. Let us see that $I = I_H$. Trivially, $I_H \subseteq I$. To be the reverse inclusion, Consider the maps

$$q_I : C^*(E) \to C^*(E)/I,$$

$$q_{I_H} : C^*(E) \to C^*(E)/I_H,$$

and

$$q_{I/I_H} : C^*(E)/I_H \to C^*(E)/I.$$

Note that $q_I = q_{I/I_H} \circ q_{I_H}$. Since $q_I(p_v) = 0$ if and only if $q_{I_H}(p_v) = 0$, and hence $q_I(s_e) = 0$ if and only if $q_{I_H}(s_e) = 0$, both $\{q_I(s_e), q_I(p_v)\}$ and $\{q_{I_H}(s_e), q_{I_H}(p_v)\}$ are Cuntz-Krieger $(E \setminus H)$-families generating the respective quotients. Let

$$\pi : C^*(E \setminus H) \to C^*(E)/I_H$$

and

$$\rho : C^*(E \setminus H) \to C^*(E)/I$$

be the corresponding universal morphisms. As ρ and $q_{I/I_H} \circ \pi$ agree on the generators of $C^*(E \setminus H)$, we conclude that $\rho = q_{I/I_H} \circ \pi$. So, ρ is injective by Lemma 3.5.4 and Theorem 3.3.10. Since π is onto, we conclude that q_{I/I_H} is injective. Thus, $I = I_H$, as claimed.

Since $I = I_{H_I}$, the map $H \mapsto I_H$ is onto. To see that it is injective, we shall prove that if $H \in \mathcal{H}_E$, then $H = \{v \in E^0 \mid p_v \in I_H\}$. Trivially,

$H \subseteq \{v \in E^0 \mid p_v \in I_H\}$. To see the reverse inclusion, let $\{t, q\}$ be the Cuntz-Krieger $(E \setminus H)$-family generating $C^*(E \setminus H)$. Notice that defining $t_e = 0$ for every $s(e) \in H$ and $q_v = 0$ for every $v \in H$, we have in fact that $\{t, q\}$ is a Cuntz-Krieger E-family. Then, by Theorem 3.3.9 there exists a morphism

$$\pi_{t,q} : C^*(E) \to C^*(E \setminus H)$$

such that $\pi_{t,q}(q_v) = 0$ for every $v \in H$. Thus, $I_H \subseteq \mathrm{Ker}\pi_{t,q}$. Then, for any $v \notin H$, as $q_v \neq 0$, we have that $\pi_{t,q}(p_v) \neq 0$, whence $p_v \notin \mathrm{Ker}\pi_{t,q}$, and thus $p_v \notin I_H$. Hence, $H = \{v \in E^0 \mid p_v \in I_H\}$, as desired.

Chapter 4

Links between $C^*(E)$ and $L_{\mathbb{C}}(E)$.

4.1 Leavitt path algebras versus graph C^*-algebras

Leavitt algebras [41] appeared in the 1960's as a universal model for non-IBN algebras. Independently Cuntz [27], in 1977, introduced an analytic version in order to model some pathologies for K-Theory of C^*-algebras. These algebras (the so-called Cuntz algebras) were generalized in [29], in connection with the classification of subshifts of finite type. Enomoto and Watatani [33] introduced the notion of C^*-algebra of a finite graph as a way of giving a presentation of Cuntz-Krieger algebras, encoding such a presentation in a finite direct graph. Fifteen years later, Kumjian, Pask and Raeburn [40] extended Enomoto & Watatani's construction to infinite row-finite graphs (extending a previous model of Cuntz-Krieger algebras given in terms of groupoids [39]). This model and its generalizations constitutes now the kernel of the study of graph C^*algebras.

From the algebraic point of view, the first generalization can be founded in [12], where a construction (fractional skew monoid rings) generalizing skew group rings to semigroup rings is introduced. This construction is a generalization of the crossed product of a C^*-algebra by an endomorphism introduced by Paschke [45], lately used by Rørdam [49] to model Cuntz-Krieger algebras; the same model holds for the algebraic construction in [12]. Around 2003, Abrams and Aranda [2], and Ara, Moreno and Pardo (independently) [15] introduced the notion of Leavitt path algebra on a row-finite graph as a generalization of both Leavitt algebras and Cuntz-Krieger graph C^*-algebras.

So, in the study of the structure of Leavitt path algebras, the properties enjoyed by graph C^*-algebras has been an inspiration source. Nevertheless, the graph C^*-algebra $C^*(E)$ is an analytic object, and even if the results obtained for $L_K(E)$ turns out to be surprisingly analog (except in two cases we will analyze later in this chapter), there is no known way of transfer directly results from the graph C^*-algebra context to the Leavitt path algebra context, and conversely; in fact, a major question for bridging the gap between both classes of algebras is to find a correct framework in which we can state and proof results for these classes of algebras with no distinctions.

From the C^*-algebraists side, one of the most surprising facts about Leavitt path algebras theory is that an algebraic analog of the Gauge Invariant Uniqueness Theorem (Theorem 3.3.9) do not play a central role (in fact, no role) in the study of Leavitt path algebras. As seen in Chapter 4, in the case of graph C^*-algebras, this theorem is essential to guarantee the existence & uniqueness of the algebra, and also the structure of the lattice of ideals. But as seen in the Chapter 1, in the case of Leavitt path algebras, its existence & uniqueness relies in properties emanating of Universal Algebra, while the structure of (graded) ideals can be determined using the nonstable K-Theory of $L_K(E)$, as shown in [15, Theorem 5.3], and then the algebraic versions of Theorems 3.3.9 & 3.3.10 (over graphs satisfying Condition (L)) are consequence of [15, Theorem 5.3] and [2, Corollary 3.3]. In fact, to determine the injectivity of Leavitt path algebra maps do not play any role in the theory before the work of classification of purely infinite simple Leavitt path algebras (see [14] for general definition, [3] for characterization of this property in Leavitt path algebras, and [1] for a first approach to the classification problem).

Because of the $\mathbb{Z}$-graded structure of $L_K(E)$ (for any field K) and [15, Theorem 5.3], "it is clear" that the natural map

$$\varphi : L_{\mathbb{C}}(E) \to C^*(E)$$

is an injective $*$-algebra map for any row-finite graph E. But is necessary to wait to Tomforde's work [53] to find a formal, general proof of this fact; the interest of Tomforde's result relies in the fact that his proof follows the C^*-algebra strategy, thus offering a first analog of G.I.U. Theorem for Leavitt path algebra (a version has been studied in Chapter 2). We will start our comparison of both kind of algebras in this point.

4.2 Tomforde's Graded Invariant Uniqueness Theorem for LPAs

The source of the section is [53]. In this paper Tomforde proves an algebraic version of the G.I.U. Theorem for Leavitt path algebras (over arbitrary graphs)

Theorem 4.2.1 [53, Theorem 4.8] *Let E by a graph, let K be a field, and let A be any $\mathbb{Z}$-graded algebra. If $\pi : L_K(E) \to A$ is a $\mathbb{Z}$-graded ring morphism such that $\pi(v) \neq 0$ for every $v \in E^0$, then π is injective.*

As a consequence, Tomforde prove a version of [15, Theorem 5.3] for arbitrary graphs [53, Theorem 5.7], linking graded ideals of $L_K(E)$ with hereditary and saturated subsets of E^0. Thus, he deduce a Cuntz-Krieger Uniqueness Theorem (again on arbitrary graphs), as follows

Theorem 4.2.2 [53, Theorem 6.8] *Let E by a graph that satisfies Condition (L), let K be a field. If $\pi : L_K(E) \to A$ is a ring morphism such that $\pi(v) \neq 0$ for every $v \in E^0$, then π is injective.*

In particular he deduces that E satisfies Condition (K) if and only if every ideal in $L_K(E)$ is graded [53, Theorem 6.16] (the row-finite version was proved by Aranda, Siles and Pardo [19, Theorem 4.5]).

So, Tomforde's paper states in the case of arbitrary graphs that there exists a coupling between graded ideals in Leavitt path algebras and gauge-invariant ideals in graph C^*-algebras, which determine the form of the algebraic versions of G.I.U.T. and C-K.U.T. (we will study that relation in a deep way a little later). One interesting application [53, Theorem 7.3] is the prove of the injectivity of the map

$$\varphi : L_{\mathbb{C}}(E) \to C^*(E).$$

Notice that the principal problem to apply [15, Theorem 5.3] and [2, Corollary 3.3] for doing that is that $C^*(E)$ is not a $\mathbb{Z}$-graded algebra. This problem was avoided by noticing that $\mathcal{A} = \varphi(L_{\mathbb{C}}(E))$ (which is a $\mathbb{Z}$-graded algebra) is a dense $*$-**subalgebra** of $C^*(E)$. Tomforde's argument ignores this assumption, and in fact allows to prove this fact as a consequence of Theorem 4.2.1. The strategy is to consider the standard gauge action γ of $\mathbb{T}$ on $C^*(E)$, and to define, for each $n \in \mathbb{Z}$,

$$\mathcal{A}_n = \left\{ a \in \mathcal{A} \mid \int_{\mathbb{T}} z^{-n} \gamma_z(a) dz = a \right\}$$

where the integral is defined with respect to the normalized Haar measure on $\mathbb{T}$. Thus, given $\lambda s_\alpha s_\beta^* \in \mathcal{A}$, we get

$$\int_{\mathbb{T}} z^{-n} \gamma_z(\lambda s_\alpha s_\beta^*) = \begin{cases} \lambda s_\alpha s_\beta^* & \text{if } |\alpha| - |\beta| = n \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\sum_{k=1}^{N} \lambda_k s_{\alpha,k} s^*_{\beta,k} \in \mathcal{A}$ if and only if $|(\alpha,k)| - |(\beta,k)| = n$ for every $1 \leq k \leq N$. Thus, $\mathcal{A} = \oplus_{n\in\mathbb{Z}} \mathcal{A}_n$ is a $\mathbb{Z}$ grading for $\mathcal{A}$. Now, since $\varphi(v) = p_v$ for every $v \in E^0$, while $\varphi(e) = s_e$ and $\varphi(e^*) = s_e^*$ for every $e \in E^1$, we conclude that φ is $\mathbb{Z}$-graded, so that Theorem 4.2.1 apply, and thus φ is injective. And then $\mathcal{A}$ is a dense $*$-subalgebra of $C^*(E)$ for free! [53, Corollary 7.5].

Also, Tomforde state a direct link between graded ideals of $L_{\mathbb{C}}(E)$ and gauge-invariant ideals of $C^*(E)$. To be concrete, under the identification of $L_{\mathbb{C}}(E)$ with $\mathcal{A} \subset \overline{\mathcal{A}}^{\|\cdot\|_0} = C^*(E)$, where for any $a \in L_{\mathbb{C}}(E)$

$$\|a\|_0 = \sup\{\|\pi(a)\| \mid \pi : L_{\mathbb{C}}(E) \to B(H) \text{ nondegenerate representation}\}$$

[53, Corollary 7.6], the map $I \mapsto \overline{I}$ defines a lattice isomorphism from graded ideals of $L_{\mathbb{C}}(E)$ to gauge-invariant ideals of $C^*(E)$, with inverse map $J \mapsto J \cap L_{\mathbb{C}}(E)$. This isomorphism identifies ideals of $L_{\mathbb{C}}(E)$ with closed ideals of $C^*(E)$ whenever E enjoys Condition (K). And certainly fails for ungraded ideals (see [53, Remark 7.8]).

For applications about injectivity of maps on Leavitt path algebras, it is also relevant a generalization of the standard action of $\mathbb{T}$ on $C^*(E)$, introduced in [1, Section 1].

Definitions 4.2.3 [1, Definitions 1.3] Let K be a field, and let A be a $\mathbb{Z}$-graded algebra over K. For $t \in K^* = K \setminus \{0\}$ and a any homogeneous element of A of degree d, set

$$\tau_t(a) = t^d a,$$

and extend τ_t to all of A by linearity. It is easy to show that τ_t is a K-algebra automorphism of A for each $t \in K^*$. Then $\tau : K^* \to \mathrm{Aut}_K(A)$ is an action of K on A, which we call the *gauge action* of K on A.

If I is an ideal of A, we say that I is *gauge-invariant* in case $\tau_t(I) = I$ for each $t \in K^*$. This condition is equivalent to requiring that $\tau_t(I) \subseteq I$ for every $t \in K^*$, since $\tau_{t^{-1}}(I) \subseteq I$ gives $I \subseteq \tau_t(I)$.

The next result establishes a relationship between graded and gauge-invariant ideals of any $\mathbb{Z}$-graded algebra.

Proposition 4.2.4 [1, Proposition 1.4] *Let K be a field, let A be a $\mathbb{Z}$-graded K-algebra, and let I be an ideal of A. Let $\tau : K^* \to \mathrm{Aut}_K(A)$ be the gauge action of K on A.*

1. *If I is generated as an ideal of A by elements of degree 0, then I is gauge-invariant.*

2. *If K is infinite, and if I is gauge-invariant, then I is graded.*

We now apply this result in the context of Leavitt path algebras. For clarity, we present here the definition of the gauge action of K on the Leavitt path algebra $L_K(E)$ of the row-finite graph E.

Definition 4.2.5 [1, Definition 1.5] Let E be a row-finite graph, and let K be a field. Then the *gauge action τ of K on* the Leavitt path algebra $L_K(E)$ (denoted sometimes by τ^E for clarity) is given by

$$\begin{array}{cccc} \tau^E: & K^* & \to & \mathrm{Aut}_K(L_K(E)) \\ & t & \mapsto & \tau_t^E \end{array}$$

as follows: for every $t \in K^*$, for every $v \in E^0$, and for every $e \in E^1$

$$\begin{array}{cccc} \tau_t^E: & L_K(E) & \to & L_K(E) \\ & v & \mapsto & v \\ & e & \mapsto & te \\ & e^* & \mapsto & t^{-1}e^* \end{array}$$

and then extend linearly and multiplicatively to all of $L_K(E)$.

For a graph E, the set of graded ideals of $A = L_K(E)$ is denoted by $\mathcal{L}_{\mathrm{gr}}$.

Proposition 4.2.6 [1, Proposition 1.6] *Let E be a row-finite graph, let K be an infinite field, and let I be an ideal of $L_K(E)$. Then $I \in \mathcal{L}_{gr}$ if and only if I is gauge-invariant.*

We note that the implication "$I \in \mathcal{L}_{\mathrm{gr}}$ implies I is gauge-invariant" holds for any field K, finite or infinite, because in this case I is generated as ideal by the set $I \cap E^0$. In contrast, we now show that the converse implication of Proposition 4.2.6 is never true for any finite field.

Proposition 4.2.7 [1, Proposition 1.7] *For any finite field K there exists a graph E such that the Leavitt path algebra $L_K(E)$ contains a non-graded ideal which is gauge-invariant.*

Proof. If we denote card(K) by $m+1$, then $t^m = 1$ for all $t \in K^*$. Let E be the graph

$$\begin{tikzcd} \bullet^{v} \arrow[loop right, "x"] \end{tikzcd}$$

so that, as noted previously, $L_K(E) \cong K[x, x^{-1}]$. In particular we have $\tau_t(1+x^m) = 1+x^m$ for all $t \in K^*$. This then yields that the ideal $I =< 1+x^m >$ of $L_K(E)$ is gauge-invariant. But it is well known (or it can be shown using an argument similar to that given in the proof of [2, Theorem 3.11]) that I is not a graded ideal of $K[x, x^{-1}]$. □

We are now in position to present the main application of these ideas.

Theorem 4.2.8 [1, Proposition 1.7] *(The Algebraic Gauge-Invariant Uniqueness Theorem.) Let E be a row-finite graph, let K be an infinite field, and let A be a K-algebra. Suppose*

$$\phi : L_K(E) \to A$$

is a K-algebra homomorphism such that $\phi(v) \neq 0$ for every $v \in E^0$. If there exists a group action $\sigma : K^* \to Aut_K(A)$ such that $\phi \circ \tau_t^E = \sigma_t \circ \phi$ for every $t \in K^*$, then ϕ is injective.

Proof. Let $I = \mathrm{Ker}(\phi)$. Then for every $a \in I$ and for every $t \in K^*$, $\phi(\tau_t^E(a)) = \sigma_t(\phi(a)) = \sigma_t(0) = 0$, whence $\tau_t^E(a) \in \mathrm{Ker}(\phi) = I$. Thus for every $t \in K^*$ we have $\tau_t^E(I) \subseteq I$, so that I is gauge-invariant. Hence $I \in \mathcal{L}_{\mathrm{gr}}$ by Proposition 4.2.6. In particular, if $I \neq \{0\}$, then $I \cap E^0 \neq \emptyset$ by [15, Proposition 5.2 and Theorem 5.3], contradicting the hypothesis that $\phi(v) \neq 0$ for every $v \in E^0$. □

When $K = \mathbb{C}$, this result apply for the exchange of K^* by $\mathbb{T}$, so that Theorem 4.2.8 is an algebraic precursor of Theorem 3.3.9. As final remark, we notice that an approach is a similar direction, due to Raeburn, appears in [20, Section 1.3].

4.3 Comparison of properties

In this section we will revisit some properties of both graph C^*-algebras and Leavitt path algebras, characterized in terms of intrinsic properties of the graphs, in order to state similarities and differences between both classes of algebras.

One of the main aspects in the comparison is the independence of results in both contexts. The reason is that to perform completions in any norm uses to "broke" properties in a dramatic way. For example, pick $\alpha = e^{i\Theta}$ for an irrational number $\Theta \in [0, 2\pi)$, and define the complex McConnel-Petit algebra

$$R_\Theta = \mathbb{C}\langle x, yx^{-1}, y^{-1} \mid xy = \alpha yx \rangle.$$

Then, R_Θ has a natural involution, and moreover, a faithful $*$-representation into $B(H)$ for a suitable countable complex Hilbert space H, so that it admits a C^*-completion A_Θ, called the irrational rotation algebra. Both are simple algebras, but R_Θ is a (Dedekind) domain, $sr(R_\Theta) = 2$ and $K_0(R_\Theta) = \mathbb{Z}$, while A_Θ is not longer a domain, $sr(A_\Theta) = 1$ and $K_0(A_\Theta) =$ is a dense subgroup of the reals! Mostly of the arguments used in the Leavitt path algebra/graph C^*-algebra context are bad-behaved in this sense.

4.3.1 Simple & Purely infinite simple algebras

In any family of rings or algebras, simplicity (i.e. the absence of proper two-sided ideals) plays a major role amongst the properties, and it is directly related to the existence of faithful representations. In the particular case of graph C^*-algebras and Leavitt path algebras, simplicity encodes in fact faithfulness of the representation of an universal object, and thus it has a “universal” character.

In the case of graph C^*-algebras, the first result of characterization of simplicity was given by Cuntz and Krieger [29] for Cuntz-Krieger algebras. In the more general setting several authors did the job for various constructions associated to (row-finite, or even arbitrary) graphs.

The singular point is that simplicity is reflected in intrinsic properties of the graph. In fact, in both classes, simplicity of $C^*(E)$ (or that of $L_K(E)$) depend only on two facts:

1. The algebra is gauge-simple (respectively graded-simple).

2. Every nonzero ideal contains a nonzero gauge- (graded-) ideal.

Because of the results in Chapters 2 and 3, gauge-simplicity (graded-simplicity) means that E^0 contains no proper hereditary and saturated subsets; this condition can be expressed (this is the classical way of) by saying that E is a cofinal graph, that is, for any vertex $v \in E^0$ and any infinite path α there exists a finite path μ such that $s(\mu) = v$, while $r(\mu)$ is a vertex in the path α [19, Lemma 2.8]. Part 2 in the above list depends only in the fact that E satisfies Condition (L) [2, Corollary 3.3]. For the case of Leavitt path algebras this result was first proved, in a direct way, by Abrams and Aranda [2, Theorem 3.11]; for the case of graph C^*-algebras the result is analog, and it was obtained by different authors, see e.g. [47, Chapter 4]

Notice that, if E is a graph such that $L_K(E)$ is simple, then either has a sink (and then E is finite, it has only one sink, it contain no cycles, and every vertex connect with the sink), or it contain no sinks, so that either is acyclic (it contains no cycles) and then is infinite, or it contains at least one cycle, so that every vertex lies at least in two different closed simple paths. In the first two cases, $L_K(E)$ turns out to be isomorphic to $M_n(K)$ for some $n \in \mathbb{N}$ (case with a sink) or locally matricial (acyclic case with no sinks); and analogously $C^*(E) \cong M_n(\mathbb{C})$ or is a AF-algebra.

The interesting question is to decide which kind of algebra is obtained in the third case. We need to define an special class of simple algebras. Recall that an idempotent (respectively projection) $e \in R$ is said to be infinite if there exist orthogonal idempotents $g, h \in R$ such that $e = g + h$, $eR \cong gR$ and $h \neq 0$.

Definition 4.3.1

1. Let A be a C^*-algebra. We say that A is purely infinite simple if it is simple and every hereditary subalgebra of A contains an infinite projection [27].

2. Let R be a ring. We say that R is purely infinite simple if it is simple and every right ideal of R contains an infinite idempotent [14].

When the ring (C^*-algebra) R has a unit, it is purely infinite and simple if and only if for every nonzero $a \in R$ there exist $x, y \in R$ such that $xay = 1$, and R is not a division ring.

The fact is that the third of the above cases for a graph E with $L_K(E)$ (resp. $C^*(E)$) simple is exactly the case of Leavitt path algebras being purely infinite simple [3, Theorem 11] (for the case of $C^*(E)$ the result is analog, and it was obtained by different authors, see e.g. [47, Chapter 4]).

So, we close this part with two remarks:

1. The results are analog in both classes, but a simple inspection of the methods used in both context reveals that this results are also independent.

2. There is a dichotomy in simple Leavitt path algebras (graph C^*-algebras): they are either ultramatricial (AF-algebras) or purely infinite simple.

4.3.2 Real rank zero versus Exchange property

Either in the context of C^*-algebras or that of rings, the abovementioned dichotomy for simple objects is associated to an interesting structural property in any of those classes.

Recall that a C^*-algebra A is sad to have real rank zero ($RR(A) = 0$ for short) if the set of self-adjoint invertible elements is dense in the set of self-adjoint elements. This notion corresponds to a sort of noncommutative dimension for C^*-algebras generalizing the covering dimension for topological spaces (in fact is exactly that in the case of commutative C^*-algebras); for a primer on this, see [25] or [57].

On the other side, a (not necessarily unital) ring R is said to be an exchange ring (see [9]) if for every element $x \in R$ the equivalent conditions in the next lemma are satisfied.

Lemma 4.3.2 [9, Lemma 1.1] *Let R be a ring and let R' be a unital ring containing R as a two-sided ideal. Then the following conditions are equivalent for an element $x \in R$:*

1. *There exists $e^2 = e \in R$ with $e - x \in R'(x - x^2)$,*
2. *there exist $e^2 = e \in Rx$ and $c \in R'$ such that $(1-e) - c(1-x) \in J(R')$,*
3. *there exists $e^2 = e \in Rx$ such that $R' = Re + R'(1-x)$,*
4. *there exists $e^2 = e \in Rx$ such that $1 - e \in R'(1-x)$,*
5. *there exist $r, s \in R$, $e^2 = e \in R$ such that $e = rx = s + x - sx$.*

(Here $J(R')$ denotes the Jacobson radical of R').

By [13, Theorem 7.2] (see [9] for nonunital rings), a C^*-algebra has real rank zero if and only if it is an exchange ring. Thus, a good test to compare graph C^*-algebras and Leavitt path algebras is to characterize the property of being an exchange ring in both contexts.

In the case of graph C^*-algebras, $RR(C^*(E)) = 0$ if and only if the graph E has no isolated closed simple paths (equivalently, no isolated cycles), i.e. E satisfies Condition (K), which is equivalent to the fact that every two-sided ideal is gauge-invariant. This characterization was shown by Jeong, Park and Shin [36, 37]. The basis of the proof is the fact that any purely infinite simple C^*-algebra has real rank zero [25, 56], and that real rank zero is preserved for extensions under mild hypotheses.

In the case of Leavitt path algebras, the result is analog [19, Theorem 4.5], and the basis of the proof is essentially the same, because of the fact that any purely infinite simple ring is an exchange ring [10].

Nevertheless, a simple inspection in the proofs for both graph C^*-algebras and Leavitt path algebras reveals that these proofs cannot be exchanged, and do not imply the result in the opposite context.

4.3.3 K-Theory

Now, we will make a brief raid on K-theoretical invariants for graph C^*-algebras and Leavitt path algebras.

Our references for K-theory for C^*-algebras are [24] and [50]. For algebraic K-theory, we refer the reader to [51]. For a unital ring R, let $M_\infty(R)$ be the directed union of $M_n(R)$ ($n \in \mathbb{N}$), where the transition maps $M_n(R) \to M_{n+1}(R)$ are given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. We define $V(R)$ to be the set of isomorphism classes (denoted $[P]$) of finitely generated projective left R-modules, and we endow $V(R)$ with the structure of a commutative monoid by imposing the operation

$$[P] + [Q] := [P \oplus Q]$$

for any isomorphism classes $[P]$ and $[Q]$. Equivalently [24, Chapter 3], $V(R)$ can be viewed as the set of equivalence classes $V(e)$ of idempotents e in $M_\infty(R)$ with the operation

$$V(e) + V(f) := V\left(\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right)$$

for idempotents $e, f \in M_\infty(R)$. The group $K_0(R)$ of a unital ring R is the universal group of $V(R)$. Recall that, as any universal group of an abelian monoid, the group $K_0(R)$ has a standard structure of partially pre-ordered abelian group. The set of positive elements in $K_0(R)$ is the image of $V(R)$ under the natural monoid homomorphism $V(R) \to K_0(R)$. Whenever A is a C^*-algebra, the monoid $V(A)$ agrees with the monoid of Murray-von Neumann equivalence classes of projections in $M_\infty(A)$; see [24, 4.6.2 and 4.6.4] or [50, Exercise 3.11]. It follows that the algebraic version of $K_0(A)$ coincides with the operator-theoretic one. Also, in the case of nonunital rings with local units (as for example graph C^*-algebras or Leavitt path algebras) the group $K_0(R)$ still is the universal group of $V(R)$.

In order to compute the monoid $V(L_K(E))$ (and so the Grothendieck group $K_0(L_K(E))$), we need to reduce the problem to the case of finite graphs. Recall that a graph homomorphism $f\colon E = (E^0, E^1) \to F = (F^0, F^1)$ is given by two maps $f^0\colon E^0 \to F^0$ and $f^1\colon E^1 \to F^1$ such that $r_F(f^1(e)) = f^0(r_E(e))$ and $s_F(f^1(e)) = f^0(s_E(e))$ for every $e \in E^1$. We say that a graph homomorphism f is *complete* in case f^0 is injective and f^1 restricts to a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(f^0(v))$ for every $v \in E^0$ such that v emits edges. Note that under the above assumptions, the map f^1 must also be injective. Let us consider the category $\mathcal{G}$ whose objects are all the row-finite graphs and whose morphisms are the complete graph homomorphisms. It is easy to check that the category $\mathcal{G}$ admits direct limits. In order to simplify notation, the K-algebra $L_K(E)$ will be sometimes denoted by $L(E)$.

Lemma 4.3.3 ([15, Lemma 3.1]) *Every row-finite graph E is a direct limit in the category $\mathcal{G}$ of a directed system of finite graphs.*

Since the map $E \mapsto L_K(E)$ is a continuous covariant functor, we get

Lemma 4.3.4 ([15, Lemma 3.2]) *The assignment $E \mapsto L_K(E)$ can be extended to a functor L_K from the category $\mathcal{G}$ of row-finite graphs and complete graph homomorphisms to the category of K-algebras and (not necessarily unital) algebra homomorphisms. The functor L_K is continuous, that is, it commutes with direct limits. It follows that every graph algebra $L_K(E)$ is the direct limit of graph algebras corresponding to finite graphs.*

and the analog for graph C^*-algebras

Lemma 4.3.5 ([15, Lemma 3.3]) *The assignment $E \mapsto C^*(E)$ can be extended to a continuous functor from the category $\mathcal{G}$ of row-finite graphs and complete graph homomorphisms to the category of C^*-algebras and $*$-homomorphisms. Every graph C^*-algebra $C^*(E)$ is the direct limit of graph C^*-algebras associated with finite graphs.*□

Let M_E be the abelian monoid given by the generators $\{a_v \mid v \in E^0\}$, with the relations:

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \qquad \text{for every } v \in E^0 \text{ that emits edges.} \tag{4.1}$$

Since the map $E \mapsto M_E$ is a continuous covariant functor, we get

Lemma 4.3.6 ([15, Lemma 3.4]) *The assignment $E \mapsto M_E$ can be extended to a continuous functor from the category $\mathcal{G}$ of row-finite graphs and complete graph homomorphisms to the category of abelian monoids. It follows that every graph monoid M_E is the direct limit of graph monoids corresponding to finite graphs.*

Now, the above reduction to the case of a finite graph E allows to compute the monoid $V(L(E))$ by using [23, Theorem 6.2]

Theorem 4.3.7 ([15, Theorem 3.5]) *Let E be a row-finite graph. Then there is a natural monoid isomorphism $V(L_K(E)) \cong M_E$.*

The interesting (and highly nontrivial) fact is that the monoid $V(C^*(E))$ is naturally isomorphic to $V(L_{\mathbb{C}}(E))$, as follows

Theorem 4.3.8 ([15, Theorem 7.1]) *Let E be a row-finite graph, and let $L(E) = L_{\mathbb{C}}(E)$ be the graph algebra over the complex numbers. Then the natural inclusion $\psi\colon L(E) \to C^*(E)$ induces a monoid isomorphism*

$$V(\psi)\colon V(L(E)) \to V(C^*(E)).$$

In particular the monoid $V(C^*(E))$ is naturally isomorphic with the monoid M_E.

Then, in terms of $K_0(L_K(E))$ we have an easy way to compute it (which coincides with the computation of $K_0(C^*(E))$, see e.g. [20, §2.3.1]) when the graph is finite. As noticed above, $K_0(L(E)) \cong \mathrm{Grot}(M_E) := G$, where $\mathrm{Grot}(M_E)$ denotes the universal group of the monoid M_E. Since M_E is finitely generated, so is its universal group G. Thus G admits a presentation $\pi : \mathbb{Z}^n \to G$ (an epimorphism). Here $\ker(\pi)$ is the subgroup of relations, which in this setting corresponds to the image of the group homomorphism $A_E^t - I : \mathbb{Z}^n \to \mathbb{Z}^n$, where A_E^t is the transpose of the incidence matrix A_E of E. Hence we get

$$K_0(L(E)) \cong G \cong \mathbb{Z}^n/\ker(\pi) = \mathbb{Z}^n/\mathrm{im}(A_E^t - I) = \mathrm{coker}(A_E^t - I).$$

Moreover, under this isomorphism the element $[1_{L(E)}]$ is represented by

$$(1, 1, ..., 1)^t + \mathrm{im}(A_E^t - I)$$

in $\mathrm{coker}(A_E^t - I)$.

This facility connects with the possibility of using K-theoretical invariant to classify Leavitt path algebras up to isomorphism. In the study of C*-algebras, an important role is played by the Classification Theorem of purely infinite simple unital nuclear C*-algebras (see e.g. [38, 46]). Specifically, Kirchberg and Phillips (independently) showed that if X and Y are purely infinite simple unital C*-algebras (satisfying certain additional conditions), then $X \cong Y$ as C*-algebras if and only if (i) $K_0(X) \cong K_0(Y)$ via an isomorphism ϕ for which $\phi([1_X]) = [1_Y]$, and (ii) $K_1(X) \cong K_1(Y)$.

As it turns out, in the more specific case of purely infinite simple unital Cuntz-Krieger graph C*-algebras, K-theoretic information is in fact encoded in the transpose A_E^t of the incidence matrix A_E of the graph E. Specifically, when E has no sinks, then by [20, Theorem 3.9]

$$K_0(C^*(E)) \cong \mathrm{coker}(A_E^t - I) \quad \text{and} \quad K_1(C^*(E)) \cong \ker(A_E^t - I),$$

where I is the identity matrix of size $n = |E^0|$.

We seek a similar result in the setting of purely infinite simple unital Leavitt path algebras. So suppose E and F are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple unital. By [3, Theorem 11] these graphs contain no sinks. By [1, Proposition 3.6] we can assume without loss of generality that E and F have the same number n of vertices and that they have no sources. Thus if $K_0(L(E)) \cong K_0(L(F))$, then using the previously established isomorphism we get $\mathrm{coker}(A_E^t - I) \cong \mathrm{coker}(A_F^t - I)$. This in turn implies (by the Fundamental Theorem of Finitely Generated Abelian Groups) the existence of invertible matrices $P, Q \in \mathrm{M}_n(\mathbb{Z})$ such that $A_F^t - I = P(A_E^t - I)Q$. Thus $\ker(A_F^t - I) \cong \ker(A_E^t - I)$ (as these are subgroups of $\mathbb{Z}^n$ having equal rank); notice that in particular, since $K_1(C^*(E)) \cong \ker(A_E^t - I)$, we have recovered the result of [20, Theorem 3.9] for graph C^*-algebras. Moreover, by using the unique unital ring map $\psi : \mathbb{Z} \to K$, we get that the PAQ-equivalence of $A_E^t - I$ and $A_F^t - I$ also holds on K. If $K^\times$ denotes the multiplicative group on nonzero elements in K, then the previous remark implies that $\mathrm{coker}(A_E^t - I : (K^\times)^n \to (K^\times)^n)$ and $\mathrm{coker}(A_F^t - I : (K^\times)^n \to (K^\times)^n)$ (where $A_E^t - I$ and $A_E^t - I$ are seen as multiplicative maps on $(K^\times)^n$) are also isomorphic. Since by [26, Theorem 3.19], for any finite graph G with n vertices with no sinks or sources we have

$$K_1(L(G)) \cong \mathrm{coker}(A_G^t - I : (K^\times)^n \to (K^\times)^n) \oplus \ker(A_G^t - I : \mathbb{Z}^n \to \mathbb{Z}^n),$$

we conclude that the hypothesis $K_0(L(E)) \cong K_0(L(F))$ in fact yields $K_1(L(E)) \cong K_1(L(F))$ as a consequence. With this observation and the aforementioned Kirchberg - Phillips result in mind, it is then natural to ask the following

> **The Classification Question for purely infinite simple unital Leavitt path algebras.** Suppose E and F are graphs for which $L(E)$ and $L(F)$ are purely infinite simple unital. If $K_0(L(E)) \cong K_0(L(F))$ via an isomorphism ϕ having $\phi([1_{L(E)}]) = [1_{L(F)}]$, must $L(E)$ and $L(F)$ be isomorphic?

Some advances in this direction, analog to those of graph C^*-algebras, are obtained in [1] (and recently improved in [8]).

4.3.4 Stable rank

Stable rank for rings is an interesting property, related to computing general linear groups on unital rings. Also it is related to interesting cancellation properties of finitely generated projective modules over rings. This is specially true in case of exchange rings [13], and it is a witness for an affirmative answer to the Separativity Problem: the values of stable rank for a separative exchange ring must be $1, 2$ or ∞.

In case of graph C^*-algebras, the amazing result is that not only the values of stable rank are $1, 2$ or ∞, but also that the occurrence of this values is determined in terms of intrinsic properties of the graph, so that a Trichotomy is obtained [31, Theorem 3.4].

The same result is enjoyed by Leavitt path algebras, except that the Trichotomy result do not coincide exactly with this of graph C^*-algebras.

First, we need to recall a bunch of definitions and results. Let S be any unital ring containing an associative ring R as a two-sided ideal. The following definitions can be found in [54]. A column vector $b = (b_i)_{i=1}^n$ is called *R-unimodular* if $b_1 - 1, b_i \in R$ for $i > 1$ and there exist $a_1 - 1, a_i \in R$ $(i > 1)$ such that $\sum_{i=1}^n a_i b_i = 1$. The *stable rank* of R (denoted by $sr(R)$) is the least natural number m for which for any R-unimodular vector $b = (b_i)_{i=1}^{m+1}$ there exist $v_i \in R$ such that the vector $(b_i + v_i b_{m+1})_{i=1}^m$ is R-unimodular. If such a natural m does not exist we say that the stable rank of R is infinite. The definition does not depend on the choice of S. Stable rank of R enjoys the following properties:

1. If $R = \prod_{\lambda\in\Lambda} R_\lambda$, then $sr(R) = \max_\lambda\{sr(R_\lambda)\}$ [54, Lemma 2].

2. For every $n \in \mathbb{N}$, $sr(M_n(R)) = 1 - \left\lfloor -\dfrac{sr(R) - 1}{n} \right\rfloor$, where $\lfloor a \rfloor$ denotes the integral part of a [54, Theorem 3].

3. For any two-sided ideal I of R,
$$\max\{sr(I), sr(R/I)\} \leq sr(R) \leq \max\{sr(I), sr(R/I) + 1\}$$
[54, Theorem 4].

It is easy to see from [54] that if $R = \lim R_n$, then $\mathrm{sr}(R) \leq \liminf\limits_{n\to\infty} \mathrm{sr}(R_n)$. Thus, from this and [54, Corollary to Theorem 3], we get that $sr(R) = 1$ for any locally matricial algebra. Also it is well-known (see e.g. [14, Proposition 2.1]) that if R is a unital purely infinite simple ring, then $sr(R) = \infty$.

Two facts that are interesting with respect to stable rank of rings are:

1. Stable rank is not a Morita invariant property: for any ring R such that $sr(R) = n > 2$, $sr(M_n(R)) = 2$, but both rings are trivially Morita equivalent.

2. Because of Evans' Theorem [34], $sr(R) = 1$ implies that $V(R)$ is a cancellative monoid. The converse is not true in general (e.g.: $sr(\mathbb{Z}) = 2$, but $V(\mathbb{Z}) = \mathbb{Z}^+$).

We recall the following definition from [48]. Let A be a C^*-algebra, and let $A^{\sim}$ be its minimal unitization. Then, the topological stable rank of A, denoted by $tsr(A)$, is the least integer n such that the set of n-tuples in $(A^{\sim})^n$ that generate A as a left ideal is dense in $(A^{\sim})^n$. If such an integer does not exist, then $tsr(A) = \infty$. Because of [35, Theorem], for any unital C^*-algebra A we have $tsr(A) = sr(A)$, so that the properties enjoyed by $tsr(A)$ [48] are consequence of these enjoyed by general rings [54].

It has an special interest the case of stable C^*-algebras. A C^*-algebra A is stable if and only if $A \cong A \otimes \mathbb{K}$, where $\mathbb{K}$ is the C^*-algebra of compact operators over a separable Hilbert space. Notice that this is equivalent to the fact that A is isomorphic to the completion of the pre-C^*-algebra $M_\infty(A)$. Thus, for any C^*-algebra A, $sr(A \otimes \mathbb{K}) = 2$ unless $sr(A) = 1$, in which case $sr(A \otimes \mathbb{K}) = 1$ [48, Theorem 6.4].

Now, the strategy for computing the stable rank of a Leavitt path algebra $L(E)$ is quite similar to that of [31], except for a technical problem about bounding the stable rank of an extension of rings, which turns out to be trivial in case of graph C^*-algebras, but not in case of Leavitt path algebras. The reason is that, if E is an isolated cycle, then $\mathrm{sr}(C^*(E)) = 1$ while $\mathrm{sr}(L(E)) = 2$, so that the reduction argument used in [31] for graph C^*-algebras (via extensions) do not work correctly for Leavitt path algebras.

In the concrete case of *exchange* Leavitt path algebras this difficulty can be avoided, as shown by Aranda, Pardo and Siles in [19]. The general case was solved by Ara and Pardo in [16]. The main result is

Theorem 4.3.9 ([16, Theorem 2.8]) *Let E be a row-finite graph. Then the values of the stable rank of $L(E)$ are:*

1. *$sr(L(E)) = 1$ if E is acyclic.*

2. *$sr(L(E)) = \infty$ if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal, contains no sinks and each cycle has an exit.*

3. *$sr(L(E)) = 2$ otherwise.*

which contrast with

Theorem 4.3.10 ([31, Theorem 3.4]) *Let E be a row-finite graph. Then the values of the stable rank of $C^*(E)$ are:*

1. *$sr(L(E)) = 1$ if every cycle of E has no exits.*

2. *$sr(L(E)) = \infty$ if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal, contains no sinks and each cycle has an exit.*

3. *$sr(L(E)) = 2$ otherwise.*

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