REMARKS ON THE AREA THEOREM IN THE THEORY OF UNIVALENT FUNCTIONS

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Abstract. We prove an integral extension of the classical area theorem for univalent functions. We give an application finding geometric conditions on the image domain of a univalent function \( f \) which imply that \( f \) belongs to the Hardy space \( H^p, 0 < p < \infty \).

1. Introduction and main results

Let \( \mathbb{D} \) denote the open unit disk of the complex plane \( \mathbb{C} \). A complex-valued function defined in \( \mathbb{D} \) is said to be univalent if it is analytic and one-to-one there. We refer to [5] and [8] for the theory of these functions. Throughout the paper, \( \mathcal{U} \) will stand for the class of all univalent functions in \( \mathbb{D} \). The classical area theorem [5, p. 29], which is a key in the proof of a good number of results in the theory of univalent functions, can be stated as follows:

**Theorem A.** If \( \varphi(z) = \sum_{n=0}^{\infty} b_n z^n, b_0 \neq 0 \), is an analytic function in \( \mathbb{D} \) such that the meromorphic function \( \varphi(z)/z \) is one-to-one in \( \mathbb{D} \), then

\[
\sum_{n=1}^{\infty} \frac{n^2|b_n|^2}{n+1} \leq \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1}
\]

or, equivalently,

\[
\int_{\mathbb{D}} |z\varphi'(z)|^2 \, dm(z) \leq \int_{\mathbb{D}} |\varphi(z)|^2 \, dm(z).
\]

Here, \( dm(z) = dx dy \) denotes the usual Lebesgue area measure.

In this paper we generalize Theorem A in the following way.

**Theorem 1.** Let \( p > 0 \). If \( \varphi \) is a function as in Theorem A and \( \varphi(z) \neq 0 \) for all \( z \in \mathbb{D} \), then

\[
(1.1) \quad \int_{\mathbb{D}} |z|^p |\varphi(z)|^{p-2} |\varphi'(z)|^2 \, dm(z) \leq \int_{\mathbb{D}} |z|^{p-2} |\varphi(z)|^p \, dm(z).
\]


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Consequently, we deduce the following theorem on univalent functions.

**Theorem 2.** If $f$ is univalent in $\mathbb{D}$, $f(0) = 0$ and $p > 0$, then

\[
\int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left(1 - \left| \frac{zf'(z)}{f(z)} \right|^2 \right) \, dm(z) \leq \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, dm(z),
\]

or equivalently

\[
\int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| \frac{zf'(z)}{f(z)} \right|^2 \, dm(z) \leq 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) \, dm(z).
\]

Given an space $X$ of analytic functions, one of the most interesting problems in the theory of univalent functions is finding geometric conditions on a domain $\Omega$ which imply that $\Omega$ is an $X$-domain, that is, any analytic function $f$ defined on $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$ belongs to $X$. This problem has been solved for a good number of spaces, such as the Bloch space, Besov spaces $B_p$, $1 < p < \infty$, . . . (see [1], [3]). However, this is an open problem for $H^p$ spaces (see [4]). We shall use Theorem 2 to find geometric conditions on the image domain of a function $f \in U$ which imply its membership in $H^p$.

For simplicity, we shall assume that $0 \in f(\mathbb{D})$.

Given a domain $\Omega \subset \mathbb{C}$ and a point $w$ in $\Omega$, we shall write $d_\Omega(w)$ to denote the (Euclidean) distance from $w$ to the boundary $\partial \Omega$. The following inequalities play an essential role in the proof of our results (see, e.g., [9], Corollary 1.4).

If $\Omega$ is a simply connected proper subdomain of $\mathbb{C}$ and $F$ is a conformal mapping from $\Delta$ onto $\Omega$ then we have

\[
d_\Omega(F(z)) \leq |F'(z)|(1 - |z|^2) \leq 4d_\Omega(F(z)), \quad z \in \mathbb{D}.
\]

The following result is proved in [2, Corollary 7].

**Corollary B.** Suppose that $1/2 \leq p < \infty$ and $f \in A^{2p} \cap U$. Set $\Omega = f(\mathbb{D})$ and suppose that $0 \in \Omega$. For $\varepsilon > 0$, set $\Omega_\varepsilon = \{w \in \Omega : |w| > \varepsilon\}$. If

\[
\int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^{2p}} \, dm(w) < \infty,
\]

for all sufficiently small $\varepsilon > 0$, then $f \in H^p$.

Here, we shall prove the following extension of this result.

**Theorem 3.** Suppose that $0 < \beta < 1$, $1 - \frac{\beta}{2} < p < \infty$, and $f \in A^{2p} \cap U$. Set $\Omega = f(\mathbb{D})$ and suppose that $0 \in \Omega$. If

\[
\int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^{2p-2 - \beta + \frac{\beta}{2} - \frac{\beta}{2}} \, dm(w) < \infty,
\]

for some $\delta$, $0 < \delta < \frac{1+p}{2}$ and all sufficiently small $\varepsilon > 0$, then $f \in H^p$. 

Moreover, we shall prove that this result is sharp in certain sense.

The paper is organized as follows. Section 2 is devoted to prove Theorem 1 and Theorem 2. Corollary 3 and some other results are proved in section 3.

2. Proof of the main results

The proof of Theorem 1 is based on the following fact due to Prawitz [10]. The proof is borrowed from [7] (see also [6]).

**THEOREM C.** Let \( f : \mathbb{D} \rightarrow \mathbb{C} \) be a univalent function and \( f(0) = 0 \), and let
\[
J_p(r) = J_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{-p} d\theta, \quad p > 0, \quad 0 < r < 1.
\]

Then
\[
2\pi J'_p(r) = -(p/r) \text{Im} \int_{\Gamma_r} |w|^{-p-2} \bar{w} dw \quad (w = u + iv)
\]
\[
= -(p/r) \int_{\Gamma_r} |w|^{-p-2}(vdv - udu)
\]
\[
< 0,
\]
for all \( r \in (0, 1) \), where \( \Gamma_r \) is the image under \( f \) of the circle \( \{ \zeta \in \mathbb{C} : |\zeta| = r \} \) and \( \Gamma_r \) is positively oriented.

**Proof.** We have
\[
2\pi J'_p(r) = -p \int_0^{2\pi} |f(re^{i\theta})|^{-p-2} \text{Re}\{\overline{f(re^{i\theta})}f'(re^{i\theta})e^{i\theta}\} d\theta
\]
\[
= -(p/r) \text{Im} \int_{|\zeta|=r} |f(\zeta)|^{-p-2} \overline{f(\zeta)}f'(\zeta) d\zeta
\]
\[
= -(p/r) \text{Im} \int_{\Gamma_r} |w|^{-p-2} \bar{w} dw \quad (w = u + iv)
\]
\[
= -(p/r) \int_{\Gamma_r} |w|^{-p-2}(vdv - udu),
\]
where \( \Gamma_r \) is the image under \( f \) of the circle \( |\zeta| = r \) and the curve \( \Gamma_r \) is positively oriented. Now we apply Green’s formula to the domain \( \Omega_{r,R} \) bounded by \( \Gamma_r \) and the circle \( |w| = R \), where \( R > \max_{|z|=r}|f(z)| \). Since
\[
\partial(|w|^{-p-2}u)/\partial u - \partial(-|w|^{-p-2}v)/\partial v = -p|w|^{-p-2},
\]
we have
\[
\int_{|w|=R} |w|^{-p-2}(vdv - udu) - \int_{\Gamma_r} |w|^{-p-2}(vdv - udu)
\]
\[
= -p \int_{\Omega_{r,R}} |w|^{-p-2} du dv.
\]
The first integral is equal to $2\pi R^{-p}$, and therefore

$$J'_p(r) = -(p/r)R^{-p} - (p^2/2\pi r)\int_{\Omega_{r,R}} |w|^{-p-2} \, du \, dv.$$  

Letting $R$ tend to $\infty$ we get

$$J'_p(r) = -(p^2/2\pi r)\int_{\Omega_r} |w|^{-p-2} \, du \, dv,$$

where $\Omega_r$ is the exterior of the curve $\Gamma_r$. This concludes the proof.  

\textbf{Proof of Theorem 1.} With the hypotheses of Theorem 1, let $g(z) = \varphi(z)/z$ and $f(z) = 1/g(z) = z/\varphi(z)$. Let

$$I_p(r) = I_p(r, g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta = J_p(r, f), \quad p > 0.$$  

The function $f$ satisfies the hypotheses of Theorem C and therefore

$$2\pi I'_p(r) = 2\pi J'_p(r) = -(p/r)\int_{\Gamma_r} |w|^{-p-2}(u \, dv - v \, du) < 0.$$  

By the change $w \mapsto 1/w$ we get

$$2\pi I'_p(r) = (p/r)\int_{\gamma_r} |w|^{-p-2}(u \, dv - v \, du) < 0,$$

where $\gamma_r$ denotes the curve $w = g(re^{it})$, $0 \leq t \leq 2\pi$, which is a curve of negative orientation. Now we try to parameterize $\gamma_r$ by $w = F(e^{it})$, where $F(e^{it}) \equiv g(re^{it})$, indeed we choose

$$F(z) = \overline{z}/r \varphi(rz).$$

Then we have

$$\int_{\gamma_r} |w|^{-p-2}(u \, dv - v \, du) = \Im \int_{|\zeta|=1} |F(\zeta)|^{-p-2} F(\zeta) \, dF(\zeta).$$

Now, we choose a circle $T_\rho$ of radius $\rho$ centered at 0 ($0 < \rho < \frac{1}{2}$), and apply Green’s formula to the annulus $A_\rho := \{z : \rho \leq |z| \leq 1\}$ to get

$$\Im \int_{|\zeta|=1} |F(\zeta)|^{-p-2} F(\zeta) \, dF(\zeta) = p \int_{A_\rho} |F|^{-p-2} J_F \, dm + \Im \int_{T_\rho} |F|^{-p-2} F \, dF,$$

where $J_F(z)$ is the Jacobian of $F$,

$$J_F(z) = |\partial F/\partial z|^2 - |\partial F/\partial \overline{z}|^2 = |z\varphi'(rz)|^2 - |\varphi(rz)|^2/r^2,$$

here the circles are positively oriented. From the properties of $\varphi$ it follows that

$$\Im \int_{T_\rho} |F|^{-p-2} F \, dF \leq C\rho^p,$$
and
\[ \int_D |F|^{p-2} |J_F| \, dm \leq C \int_D |z|^{p-2} \, dm(z) < \infty. \]
Hence, by passing to the limit as \( \rho \to 0 \), we get
\[ \text{Im} \int_{|z|=1} |F(\zeta)|^{p-2} \overline{F(\zeta)} \, dF(\zeta) = p \int_D |F|^{p-2} J_F \, dm. \]
From this, (2.1) and (2.2) it follows that
\[ \int_D |F(z)|^{p-2} J_F(z) \, dm(z) < 0, \]
that is
\[
\int_D |z\varphi(rz)|^{p-2} |z\varphi'(rz)|^2 \, dm(z) < r^{-2} \int_D |z|^{p-2} \varphi(rz) \, dm(z) \tag{2.3}
\]
\[ = 2\pi r^{-2} \int_0^1 t^{p-1} I_p(rt, \varphi) \, dt, \]
where, as above,
\[ I_p(s, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(se^{i\theta})|^p \, d\theta. \]
As it is well known, \( I_p(s, \varphi) \) increases with \( s \) and so we can apply the monotone convergence theorem to get
\[ \lim_{r \to 1^-} \int_0^1 t^{p-1} I_p(rt, \varphi) \, dt = \int_0^1 t^{p-1} \lim_{r \to 1^-} I_p(rt, \varphi) \, dt \]
\[ = \int_0^1 t^{p-1} I_p(t, \varphi) \, dt. \]
From this and (2.3), via Fatou’s lemma, we obtain (1.1). This completes the proof of Theorem 1.

**Proof of Theorem 2.** If \( f \) is univalent in \( \mathbb{D} \), \( f(0) = 0 \), then let \( \varphi(z) = z/f(z) \). Since
\[ \varphi'(z) = \frac{f(z) - zf'(z)}{f(z)^2}, \]
we see, by Theorem 1, that
\[ \int_D |z|^{2p-2} |f(z)|^{2-p} \left| \frac{f(z) - zf'(z)}{f(z)^2} \right|^2 \, dm(z) \leq \int_D |z|^{p-2} \left| \frac{z}{f(z)} \right|^p \, dm(z). \]
The result follows.

**Remark 1.** Equality is possible in (1.1), (1.2) and (1.3). In the case of (1.1) we take
\[ \varphi(z) = (1 - e^{-i\theta} z)^2, \quad \theta \in [0, 2\pi]. \]
Then
\[ \text{Im} \int_{|\zeta|=1} |F(\zeta)|^{p-2} \overline{F(\zeta)} \, dF(\zeta) = 0, \]
where
\[ F(z) = z\varphi(z) = \bar{z}(1 - e^{-i\theta}z)^2. \]

Now an application of Green’s formula as above shows that
\[ \int_{D} |F|^{p-2} J_{F} \, dm = 0, \]
which implies (1.1).

The above deduction of Theorem 2 from Theorem 1 shows that equality in (1.2) and (1.3) is attained if \( f_\theta \) is any rotation of Koebe function:
\[ f_\theta(z) = \frac{z}{(1 - e^{-i\theta}z)^2}. \]

3. Applications

First, we present the following simple but useful lemma.

**Lemma 1.** If \( f \) is univalent in \( \mathbb{D} \), \( f(0) = 0 \) and \( p > 0 \), then
\[ \int_{\mathbb{D}} |z|^{2p} |f(z)|^{-p} \left| \frac{f'(z)}{f(z)} \right|^2 \, dm(z) < \infty. \]

**Proof.**
Observe that if \( f \in \mathcal{U} \) with \( f(0) = 0 \), then
\[ \int_{\mathbb{D}} |z|^{2p} |f(z)|^{-p} \, dm(z) < \infty \quad \text{for any } p > 0, \]
which together with Theorem 2 gives that
\[
\begin{align*}
&\int_{\mathbb{D}} |z|^{2p} |f(z)|^{-p} \left| \frac{f'(z)}{f(z)} \right|^2 \, dm(z) \\
&\leq 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| 1 - \frac{zf'(z)}{f(z)} \right|^2 \, dm(z) + 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, dm(z) \\
&\leq 4 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, dm(z) \quad \text{< } \infty.
\end{align*}
\]
This finishes the proof. \( \square \)

Now, we are ready to obtain our result on \( H^p \)-univalent functions.

**Proof of Theorem 3.** Bearing in mind [2, Theorem 1] it is enough to prove that
\[ \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) \quad < \infty. \]

Take \( \eta > 0 \) such that \( \{|w| < \eta\} \subset \Omega \) and take \( \varepsilon \) with \( 0 < \varepsilon < \eta \) and set \( \mathbb{D}_\varepsilon = f^{-1}(\Omega_\varepsilon) \).
By Hölder’s inequality (with conjugate exponents $\frac{2}{1+\beta} > 1$ and $\frac{2}{1-\beta}$), and Lemma 1, we deduce that

\[(3.2)\]
\[\int_{D_{\epsilon}} |f'(z)|^{p} (1 - |z|^2)^{p-1} \, dm(z) \]
\[= \int_{D_{\epsilon}} \left[ (1 - |z|^2) |f'(z)|^p \right]^{p-1} \left| f'(z) \right|^{\beta} |z|^{-2\delta} |f(z)|^{1-\beta + \delta} \left[ |f(z)|^{-\delta} |z|^{2\delta} \frac{f'(z)}{f(z)} \right]^{1-\beta} \, dm(z) \]
\[\leq \left( \int_{D_{\epsilon}} \left[ (1 - |z|^2) |f'(z)| \right]^{\frac{2(p-1)}{1+\beta}} \left| f'(z) \right|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} \, dm(z) \right)^{\frac{1+\beta}{2}} \]
\[\times \left( \int_{D_{\epsilon}} \left| f'(z) \right|^{\frac{2\delta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} \frac{f'(z)}{f(z)}^2 \, dm(z) \right)^{\frac{1-\beta}{2}} \]
\[\leq C \left( \int_{D_{\epsilon}} \left[ (1 - |z|^2) |f'(z)| \right]^{\frac{2(p-1)}{1+\beta}} \left| f'(z) \right|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} \, dm(z) \right)^{\frac{1+\beta}{2}}.
\]

On the other hand, if $0 < \delta < \frac{1+p}{2}$, using Hölder’s inequality (with conjugate exponents $\frac{1+\beta}{\beta}$ and $1+\beta$), making the change of variable $w = f(z)$ and bearing in mind (1.4) and that $f \in A^{2p}$, we deduce that

\[\int_{D_{\epsilon}} \left[ (1 - |z|^2) |f'(z)| \right]^{\frac{2(p-1)}{1+\beta}} \left| f'(z) \right|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} \, dm(z) \]
\[= \int_{D_{\epsilon}} \frac{\left[ (1 - |z|^2) |f'(z)| \right]^{\frac{2(p-1)}{1+\beta}}}{\left| f(z) \right|^{\frac{2(p+\beta-1-\delta)}{1+\beta}}} \left| f'(z) \right|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2p}{1+\beta}} \, dm(z) \]
\[= \left( \int_{D_{\epsilon}} \frac{\left[ (1 - |z|^2) |f'(z)| \right]^{\frac{2(p-1)}{1+\beta}}}{\left| f(z) \right|^{\frac{2(p+\beta-1-\delta)}{1+\beta}}} \left| f'(z) \right|^2 \, dm(z) \right)^{\frac{1}{1+\beta}} \]
\[\times \left( \int_{D} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{2p} \, dm(z) \right)^{\frac{1}{1+\beta}} \]
\[\leq C \left( \int_{\Omega} \frac{d\Omega(z)^{2p-2}}{|w|^{\frac{2p-2}{\beta} + 2\frac{\beta}{1+\beta}}} \, dm(z) \right)^{\frac{1}{1+\beta}} < \infty,
\]

which together with (3.1) and (3.2), finishes the proof.

Moreover, we are able to prove that this result is sharp in the following sense.

**Theorem 4.** If $0 < \beta < 1$ and $1 - \frac{2}{2p} < p < \infty$, then there exists a univalent function $g \in A^{2p} \setminus H^p$ with $g(0) = 0$ and such that, setting $\Omega = g(\mathbb{D})$ and
\( \Omega_\varepsilon = \{ w \in \Omega : |w| > \varepsilon \} \),

\[
\int_{\Omega_\varepsilon} \frac{d\Omega(w)^{2p-2}}{|w|^{2p-2 + 2 + \kappa}} \, dm(w) < \infty, \quad \varepsilon > 0,
\]

for every \( \kappa > 0 \).

**Proof of Theorem 4.** We shall follow the lines of the proof of [2, Thorem 8]. Take \( p \in (1/2, \infty) \) and let \( f \) be the function defined in the proof of [2, Theorem 3], that is,

\[
f(z) = \left[ \frac{1}{(1 - z) \log \frac{2e}{1-z}} \right]^{\frac{1}{p}}, \quad z \in \mathbb{D}.
\]

Set

\[
g(z) = f(z) - f(0), \quad z \in \mathbb{D}.
\]

Then \( g \) is univalent, \( g(0) = 0 \) and \( g \in A^{2p} \setminus H^p \). Finally, we shall see that (3.3) holds.

Take \( \varepsilon > 0 \). Since \( g(0) = 0 \), there exists \( \eta \) with \( 0 < \eta < 1 \) such that

\[
g^{-1}(\Omega_\varepsilon) \subset \mathbb{D}_\eta = \{ z \in \mathbb{D} : |z| > \eta \}.
\]

We have that

\[
g'(z) = \frac{1}{p(1 - z)^{1 + \frac{1}{2}}} \left[ \left( \frac{1}{\log \frac{2e}{1-z}} \right)^{\frac{1}{p}} \left( 1 - \frac{1}{\log \frac{2e}{1-z}} \right) \right], \quad z \in \mathbb{D},
\]

and that there exists a positive constant \( C \) such that

\[
|g(z)| \geq C \left| \frac{1}{(1 - z) \log \frac{2e}{1-z}} \right|^{\frac{1}{p}} \quad z \in \mathbb{D}_\eta.
\]

So, using (1.4), and assuming without loss of generality that \( \kappa < p \left( \frac{2p-2}{p^2} + 1 \right) \), we obtain

\[
\int_{\Omega_\varepsilon} \frac{d\Omega(w)^{2p-2}}{|w|^{2p-2 + 2 + \kappa}} \, dm(w)
\]

\[
\leq C \int_{\mathbb{D}_\eta} (1 - |z|^2)^{\frac{2p-2}{p}} \left| g'(z) \right|^{\frac{2p-2 + 2}{p}} \left| \log \frac{2e}{1-z} \right|^{\frac{2p-2 + 2 + \kappa}{p}} \, dm(z)
\]

\[
\leq C \int_{\mathbb{D}_\eta} (1 - |z|^2)^{\frac{2p-2}{p}} \left| (1 - z) \log \frac{2e}{1-z} \right|^{\frac{2p-2 + 2 + \kappa}{p}} \left| \log \frac{2e}{1-z} \right|^{\frac{2p-2 + 2 + \kappa}{p}} \, dm(z)
\]

\[
\times \left| 1 - \frac{1}{\log \frac{2e}{1-z}} \right|^{\frac{2p-2 + 2}{p}} \, dm(z)
\]
\[
\leq C \int_{D_\rho} (1 - |z|^2)^\frac{2p-2}{\beta} \left| \frac{1}{1-z} \right|_p \frac{\log \left| \frac{2e^{1-z}}{1-z} \right|}{1 - z^2} \ dm(z) \\
\leq C \int_0^1 (1 - r)^{-1 + \frac{2p}{\beta}} \left( \log \left| \frac{2e^{1-r}}{1-r} \right| \right) \frac{1}{p} \ dr < \infty.
\]
This finishes the proof.

REFERENCES


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