Weighted Bergman Spaces induced by rapidly increasing weights

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Abstract

This monograph is devoted to the study of the weighted Bergman space $A^p_\omega$ of the unit disc $D$ that is induced by a radial continuous weight $\omega$ satisfying

$$\lim_{r\to 1^-} \frac{\int_r^1 \omega(s) \, ds}{\omega(r)(1-r)} = \infty.$$

Every such $A^p_\omega$ lies between the Hardy space $H^p$ and every classical weighted Bergman space $A^p_\alpha$. Even if it is well known that $H^p$ is the limit of $A^p_\alpha$, as $\alpha \to -1$, in many respects, it is shown that $A^p_\omega$ lies “closer” to $H^p$ than any $A^p_\alpha$, and that several finer function-theoretic properties of $A^p_\alpha$ do not carry over to $A^p_\omega$.

As to concrete objects to be studied, positive Borel measures $\mu$ on $D$ such that $A^p_\omega \subset L^q(\mu)$, $0 < p \leq q < \infty$, are characterized in terms of a neat geometric condition involving Carleson squares. In this characterization Carleson squares can not be replaced by (pseudo)hyperbolic discs. These measures are shown to coincide with those for which a Hörmander-type maximal function from $L^p_\omega$ to $L^q(\mu)$ is bounded. It is also proved that each $f \in A^p_\omega$ can be represented in the form $f = f_1 \cdot f_2$, where $f_1 \in A^{p_1}_\omega$, $f_2 \in A^{p_2}_\omega$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Because of the tricky nature of $A^p_\omega$ several new concepts are introduced. In particular, the use of a certain equivalent norm involving a square area function and a non-tangential maximal function related to lens type regions with vertexes at points in $D$, gives raise to a some what new approach to the study of the integral operator

$$T_\phi(f)(z) = \int_0^z f(\zeta) \phi'(\zeta) \, d\zeta.$$

This study reveals the fact that $T_\phi : A^p_\omega \to A^p_\omega$ is bounded if and only if $\phi$ belongs to a certain space of analytic functions that is not conformally invariant. The lack of this invariance is one of the things that cause difficulties in the proof leading the above-mentioned new concepts, and thus further illustrates the significant difference between $A^p_\omega$ and the standard weighted Bergman space $A^p_\alpha$. The symbols $\phi$ for which $T_\phi$ belongs to the Schatten $p$-class $S_p(A^2_\alpha)$ are also described. Furthermore, techniques developed are applied to the study of the growth and the oscillation of analytic solutions of (linear) differential equations.

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Preface

This work concerns the weighted Bergman space $A^p_\omega$ of the unit disc $\mathbb{D}$ that is induced by a radial continuous weight $\omega : [0,1) \to (0, \infty)$ such that

\[
\lim_{r \to 1^-} \int_r^1 \frac{\omega(s) ds}{\omega(r)(1-r)} = \infty.
\]

A radial continuous weight $\omega$ with the property (\(\ddagger\)) is called rapidly increasing and the class of all such weights is denoted by $I$. Each $A^p_\omega$ induced by $\omega \in I$ lies between the Hardy space $H^p$ and every classical weighted Bergman space $A^p_\alpha$ of $\mathbb{D}$.

In many respects the Hardy space $H^p$ is the limit of $A^p_\alpha$, as $\alpha \to -1$, but it is well known to specialists that this is a very rough estimate since none of the finer function-theoretic properties of the classical weighted Bergman space $A^p_\alpha$ is carried over to the Hardy space $H^p$ whose (harmonic) analysis is much more delicate. One of the main motivations for us to write this monograph is to study the spaces $A^p_\omega$, induced by rapidly increasing weights, that indeed lie “closer” to $H^p$ than any $A^p_\alpha$ in the above sense and explore the change of these finer properties related to harmonic analysis. We will see that the “transition” phenomena from $H^p$ to $A^p_\omega$ does appear in the context of $A^p_\omega$ with rapidly increasing weights and raises a number of new interesting problems, some of which are addressed in this monograph.

Chapter 1 is devoted to proving several basic properties of the rapidly increasing weights that will be used frequently in the monograph. Also examples are provided to show that, despite of their name, rapidly increasing weights are by no means necessarily increasing and may admit a strong oscillatory behavior. Most of the presented results remain true or have analogues for those weights $\omega$ for which the quotient in the left hand side of (\(\ddagger\)) is bounded and bounded away from zero. These weights are called regular and the class of all such weights is denoted by $R$. Each standard weight $\omega(r) = (1-r^2)^{\alpha}$ is regular for all $-1 < \alpha < \infty$. Chapter 1 is instrumental for the rest of the monograph.

Many conventional tools used in the theory of the classical Bergman spaces fail to work in $A^p_\omega$ that is induced by a rapidly increasing weight $\omega$. For example, one can not find a weight $\omega' = \omega'(p)$ such that $\|f\|_{A^p_\omega} \asymp \|f'\|_{A^{p'}_\omega}$ for all analytic functions $f$ in $\mathbb{D}$ with $f(0) = 0$, because such a Littlewood-Paley type formula does not exist unless $p = 2$. Moreover, neither $q$-Carleson measures for $A^p_\omega$ can be characterized by a simple condition on pseudohyperbolic discs, nor the Riesz projection is necessarily bounded on $A^p_\omega$ if $0 < p \leq 1$. However, it is shown in Chapter 2 that the embedding $A^p_\omega \subset L^q(\mu)$ can be characterized by a geometric condition on Carleson squares $S(I)$ when $\omega$ is rapidly increasing and $0 < p \leq q < \infty$. 

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In these considerations we will see that the weighted maximal function
\[ M_{\omega}(\varphi)(z) = \sup_{I: z \in S(I)} \frac{1}{\omega(S(I))} \int_{S(I)} |\varphi(\xi)|\omega(\xi)\,dA(\xi), \quad z \in \mathbb{D}, \]
introduced by Hörmander [45], plays a role on \( A^p_\omega \) similar to that of the Hardy-Littlewood maximal function on the Hardy space \( H^p \). Analogously, the conventional norm in \( A^p_\omega \) is equivalent to a norm expressed in terms of certain square area functions. These results illustrate in a very concrete manner the significant difference between the function-theoretic properties of the classical weighted Bergman space \( A^p_\omega \) and those of \( A^p_\omega \) induced by a rapidly increasing weight \( \omega \).

We will put an important part of our attention to the integral operator
\[ T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta)\,d\zeta, \quad z \in \mathbb{D}, \]
induced by an analytic function \( g \) on \( \mathbb{D} \). This operator will allow us to further underscore the transition phenomena from \( A^p_\omega \) to \( H^p \) through \( A^p_\omega \) with \( \omega \in \mathcal{I} \). The choice \( g(z) = z \) gives the usual Volterra operator and the Cesàro operator is obtained when \( g(z) = -\log(1-z) \). The study of integral operators on spaces of analytic functions merges successfully with other areas of mathematics, such as the theory of univalent functions, factorization theorems, harmonic analysis and differential equations. Pommerenke was probably one of the first authors to consider the operator \( T_g \) [68]. However, an extensive study of this operator was initiated by the seminal works by Aleman, Cima and Siskakis [7, 10, 11]. In all these works classical function spaces such as BMOA and the Bloch space \( \mathcal{B} \) arise in a natural way. It is known that embedding-type theorems and equivalent norms in terms of the first derivative have been key tools in the study of the integral operator. Therefore the study of \( T_g \) has also lead developments that evidently have many applications in other branches of the operator theory on spaces of analytic functions. Recently, the spectrum of \( T_g \) on the Hardy space \( H^p \) [9] and the classical weighted Bergman space \( A^p_\omega \) [8] has been studied. The approach used in these works reveals, in particular, a strong connection between a description of the resolvent set of \( T_g \) on \( H^p \) and the classical theory of the Muckenhoupt weights. In the case of the classical weighted Bergman space \( A^p_\omega \), the Bekollé-Bonami weights take the role of Muckenhoupt weights.

The approach we take to the study of the boundedness of the integral operator \( T_g \) requires, among other things, a factorization of \( A^p_\omega \)-functions. In Chapter 3 we establish the required factorization by using a probabilistic method introduced by Horowitz [47]. We prove that if \( \omega \) is a weight (not necessarily radial) such that
\[ \omega(z) \asymp \omega(\zeta), \quad z \in \Delta(\zeta, r), \quad \zeta \in \mathbb{D}, \]
where \( \Delta(\zeta, r) \) denotes a pseudohyperbolic disc, and polynomials are dense in \( A^p_\omega \), then each \( f \in A^p_\omega \) can be represented in the form \( f = f_1 \cdot f_2 \), where \( f_1 \in A^{p_1}_\omega \), \( f_2 \in A^{p_2}_\omega \) and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \), and the following norm estimates hold
\[ \|f_1\|_{A^{p_1}_\omega} \cdot \|f_2\|_{A^{p_2}_\omega} \leq \frac{p}{p_1} \|f_1\|_{A^{p_1}_\omega}^{p_1} + \frac{p}{p_2} \|f_2\|_{A^{p_2}_\omega}^{p_2} \leq C(p_1, p_2, \omega)\|f\|_{A^p_\omega}^p. \]
These estimates achieve particular importance when we recognize that under certain additional hypothesis on the parameters \( p, p_1, \) and \( p_2 \), the constant in \((\xi)\) only depends on \( p_1 \). This allows us to describe those analytic symbols \( g \) such that \( T_g : A^p_\omega \rightarrow A^p_\omega \) is bounded, provided \( 0 < q < p < \infty \) and \( \omega \in \mathcal{I} \) satisfies \((\xi)\).
By doing this we avoid the use of interpolation theorems and arguments based on
Kinchine’s inequality, that are often employed when solving this kind of problems
on classical spaces of analytic functions on \( \mathbb{D} \). The techniques used to establish
the above-mentioned factorization result permit us to show that each subset of an
\( A_p^\omega \)-zero set is also an \( A_p^\omega \)-zero set. Moreover, we will show that the \( A_p^\omega \)-zero sets
depend on \( p \) whenever \( \omega \in \mathcal{I} \cup \mathcal{R} \). This will be done by estimating the growth of
the maximum modulus of certain infinite products whose zero distribution depends
on both \( p \) and \( \omega \). We will also briefly discuss the zero distribution of functions in
the Bergman-Nevanlinna class \( BN_\omega \) that consists of those analytic functions in \( \mathbb{D} \) for which
\[
\int_{\mathbb{D}} \log^+ |f(z)| \omega(z) \, dA(z) < \infty.
\]
Results related to this discussion will be used in Chapter 7 when the oscillation of
solutions of linear differential equations in the unit disc is studied.

In Chapter 4 we first equip \( A_p^\omega \) with several equivalent norms inherited from
different \( H^p \)-norms through integration. Those ones that are obtained via the
classical Fefferman-Stein estimate or in terms of a non-tangential maximal function
related to lens type regions with vertexes at points in \( \mathbb{D} \), appear to be the most
useful for our purposes. Here, we also prove that it is not possible to establish a
Littlewood-Paley type formula if \( \omega \in \mathcal{I} \) unless \( p = 2 \). In Section 4.2 we characterize
those analytic symbols \( g \) on \( \mathbb{D} \) such that \( T_g : A_p^\omega \to A_q^\omega \), \( 0 < p, q < \infty \), is bounded
or compact. The case \( q > p \) does not give big surprises because essentially standard
techniques work yielding a condition on the maximum modulus of \( g' \). This is no
longer true if \( q = p \). Indeed, we will see that \( T_g : A_p^\omega \to A_p^\omega \) is bounded exactly
when \( g \) belongs to the space \( C^1(\omega^*) \) that consists of those analytic functions on \( \mathbb{D} \) for which
\[
||g||_{C^1(\omega^*)}^2 = |g(0)|^2 + \sup_{I \subseteq \mathbb{T}} \int_{S(I)} |g'(z)|^2 \omega^*(z) \, dA(z) \frac{\omega(S(I))}{\omega(I)} < \infty,
\]
where
\[
\omega^*(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} s \, ds, \quad z \in \mathbb{D} \setminus \{0\}.
\]
Therefore, as in the case of \( H^p \) and \( A_p^\omega \), the boundedness (and the compactness)
is independent of \( p \). It is also worth noticing that the above \( C^1(\omega^*) \)-norm has
all the flavor of the known Carleson measure characterizations of BMOA and \( \mathcal{B} \).
Both of these spaces admit the important and very powerful property of conformal
invariance. In fact, this invariance plays a fundamental role in the proofs of the
descriptions of when \( T_g \) is bounded on either \( H^p \) or \( A_p^\omega \). In contrast to BMOA
and \( \mathcal{B} \), the space \( C^1(\omega^*) \) is not necessarily conformally invariant if \( \omega \) is rapidly
increasing, and therefore we will employ different techniques. In Section 4.3 we
will show in passing that the methods used are adaptable to the Hardy spaces,
and since they also work for \( A_p^\omega \) when \( \omega \) is regular, we consequently will obtain
as a by-product a unified proof for the classical results on the boundedness and
compactness of \( T_g \) on \( H^p \) and \( A_p^\omega \). Chapter 5 is devoted to the study of the space
\( C^1(\omega^*) \) and its “little oh” counterpart \( C^0_\omega(\omega^*) \). In particular, here we will prove that
if \( \omega \in \mathcal{I} \) admits certain regularity, then \( C^1(\omega^*) \) is not conformally invariant, and
further, the strict inclusions
\[
\text{BMOA} \subsetneq C^1(\omega^*) \subsetneq \mathcal{B}
\]
are valid. Moreover, we will show, among other things, that $C^1_0(\omega^*)$ is the closure of polynomials in $C^1(\omega^*)$.

Chapter 6 offers a complete description of those analytic symbols $g$ in $\mathbb{D}$ for which the integral operator $T_g$ belongs to the Schatten $p$-class $S_p(A^2_\omega)$, where $\omega \in \mathcal{I} \cup \mathcal{R}$. If $p > 1$, then $T_g \in S_p(A^2_\omega)$ if and only if $g$ belongs to the analytic Besov space $B_p$, and if $0 < p \leq 1$, then $T_g \in S_p(A^2_\omega)$ if and only if $g$ is constant. It is appropriate to mention that these results are by no means unexpected. This is due to the fact that the operators $T_g$ in both $S_p(H^2)$ and $S_p(A^2_\omega)$ are also characterized by the condition $g \in B_p$, provided $p > 1$. What makes this chapter interesting is the proofs which are carried over in a much more general setting. Namely, we will study the Toeplitz operator, induced by a complex Borel measure and a reproducing kernel, in certain Dirichlet type spaces that are induced by $\omega^*$, in the spirit of Luecking [57]. Our principal findings on this operator are gathered in a single theorem at the end of Chapter 6.

In Chapter 7 we will study linear differential equations with solutions in either the weighted Bergman space $A^p_\omega$ or the Bergman-Nevalinna class $BN_\omega$. Our primary interest is to relate the growth of coefficients to the growth and the zero distribution of solutions. In Section 7.1 we will show how results and techniques developed in the presiding chapters can be used to find a set of sufficient conditions for the analytic coefficients of a linear differential equation of order $k$ forcing all solutions to the weighted Bergman space $A^p_\omega$. Since the zero distribution of functions in $A^p_\omega$ is studied in Chapter 3, we will also obtain new information on the oscillation of solutions. In Section 7.2 we will see that it is natural to measure the growth of the coefficients by the containment in the weighted Bergman spaces depending on $\omega$, when all solutions belong to the Bergman-Nevalinna class $BN_\omega$. In particular, we will establish a one-to-one correspondence between the growth of coefficients, the growth of solutions and the zero distribution of solutions whenever $\omega$ is regular. In this discussion results from the Nevanlinna value distribution theory are explicitly or implicitly present in many instances. Apart from tools commonly used in the theory of complex differential equations in the unit disc, Chapter 7 also relies strongly on results and techniques from Chapters 1–5, and is therefore unfortunately a bit hard to read independently.

Chapter 8 is devoted to further discussion on topics that this monograph does not cover. We will briefly discuss $q$-Carleson measures for $A^p_\omega$ when $q < p$, generalized area operators as well as questions related to differential equations and the zero distribution of functions in $A^p_\omega$. We include few open problems that are particularly related to the special features of the weighted Bergman spaces $A^p_\omega$ induced by rapidly increasing weights.

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CHAPTER 1

Basic Notation and Introduction to Weights

In this chapter we first define the weighted Bergman spaces and the classical Hardy spaces of the unit disc and fix the basic notation. Then we introduce the classes of radial and non-radial weights that are considered in the monograph, show relations between them, and prove several lemmas on weights that are instrumental for the rest of the monograph.

1.1. Basic notation

Let \( H(D) \) denote the algebra of all analytic functions in the unit disc \( D = \{ z : |z| < 1 \} \) of the complex plane \( \mathbb{C} \). Let \( T \) be the boundary of \( D \), and let \( D(a, r) = \{ z : |z - a| < r \} \) denote the Euclidean disc of center \( a \in \mathbb{C} \) and radius \( r \in (0, \infty) \). A function \( \omega : D \to (0, \infty) \), integrable over \( D \), is called a weight function or simply a weight. It is radial if \( \omega(z) = \omega(|z|) \) for all \( z \in D \). For \( 0 < p < \infty \) and a weight \( \omega \), the weighted Bergman space \( A^p_\omega \) consists of those \( f \in H(D) \) for which

\[
\|f\|_{A^p_\omega} = \int_D |f(z)|^p \omega(z) \, dA(z) < \infty,
\]

where \( dA(z) = \frac{dx \, dy}{\pi} \) is the normalized Lebesgue area measure on \( D \). As usual, we write \( A^p_\omega \) for the classical weighted Bergman space induced by the standard radial weight \( \omega(z) = (1 - |z|^2)^\alpha \), \(-1 < \alpha < \infty\). For \( 0 < p \leq \infty \), the Hardy space \( H^p \) consists of those \( f \in H(D) \) for which

\[
\|f\|_{H^p} = \lim_{r \to 1^-} M_p(r, f) < \infty,
\]

where

\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty,
\]

and

\[
M_\infty(r, f) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.
\]

For the theory of the Hardy and the classical weighted Bergman spaces, see [28, 29, 31, 41, 84].

Throughout the monograph, the letter \( C = C(\cdot) \) will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation \( a \lesssim b \) if there exists a constant \( C = C(\cdot) > 0 \) such that \( a \leq Cb \), and \( a \gtrsim b \) is understood in an analogous manner. In particular, if \( a \lesssim b \) and \( a \gtrsim b \), then we will write \( a \asymp b \).
1.2. Regular and rapidly increasing weights

The distortion function of a radial weight \( \omega : [0, 1) \to (0, \infty) \) is defined by
\[
\psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(s) \, ds, \quad 0 \leq r < 1,
\]
and was introduced by Siskakis in \([80]\). A radial weight \( \omega \) is called regular, if \( \omega \) is continuous and its distortion function satisfies
\[
\psi_\omega(r) \asymp (1 - r), \quad 0 \leq r < 1. \tag{1.1}
\]
The class of all regular weights is denoted by \( \mathcal{R} \). We will show in Section 1.4 that if \( \omega \in \mathcal{R} \), then for each \( s \in [0, 1) \) there exists a constant \( C = C(s, \omega) > 1 \) such that
\[
C^{-1} \omega(t) \leq \omega(r) \leq C \omega(t), \quad 0 \leq r \leq t \leq r + s(1 - r) < 1. \tag{1.2}
\]
It is easy to see that (1.2) implies
\[
\psi_\omega(r) \geq C(1 - r), \quad 0 \leq r < 1, \tag{1.3}
\]
for some constant \( C = C(\omega) > 0 \). However, (1.2) does not imply the existence of \( C = C(\omega) > 0 \) such that
\[
\psi_\omega(r) \leq C(1 - r), \quad 0 \leq r < 1, \tag{1.4}
\]
as is seen by considering the weights
\[
v_\alpha(r) = \left( (1 - r) \left( \log \frac{e}{1 - r} \right)^\alpha \right)^{-1}, \quad 1 < \alpha < \infty. \tag{1.5}
\]
Putting the above observations together, we deduce that the regularity of a radial continuous weight \( \omega \) is equivalently characterized by the conditions (1.2) and (1.4).

As to concrete examples, we mention that every \( \omega(r) = (1 - r)^\alpha, -1 < \alpha < \infty \), as well as all the weights in \([11, (4.4)-(4.6)]\) are regular.

The good behavior of regular weights can be broken in different ways. On one hand, the condition (1.2) implies, in particular, that \( \omega \) can not decrease very fast. For example, the exponential type weights
\[
\omega_{\gamma, \alpha}(r) = (1 - r)^\gamma \exp\left( \frac{-c}{(1 - r)^\alpha} \right), \quad \gamma \geq 0, \quad \alpha > 0, \quad c > 0, \tag{1.6}
\]
satisfy neither (1.2) nor (1.3). Meanwhile the weights \( \omega_{\gamma, \alpha} \) are monotone near 1, the condition (1.2) clearly also requires local smoothness and therefore the regular weights can not oscillate too much. We will soon come back to such oscillatory weights. On the other hand, we will say that a radial weight \( \omega \) is rapidly increasing, denoted by \( \omega \in \mathcal{I} \), if it is continuous and
\[
\lim_{r \to 1^-} \psi_\omega(r) = \infty. \tag{1.7}
\]
It is easy to see that if \( \omega \) is a rapidly increasing weight, then \( A^\beta_p \subset A^\beta_p \) for any \( \beta > -1 \), see Section 1.4. Typical examples of rapidly increasing weights are \( v_\alpha \), defined in (1.5), and
\[
\omega(r) = \left( (1 - r) \prod_{n=1}^N \log_n \frac{\exp_n 0}{1 - r} \left( \frac{\exp N+1 0}{1 - r} \right)^\alpha \right)^{-1} \tag{1.8}
\]
for all \( 1 < \alpha < \infty \) and \( N \in \mathbb{N} = \{1, 2, \ldots\} \). Here, as usual, \( \log_n x = \log(\log_{n-1} x) \), \( \log_1 x = \log x \), \( \exp_n x = \exp(\exp_{n-1} x) \) and \( \exp_1 x = e^x \). It is worth noticing that
if \( \omega \in \mathcal{I} \), then \( \sup_{r \leq t} \psi_{\omega}(r)/(1-r) \) can grow arbitrarily fast as \( t \to 1^- \). See the weight defined in (1.10) and Lemma 1.5 in Section 1.4.

In this study we are particularly interested in the weighted Bergman space \( A^p_\omega \) induced by a rapidly increasing weight \( \omega \), although most of the results are obtained under the hypotheses “\( \omega \in \mathcal{I} \cup \mathcal{R} \)”. However, there are proofs in which more regularity is required for \( \omega \in \mathcal{I} \). This is due to the fact that rapidly increasing weights may admit a strong oscillatory behavior. Indeed, consider the weight

\[
\omega(r) = \left| \sin \left( \log \frac{1}{1-r} \right) \right| v_\alpha(r) + 1, \quad 1 < \alpha < \infty.
\]

(1.9)

It is clear that \( \omega \) is continuous and \( 1 \leq \omega(r) \leq v_\alpha(r) + 1 \) for all \( r \in [0, 1] \). Moreover, if

\[
1 - e^{-\left( \frac{\pi}{4} + n\pi \right)} \leq r \leq 1 - e^{-\left( \frac{3\pi}{4} + n\pi \right)}, \quad n \in \mathbb{N} \cup \{0\},
\]

then \( -\log(1-r) \in \left[ \frac{\pi}{4} + n\pi, \frac{3\pi}{4} + n\pi \right] \), and thus \( \omega(r) \simeq v_\alpha(r) \) in there. Let now \( r \in (0, 1) \), and fix \( N = N(r) \) such that \( 1 - e^{-\left( N - 1 \right)\pi} < r \leq 1 - e^{-N\pi} \). Then

\[
\int_r^1 \omega(s) \, ds \gtrsim \sum_{n=N}^{\infty} \frac{1 - e^{-n\pi - \frac{n\pi}{2}}}{1 - e^{-n\pi}} \int_r^1 v_\alpha(s) \, ds = \sum_{n=N}^{\infty} \frac{1}{(\frac{\pi}{4} + n\pi)^{\alpha-1}} - \frac{1}{(\frac{3\pi}{4} + n\pi)^{\alpha-1}} \int_r^1 v_\alpha(s) \, ds,
\]

and it follows that \( \omega \in \mathcal{I} \). However,

\[
\frac{\omega(1 - e^{-n\pi - \frac{n\pi}{2}})}{\omega(1 - e^{-n\pi})} = v_\alpha(1 - e^{-n\pi - \frac{n\pi}{2}}) + 1 \to \infty, \quad n \to \infty,
\]

yet

\[
\frac{1 - (1 - e^{-n\pi - \frac{n\pi}{2}})}{1 - (1 - e^{-n\pi})} = e^{-\frac{n\pi}{2}} \in (0, 1)
\]

for all \( n \in \mathbb{N} \). Therefore \( \omega \) does not satisfy (1.2). Another bad-looking example in the sense of oscillation is

(1.10)

\[
\omega(r) = \left| \sin \left( \log \frac{1}{1-r} \right) \right| v_\alpha(r) + \frac{1}{e^{\alpha^{1/\beta}}}, \quad 1 < \alpha < \infty,
\]

which belongs to \( \mathcal{I} \), but does not satisfy (1.2) by the reasoning above. Moreover, by passing through the zeros of the \( \sin \) function, we see that

\[
\liminf_{r \to 1^-} \omega(r) e^{\frac{1}{1-r}} = 1.
\]

Our last example on oscillatory weights is

(1.11)

\[
\omega(r) = \left| \sin \left( \log \frac{1}{1-r} \right) \right| (1-r)^\alpha + (1-r)^\beta,
\]

where \(-1 < \alpha < \beta < \infty\). Obviously, \((1-r)^\beta \lesssim \omega(r) \lesssim (1-r)^\alpha\), so \( A^p_\alpha \subset A^p_\omega \subset A^p_\beta \). However, \( \omega \notin \mathcal{I} \) because the limit in (1.7) does not exist, and \( \omega \notin \mathcal{R} \) because \( \omega \) neither satisfies (1.2) nor (1.4), yet \( \omega \) obeys (1.3).

In the case when more local regularity is required for \( \omega \in \mathcal{I} \) in the proof, we will consider the class \( \mathcal{I}_\star \) of those \( \omega \in \mathcal{I} \) that satisfy (1.2).
1.3. Bekollé-Bonami and invariant weights

The Carleson square $S(I)$ associated with an interval $I \subset \mathbb{T}$ is the set $S(I) = \{re^{i\theta} \in \mathbb{D} : e^{i\theta} \in I$, $1 - |I| \leq r < 1\}$, where $|E|$ denotes the Lebesgue measure of the measurable set $E \subset \mathbb{T}$. For our purposes it is also convenient to define for each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{e^{i\theta} : |\arg(ae^{-i\theta})| \leq \frac{1 - |a|}{2}\}$, and denote $S(a) = S(I_a)$. Let $1 < p_0, p'_0 < \infty$ such that $\frac{1}{p_0} + \frac{1}{p'_0} = 1$, and let $\eta > -1$. A weight $\omega : \mathbb{D} \to (0, \infty)$ satisfies the Bekollé-Bonami $B_{p_0}(\eta)$-condition, denoted by $\omega \in B_{p_0}(\eta)$, if there exists a constant $C = C(p_0, \eta, \omega) > 0$ such that

\begin{equation}
\left( \int_{S(I)} \omega(z)(1 - |z|)^\eta \, dA(z) \right)^{\frac{1}{p_0}} \leq C|I|^{(2+\eta)p_0}
\end{equation}

for every interval $I \subset \mathbb{T}$. Bekollé and Bonami introduced these weights in [16, 17], and showed that $\omega \in B_{p_0}(\eta)$ if and only if the Bergman projection

$$P_\eta(f)(z) = (\eta + 1) \int_{\mathbb{D}} \frac{f(\xi)}{(1 - \xi \overline{z})^{2+\eta}}(1 - |\xi|^2)^\eta \, dA(\xi)$$

is bounded from $L_{\omega}^{p_0}$ to $A_{\omega}^{p_0}$ [17]. This equivalence allows us to identify the dual space of $A_{\omega}^{p_0}$ with $A_{\omega}^{0'}$. In the next section we will see that if $\omega \in \mathcal{R}$, then for each $p_0 > 1$ there exists $\eta = \eta(p_0, \omega) > -1$ such that $\frac{\omega(z)}{(1 - |z|)^\eta}$ belongs to $B_{p_0}(\eta)$. However, this is no longer true if $\omega \in \mathcal{I}$.

There is one more class of weights that we will consider. To give the definition, we need to recall several standard concepts. For $a \in \mathbb{D}$, define $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$. The automorphism $\varphi_a$ of $\mathbb{D}$ is its own inverse and interchanges the origin and the point $a \in \mathbb{D}$. The pseudohyperbolic and hyperbolic distances from $z$ to $w$ are defined as $d(z, w) = |\varphi_z(w)|$ and

$$d_h(z, w) = \frac{1}{2} \log \frac{1 + d(z, w)}{1 - d(z, w)}, \quad z, w \in \mathbb{D},$$

respectively. The pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius $r \in (0, 1)$ is denoted by $\Delta(a, r) = \{z : d(a, z) < r\}$. It is clear that $\Delta(a, r)$ coincides with the hyperbolic disc $\Delta_h(a, R) = \{z : d_h(a, z) < R\}$, where $R = \frac{1}{2} \log \frac{1 + r}{1 - r} \in (0, \infty)$.

The class $\mathcal{I}nv$ of invariant weights consists of those weights $\omega$ (that are not necessarily radial neither continuous) such that for some (equivalently for all) $r \in (0, 1)$ there exists a constant $C = C(r) \geq 1$ such that

$$C^{-1} \omega(a) \leq \omega(z) \leq C \omega(a)$$

for all $z \in \Delta(a, r)$. In other words, $\omega \in \mathcal{I}nv$ if $\omega(z) \propto \omega(a)$ in $\Delta(a, r)$. It is immediate that radial invariant weights are neatly characterized by the condition (1.2), and thus $\mathcal{I} \cap \mathcal{I}nv = \mathcal{I}$ and $\mathcal{R} \cap \mathcal{I}nv = \mathcal{R}$. To see an example of a radial weight that just fails to satisfy (1.2), consider

$$\omega(r) = (1 - r) \log_n \left( \frac{\exp_n 0}{1 - r} \right) = \exp \left( - \log \left( \frac{1}{1 - r} \right) \cdot \log_n \left( \frac{\exp_n 0}{1 - r} \right) \right), \quad n \in \mathbb{N},$$
It is easy to see that
\[
\frac{\psi_\omega(r)}{1 - r} \approx \frac{1}{\log_n \left( \frac{1}{1 - r} \right)}, \quad r \to 1^-,
\]
and hence \( \omega \notin I \cup R \). For an example of a weight which is not rapidly decreasing neither belongs to \( I \cup R \), see (3.28). In Chapter 3 we will see that the class of those invariant weights \( \omega \) for which polynomials are dense in \( A_p^\omega \), form a natural setting for the study of factorization of functions in \( A_p^\omega \).

### 1.4. Lemmas on weights

In this section we will prove several lemmas on different weights. These results do not only explain the relationships between different classes of weights, but are also instrumental for the rest of the monograph. The first three lemmas deal with the integrability of regular and rapidly increasing weights.

**Lemma 1.1.**

(i) Let \( \omega \in R \). Then there exist constants \( \alpha = \alpha(\omega) > 0 \) and \( \beta = \beta(\omega) \geq \alpha \) such that

\[
(1.13) \quad \left( \frac{1 - r}{1 - t} \right)^\alpha \int_t^1 \omega(s) \, ds \leq \int_t^1 \omega(s) \, ds \leq \left( \frac{1 - r}{1 - t} \right)^\beta \int_t^1 \omega(s) \, ds
\]

for all \( 0 \leq r \leq t < 1 \).

(ii) Let \( \omega \in I \). Then for each \( \beta > 0 \) there exists a constant \( C = C(\beta, \omega) > 0 \) such that

\[
(1.14) \quad \int_r^1 \omega(s) \, ds \leq C \left( \frac{1 - r}{1 - t} \right)^\beta \int_t^1 \omega(s) \, ds, \quad 0 \leq r \leq t < 1.
\]

**Proof.** (i) Let \( \omega \in R \), and let \( C_1 = C_1(\omega) > 0 \) and \( C_2 = C_2(\omega) \geq C_1 \) such that

\[
(1.15) \quad C_1(1 - r) \leq \psi_\omega(r) \leq C_2(1 - r), \quad 0 \leq r < 1.
\]

Then a direct calculation based on the first inequality in (1.15) shows that the differentiable function \( h_{C_1}(r) = \int_r^1 \omega(s) \, ds \) is increasing on \( [0, 1) \). The second inequality in (1.13) with \( \beta = 1/C_1 \) follows from this fact. The first inequality in (1.13) with \( \alpha = 1/C_2 \) can be proved in an analogous manner by showing that \( h_{C_2}(r) \) is decreasing on \( [0, 1) \).

(ii) Let now \( \omega \in I \) and \( \beta > 0 \). By (1.7), there exists \( r_0 \in (0, 1) \) such that \( \psi_\omega(r)/(1-r) \geq \beta^{-1} \) for all \( r \in [r_0, 1) \). As above we deduce that \( h_{1/\beta}(r) \) is increasing on \( [r_0, 1) \), and (1.14) follows.

We make several observations on Lemma 1.1 and its proof.

(i) If \( \omega \in R \) and \( 0 \leq r \leq t \leq r + s(1 - r) < 1 \), then Part (i) implies

\[
\frac{\omega(r)}{\omega(t)} \approx \frac{1 - t}{1 - r} \cdot \frac{\int_r^1 \omega(v) \, dv}{\int_t^1 \omega(v) \, dv} \approx \frac{\int_r^1 \omega(v) \, dv}{\int_t^1 \omega(v) \, dv} \approx 1,
\]

where the constants of comparison depend only on \( s \in [0, 1) \) and \( \beta = \beta(\omega) > 0 \). This proves the important local smoothness (1.2) of regular weights.
(ii) If \( \omega \in \mathcal{R} \), then \( h_{C_1}(r) \) is increasing and \( h_{C_2}(r) \) is decreasing by the proof of Part (i). Therefore

\[
(1.16) \quad (1 - r)^{1/C_1} \lesssim \int_r^1 \omega(s) \, ds = \omega(r) \psi_\omega(r) \lesssim (1 - r)^{1/C_2},
\]

and since \( \psi_\omega(r) \asymp (1 - r) \), we deduce \( A_\omega^p \subset A_\alpha^p \subset A_\beta^p \) for \( \alpha = C_2^{-1} - 1 \) and \( \beta = C_1^{-1} - 1 \). This means that each weighted Bergman space \( A_\omega^p \), induced by a regular weight \( \omega \), lies between two classical weighted Bergman spaces. Of course, if \( \omega \) is a radial continuous weight such that the chain of inclusions above is satisfied for some \(-1 < \alpha < \beta < \infty\), then \( \omega \) does not need to be regular as is seen by considering the oscillatory weight given in (1.11). It is also worth noticing that \( \omega(r)(1 - r)^{-\gamma} \) is a weight for each \( \gamma < 1/C_2 \) by (1.16). Moreover, an integration by parts gives

\[
(1.17) \quad \int_r^1 \omega(s)(1 - s)^{-\gamma} \, ds = (1 - r)^{-\gamma} \int_r^1 \omega(s) \, ds + \gamma \int_r^1 \left( \int_s^1 \omega(t) \, dt \right) (1 - s)^{-\gamma - 1} \, ds
\]

so, by choosing \( \gamma > 0 \) sufficiently small and reorganizing terms, we obtain (1.1) for the weight \( \omega(r)(1 - r)^{-\gamma} \), that is, \( \omega(r)(1 - r)^{-\gamma} \in \mathcal{R} \).

(iii) Let now \( \omega \in \mathcal{I} \), and let \( \beta > -1 \) be fixed. The proof of Part (ii) shows that there exists \( r_0 = r_0(\alpha) \in (\frac{1}{2},1) \) such that \( h_{2/(1+\beta)} \) is increasing on \( [r_0,1) \). Therefore \( \int_r^1 \omega(s) \, ds \gtrsim (1 - r)^{\frac{1+\beta}{2}} \) on \([r_0,1)\). If \( f \in A_\omega^p \), then

\[
\|f\|_{A_\omega^p}^p \gtrsim \int_{D\setminus D(0,r)} |f(z)|^p \omega(z) \, dA(z) \gtrsim M_\omega^p(r,f) \int_r^1 \omega(s) \, ds, \quad r \geq \frac{1}{2},
\]

and hence

\[
\|f\|_{A_\omega^p}^p \lesssim \|f\|_{A_\alpha^p}^p \int_0^1 \frac{(1 - r)^{\beta}}{\int_r^1 \omega(s) \, ds} \, dr \lesssim \int_0^{r_0} \frac{(1 - r)^{\beta}}{\int_r^1 \omega(s) \, ds} \, dr + \int_{r_0}^1 \frac{dr}{(1 - r)^{\frac{1+\beta}{2}}} < \infty.
\]

Therefore \( f \in A_\beta^p \), and we obtain the inclusion \( A_\omega^p \subset A_\beta^p \) for all \( \beta > -1 \). Moreover, by combining the equality in (1.17) and the estimate \( \int_r^1 \omega(s) \, ds \gtrsim (1 - r)^{\frac{1+\beta}{2}} \), established above, for \( \beta = 2\gamma - 1 \), we see that \( \omega(r)(1 - r)^{-\gamma} \) is not a weight for any \( \gamma > 0 \).

(iv) If \( \omega \in \mathcal{I} \) is differentiable and

\[
(1.18) \quad \lim_{r \to 1^-} \frac{\omega'(r)}{\omega^2(r)} \int_r^1 \omega(s) \, ds
\]

exists, then this limit is equal to \( \infty \) by Bernoulli-l'Hôpital theorem. This in turn implies that \( \omega \) is essentially increasing on \([0,1)\), that is, there exists a constant \( C \geq 1 \) such that \( \omega(r) \leq C \omega(s) \) for all \( 0 \leq r \leq s < 1 \). It is also known that if the limit (superior) in (1.18) is finite, then a
Littlewood-Paley type formula exists for all $0 < p < \infty$ \cite{67, 80}, that is, $\|f\|_{L_p^p} \sim \|f\|_{L_p^p}$ for all $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$.

(v) It is easy to see that a radial continuous weight $\omega$ is regular if and only if there exist $-1 < a < b < \infty$ and $r_0 \in (0, 1)$ such that

\[
\omega(r) \left(\frac{1}{1-r}\right)^b \not\to \infty, \quad r \geq r_0, \quad \text{and} \quad \omega(r) \left(\frac{1}{1-r}\right)^a \not\to 0, \quad r \geq r_0.
\]

These weights without the continuity assumption were first studied by Shields and Williams \cite{79} in the range $0 < a < b < \infty$, and they are known as normal weights. Each weight $v_n \in I$ satisfies (1.19) if we allow the number $a$ to attain the value $-1$, which is usually excluded in the definition.

\[\text{Lemma 1.2.} \]

(i) Let $\omega \in \mathcal{R}$. Then there exist constants $\gamma = \gamma(\omega) > 0$, $C_1 = C_1(\gamma, \omega) > 0$ and $C_2 = C_2(\gamma, \omega) > 0$ such that

\[
C_1 \int_t^1 \omega(s) ds \leq \int_0^t \left(\frac{1-t}{1-s}\right)^\gamma \omega(s) ds \leq C_2 \int_t^1 \omega(s) ds, \quad 0 < t < 1.
\]

More precisely, the first inequality is valid for all $\gamma > 0$, and the second one for all $\gamma > \beta$, where $\beta$ is from (1.13).

(ii) Let $\omega \in I$. Then for each $\gamma > 0$ there exists a constant $C = C(\gamma, \omega) > 0$ such that

\[
\int_0^t \left(\frac{1-t}{1-s}\right)^\gamma \omega(s) ds \leq C \int_t^1 \omega(s) ds, \quad 0 < t < 1.
\]

**Proof.** Let $t \in (0, 1)$, and let $N$ be the smallest natural number such that $(1-t)2^N > 1$. Set $t_N = 0$ and $t_n = 1 - 2^n(1-t)$ for $n = 0, \ldots, N-1$. Also, set $\omega(s) = 0$ for $s < 0$.

(i) Let $\omega \in \mathcal{R}$ and $\gamma > 0$. Then, by Lemma 1.1(i), there exists $\alpha = \alpha(\omega) > 0$ such that

\[
\int_0^t \left(\frac{1-t}{1-s}\right)^\gamma \omega(s) ds = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(\frac{1-t}{1-s}\right)^\gamma \omega(s) ds
\]

\[
\geq \sum_{n=0}^{N-1} 2^{-(n+1)\gamma} \left(\int_{t_{n+1}}^1 \omega(s) ds - \int_{t_n}^1 \omega(s) ds\right)
\]

\[
\geq \sum_{n=0}^{N-1} 2^{-(n+1)\gamma} (2^\alpha - 1) \int_{t_n}^1 \omega(s) ds
\]

\[
\geq \frac{2^\alpha - 1}{2^\gamma} \int_t^1 \omega(s) ds,
\]

and thus the first inequality in (1.20) with $C_1 = \frac{2^\alpha - 1}{2^\gamma}$ is proved. To see the second one, we may argue as above to obtain

\[
\int_0^t \left(\frac{1-t}{1-s}\right)^\gamma \omega(s) ds \leq (2^\beta - 1) \sum_{n=0}^{N-1} 2^{-n\gamma} \int_{t_n}^1 \omega(s) ds
\]

\[
\leq (2^\beta - 1) \sum_{n=0}^{\infty} 2^{(\beta-\gamma)} \int_t^1 \omega(s) ds.
\]
This gives the second inequality in (1.20) for all $\gamma > \beta$.

(ii) This can be proved by arguing as above and by using Lemma 1.1(ii). \hfill \Box

**Lemma 1.3.** If $\omega \in I \cup R$, then

$$\int_0^1 s^\gamma \omega(s) \, ds \asymp \int_{1-\frac{1}{x}}^1 \omega(s) \, ds, \quad x \in [1, \infty).$$

**Proof.** If $x = 1$, then the assertion follows by the inequalities

$$\int_0^1 s \omega(s) \, ds \leq \int_0^1 s \omega(s) \, ds \leq \left( 2 + \frac{\int_0^\frac{1}{2} \omega(s) \, ds}{\int_0^\frac{1}{2} s \omega(s) \, ds} \right) \int_0^1 s \omega(s) \, ds.$$  \hfill (1.22)

For $x > 1$ it suffices to prove

$$\int_0^1 s^\gamma \omega(s) \, ds \lesssim \int_{1-\frac{1}{x}}^1 \omega(s) \, ds.$$

To see this, let $\gamma = \gamma(\omega) > 0$ be the constant in Lemma 1.2. A simple calculation shows that

$$s^{\gamma-1}(1-s)^\gamma \leq \left( \frac{x-1}{x-1+\gamma} \right)^{\gamma-1} \left( \frac{\gamma}{x-1+\gamma} \right)^\gamma \leq \left( \frac{x-1}{\gamma} \right)^\gamma$$

for all $s \in [0, 1]$. Therefore Lemma 1.2, with $t = 1 - \frac{1}{x}$, yields

$$\int_0^1 s^\gamma \omega(s) \, ds \leq \left( \frac{x-1}{x-1+\gamma} \right)^\gamma \int_0^1 \frac{\omega(s)}{x^\gamma(1-s)^\gamma} \, ds \lesssim \int_{1-\frac{1}{x}}^1 \omega(s) \, ds.$$  \hfill (ii)

This finishes the proof. \hfill \Box

The next lemma shows that a continuous radial weight $\omega$ that satisfies (1.2) is regular if and only if it is a Bekollé-Bonami weight. Moreover, Part (iii) quantifies in a certain sense the self-improving integrability of radial weights.

**Lemma 1.4.**

(i) If $\omega \in R$, then for each $p_0 > 1$ there exists $\eta = \eta(p_0, \omega) > -1$ such that $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$.

(ii) If $\omega$ is a continuous radial weight such that (1.2) is satisfied and $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$ for some $p_0 > 0$ and $\eta > -1$, then $\omega \in R$.

(iii) For each radial weight $\omega$ and $0 < \alpha < 1$, define

$$\tilde{\omega}(r) = \left( \int_r^1 \omega(s) \, ds \right)^{-\alpha} \omega(r), \quad 0 \leq r < 1.$$

Then $\tilde{\omega}$ is also a weight and $\psi_{\tilde{\omega}}(r) = \frac{1}{1-\alpha} \psi_\omega(r)$ for all $0 \leq r < 1$.

**Proof.** (i) Since each regular weight is radial, it suffices to show that there exists a constant $C = C(p, \eta, \omega) > 0$ such that

$$\left( \int_{1-|I|} \omega(t) \, dt \right)^{\frac{p_0}{\log C}} \lesssim \left( \int_{1-|I|} \omega(t)^\eta \, dt \right)^{\frac{p_0}{\log C}} \leq C|I|^{(1+\eta)p_0}$$

for every interval $I \subset T$. To prove (1.22), set $s_0 = 1 - |I|$ and $s_{n+1} = s_n + s(1-s_n)$, where $s \in (0, 1)$ is fixed. Take $p_0$ and $\eta$ such that $\eta > \frac{\log C}{p_0 \log \frac{1}{p_0} > 0}$, where the
constant $C = C(s, \omega) > 1$ is from (1.2). Then (1.2) yields
\[
\int_{1-|I|}^{1} \omega(t) \frac{p_0}{\eta} (1-t)^{\eta} dt \leq \sum_{n=0}^{\infty} (1-s_n) \int_{s_n}^{1} \omega(t) \frac{p_0}{\eta} dt \leq \sum_{n=0}^{\infty} (1-s_n) \int_{s_n}^{1} \omega(t) \frac{p_0}{\eta} dt \leq |I|^{\eta+1} \omega(1-|I|) \frac{p_0}{\eta} \sum_{n=0}^{\infty} (1-s_n) \int_{s_n+1}^{s_n+1} \omega(t) dt \leq C \int_{0}^{1} \omega(t) \frac{p_0}{\eta} dt \leq C(p_0, s, \omega, |I|^{\eta+1} \omega(1-|I|) \frac{p_0}{\eta})}
\]
which together with (1.4) gives (1.22).

(ii) The asymptotic inequality \( \psi_\omega(r) \lesssim (1-r) \) follows by (1.22) and further appropriately modifying the argument in the proof of (i). Since the assumption (1.2) gives \( \psi_\omega(r) \gtrsim (1-r) \), we deduce \( \omega \in \mathcal{R} \).

(iii) If \( 0 \leq r < t < 1 \), then an integration by parts yields
\[
\int_{t}^{r} \omega(s) \frac{\alpha}{(\int_{s}^{1} \omega(v) dv)^{\alpha}} ds = \left( \int_{t}^{1} \omega(v) dv \right)^{1-\alpha} - \left( \int_{1}^{r} \omega(v) dv \right)^{1-\alpha} + \alpha \int_{t}^{r} \omega(s) \frac{\alpha}{(\int_{s}^{1} \omega(v) dv)^{\alpha}} ds,
\]
from which the assertion follows by letting \( t \to 1^- \). \qed

We note that a routine calculation based on (1.22) shows that \( \frac{\psi_\omega(z)}{|1-|z||}, 1 < \alpha < \infty \), is not a \( B_{p_0}(\eta) \) weight for any \( p_0 > 1 \) and \( \eta > -1 \). Later on we will see that this is actually true for each \( \omega \in \mathcal{I} \).

The next lemma shows that we can find a radial weight \( \omega \) such that the growth of the quotient \( \psi_\omega(r)/(1-r) \) is of any given admissible rate. Here one should observe that the assumptions on the auxiliary function \( h \) are not restrictions because they are necessary conditions for (1.24) to hold, as \( \omega_\lambda \) is a weight and thus satisfies (1.23).

**Lemma 1.5.** Let \( \omega \) be a radial weight. Then
\[
(1.23) \quad \int_{0}^{1} \frac{dr}{\psi_\omega(r)} = \infty.
\]
Moreover, for each function \( \lambda : [0, 1) \to (0, \infty) \) such that
\[
h(r) = \int_{0}^{r} \frac{ds}{\lambda(s)(1-s)}
\]
eexists, for all \( 0 < r < 1 \) and \( \lim_{r \to 1^-} h(r) = \infty \), there exists a radial weight \( \omega_\lambda \) such that
\[
(1.24) \quad \frac{\psi_\omega_\lambda(r)}{(1-r)} = \lambda(r), \quad 0 < r < 1.
\]
Proof. Clearly,
\[
\int_0^1 \frac{dr}{\psi_\omega(r)} = \int_0^1 \frac{\omega(r)}{\int_r^1 \omega(s) \, ds} \, dr
= \lim_{r \to 1^-} \log \frac{1}{\int_r^1 \omega(s) \, ds} - \log \frac{1}{\int_0^1 \omega(s) \, ds} = \infty
\]
for all radial weights \(\omega\). Moreover, if \(\lambda\) satisfies the hypothesis, then each radial weight

\[
\omega_\lambda(r) = C \exp \left( -\int_0^r \frac{ds}{\lambda(s)(1-s)} \right), \quad C > 0,
\]

solves the integral equation (1.24).

Another way to interpret the last assertion in Lemma 1.5 is to notice that

\[
\omega(r) = \left( \int_0^1 \omega(s) \, ds \right) \cdot \exp \left( -\int_0^r \frac{ds}{\psi_\omega(s)} \psi_\omega(r) \right),
\]

provided \(\omega\) is a radial weight.

For each radial weight \(\omega\), we define its associated weight \(\omega^\ast\) by

\[
\omega^\ast(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} \, ds, \quad z \in \mathbb{D} \setminus \{0\}.
\]

Even though \(\omega \in \mathcal{I}\) might have a bad oscillatory behavior, as we saw in Section 1.2, its associated weight \(\omega^\ast\) is regular by Lemma 1.7 below. We will also consider the non-tangential regions

(1.25) \[ \Gamma(u) = \left\{ z \in \mathbb{D} : |\theta - \arg z| < \frac{1}{2} \left( 1 - \frac{|z|}{r} \right) \right\}, \quad u = re^{i\theta} \in \mathbb{D} \setminus \{0\}, \]

and the tents

(1.26) \[ T(z) = \{ u \in \mathbb{D} : z \in \Gamma(u) \}, \quad z \in \mathbb{D}, \]

which are closely interrelated. Figure 1 illustrates how the sets \(\Gamma(z)\) and \(T(z)\) change when \(z\) varies.

Lemma 1.6 shows that \(\omega(S(z)) \asymp \omega(T(z)) \asymp \omega^\ast(z)\), as \(|z| \to 1^-\), provided \(\omega \in \mathcal{I} \cup \mathcal{R}\). Here, as usual, \(\omega(E) = \int_E \omega(z) \, dA(z)\) for each measurable set \(E \subset \mathbb{D}\).

Lemma 1.6. (i) If \(\omega\) is a radial weight, then

(1.27) \[ \omega(T(z)) \asymp \omega^\ast(z), \quad |z| \geq \frac{1}{2}. \]

(ii) If \(\omega \in \mathcal{I} \cup \mathcal{R}\), then

(1.28) \[ \omega(T(z)) \asymp \omega(S(z)), \quad z \in \mathbb{D}. \]

Proof. The first assertion (1.27) is clear by the definitions of the region \(T(z)\) and the associated weight \(\omega^\ast\). Moreover, a routine application of Bernoulli’s-Hôpital theorem, exploiting the relation \(\psi_\omega(r) \asymp (1 - r)\) for \(\omega \in \mathcal{R}\) and the definition (1.7) for \(\omega \in \mathcal{I}\), shows (1.28).

For \(\alpha \in \mathbb{R}\) and a radial weight \(\omega\), set \(\omega_\alpha(z) = (1 - |z|)^\alpha \omega(z)\). We will see next that \(\omega_\alpha^\ast(z) = (1 - |z|)^\alpha \omega^\ast(z)\) is regular whenever \(\alpha > -2\).
Lemma 1.7. If $0 < \alpha < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$, then $\omega_{\alpha-2}^* \in \mathcal{R}$ and $(\omega_{\alpha-2}^*)^*(z) \asymp \omega_{\alpha}^*(z)$ for all $|z| \geq \frac{1}{2}$.

Proof. It suffices to show that $\omega_{\alpha-2}^* \in \mathcal{R}$, since for each $\omega \in \mathcal{R}$ we have

$$\omega^*(r) \asymp \omega(S(r)) = \frac{(1-r)}{\pi} \int_r^1 \omega(s) s \, ds \asymp (1-r)^2 \omega(r), \quad r \geq \frac{1}{2},$$

by Lemma 1.6 and $\psi_\omega(r) \asymp (1-r)$.

To prove $\omega_{\alpha-2}^* \in \mathcal{R}$, note first that the relation $\omega^*(r) \asymp \omega(S(r)) \asymp (1-r) \int_r^1 \omega(s) s \, ds$ and Lemma 1.1 with standard arguments give (1.2) for $\omega_{\alpha-2}^*$. Further, since $\alpha > 0$, we deduce

$$\frac{\int_r^1 \omega_{\alpha-2}^*(s) \, ds}{\omega_{\alpha-2}^*(r)} \asymp \frac{\int_r^1 (1-s)^{\alpha-1} \left( \int_t^1 \omega(t) t \, dt \right) \, ds}{(1-r)^{\alpha-1} \int_r^1 \omega(t) t \, dt} \leq \frac{\int_r^1 (1-s)^{\alpha-1} \, ds}{(1-r)^{\alpha-1} \alpha} = \frac{1-r}{\alpha},$$

and so $\omega_{\alpha-2}^*$ satisfies also (1.4). Thus $\omega_{\alpha-2}^* \in \mathcal{R}$. \hfill \Box

The last lemma is related to invariant weights.

Lemma 1.8. If $\omega \in \mathcal{I}_{\text{nv}}$, then there exists a function $C : \mathbb{D} \rightarrow [1, \infty)$ such that

$$\omega(u) \leq C(z) \omega(\varphi_u(z)), \quad u, z \in \mathbb{D},$$

and

$$\int_{\mathbb{D}} \log C(z) \, dA(z) < \infty.$$
1. BASIC NOTATION AND INTRODUCTION TO WEIGHTS

Let first

\[ C^{-1}\omega(a) \leq \omega(z) \leq C\omega(a) \]

for all \( z \) in the hyperbolic disc \( \Delta_h(a,1) \). For each \( z, u \in \mathbb{D} \), the hyperbolic distance between \( u \) and \( \varphi_u(z) \) is

\[ \varrho_h(u, \varphi_u(z)) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} \]

By the additivity of the hyperbolic distance on the geodesic joining \( u \) and \( \varphi_u(z) \), and (1.32) we deduce

\[ \omega(u) \leq C^{E(\varrho_h(u, \varphi_u(z))) + 1} \omega(\varphi_u(z)) \leq C \left( \frac{1 + |z|}{1 - |z|} \right)^{\log C} \omega(\varphi_u(z)), \]

where \( E(x) \) is the integer such that \( E(x) \leq x < E(x) + 1 \). It follows that (1.30) and (1.31) are satisfied.

Conversely, let \( \omega \) be a weight satisfying (1.30) such that the function \( C \) is uniformly bounded in compact subsets of \( \mathbb{D} \). Then, for each \( r \in (0,1) \), there exists a constant \( C = C(r) > 0 \) such that \( \omega(u) \leq C(r)\omega(z) \) whenever \( |\varphi_u(z)| < r \). Thus \( \omega \in \mathcal{I}_{nuv} \).

Recently, Aleman and Constantin [8] studied the weighted Bergman spaces \( A^p_\omega \) induced by the differentiable weights \( \omega \) that satisfy

\[ |\nabla \omega(z)|(1 - |z|^2) \leq C\omega(z), \quad z \in \mathbb{D}, \]

for some \( C = C(\omega) > 0 \). By using [8, Proposition 3.1] and Lemma 1.8, we see that every such weight is invariant. Moreover, \( v_\alpha \) gives an example of an invariant rapidly increasing weight that does not satisfy (1.33).

1.5. Density of polynomials in \( A^p_\omega \)

It is known that the polynomials are not necessarily dense in the weighted Bergman space \( A^p_\omega \) induced by a weight \( \omega : \mathbb{D} \rightarrow (0, \infty) \), see [39, p. 120], [60, p. 134] or [29, p. 138]). Questions related to polynomial approximation have attracted (and still do) a considerable amount of attention during the last decades, and as far as we know, the problem of characterizing the weights \( \omega \) for which the polynomials are dense in \( A^p_\omega \) remains unsolved. Our interest in this problem comes from the study of factorization of functions in \( A^p_\omega \). The main result of this monograph in that direction is Theorem 3.1 in which the density of the polynomials in \( A^p_\omega \) is taken as an assumption. This because the proof relies on the existence of a dense set of \( A^p_\omega \) whose elements have finitely many zeros only. In this section we discuss known results on approximation by the polynomials in \( A^p_\omega \) and their relations to the classes of weights considered in this monograph.

We begin with recalling that the polynomials are dense in \( A^p_\omega \), whenever \( \omega \) is a radial weight, see [60, p. 134] or [29, p. 138]. This is due to the fact that in this case the dilatation \( f_r(z) = f(rz) \) of \( f \) satisfies

\[ \lim_{r \to 1^-} \|f_r - f\|_{A^p_\omega} = 0, \quad f \in A^p_\omega. \]

Another neat condition which ensures the approximation in \( A^p_\omega \) by dilatations, and therefore by polynomials, is

\[ \omega(z) \leq C(\omega)\omega(rz), \quad r_0 \leq r < 1, \quad r_0 \in (0,1), \]

\[ \lim_{r \to 1^-} \|f_r - f\|_{A^p_\omega} = 0, \quad f \in A^p_\omega. \]
Recall that $M \subset A^p$ is called invariant if $zM \subset M$, and that $f \in A^p$ is a cyclic element if the closure of the set of polynomial multiples of $f$ is the whole space $A^p$. If now $Q$ is a cyclic element of $A^p$ and $\omega$ is defined by $\omega(z) = |Q(z)|^p$, then the polynomials are dense in $A^p_\omega$ by the cyclicity. A natural way to construct cyclic elements of $A^p$ is to choose a function $\phi \in L^q(T)$, where $q \geq \max\{p/2, 1\}$, such that $\log \phi \in L^1(T)$, and consider the associated $H^q$-outer function

\begin{equation}
Q(z) = \exp \left( \frac{1}{2\pi} \int_T \frac{z+\zeta}{\zeta-z} \log \phi(\zeta) \, d\zeta \right).
\end{equation}

Then Beurling’s theorem [18] ensures that $Q$ is cyclic in $H^q$, and hence in $A^p$ as well. However, despite of the Hardy space case there are singular inner functions (in the classical sense) that are cyclic in $A^p$. For a detailed discussion involving Beurling’s theorem and cyclic elements in $A^p$ the reader is invited to see [29, p. 245–260] and the references therein.

The results of Hedberg [39] can also be used to construct explicit examples of non-radial weights $\omega$ such that the polynomials are dense in $A^p_\omega$. We show two direct applications that are stated as lemmas. As usual, an analytic function $f$ is called univalent if it is injective.

**Lemma 1.9.** Let $f$ be a non-vanishing univalent function in $\mathbb{D}$, $0 < \gamma < 1$ and $\omega = |f|^\gamma$. Then the polynomials are dense in $A^p_\omega$ for all $p \geq 1$.

**Proof.** Since $f$ is univalent and zero-free, so is $1/f$, and hence both $f$ and $1/f$ belong to $A^p$ for all $0 < p < 1$. By choosing $\delta > 0$ such that $\gamma(1 + \delta) < 1$ we deduce that both $\omega$ and $\frac{1}{\omega}$ belong to $L^{1+\delta}$. Therefore the polynomials are dense in $A^p_\omega$ by [39, Theorem 2].

**Lemma 1.10.** Let $f$ be a univalent function in $\mathbb{D}$ such that $f(0) = 0$, and $0 < \gamma < \infty$. Let the weight $\omega$ be defined by $\omega(z) = \left| \frac{z}{f(z)} \right|^\gamma$ for all $z \in \mathbb{D}$. Then the polynomials are dense in $A^p_\omega$ for all $p \geq 1$.

**Proof.** The function $\int_0^{2\pi} |f(re^{it})|^{-p} \, dt$ is a decreasing function of $r$ on $(0, 1)$ for all $0 < p < \infty$ by [66, Theorem 4.3.1]. It follows that $\omega \in L^\alpha$ for all $0 < \alpha < \infty$. Moreover, $f(z)/z$ is univalent, and so $\frac{1}{\omega} \in L^\alpha$ for all $\alpha < 1/\gamma$. Therefore the polynomials are dense in $A^p_\omega$ by [39, Theorem 2].

Motivated by factorization of functions in $A^p_\omega$, we are mainly interested in setting up invariant weights $\omega$ such that the polynomial approximation is possible in $A^p_\omega$. Obviously, the radial weights $\omega$ in this class are precisely those that satisfy (1.2). Furthermore, it is easy to see that, for each $0 < \alpha < 2$ and $\xi \in \mathbb{T}$, the non-radial invariant weight $\omega_\xi(z) = \left| \frac{z-\xi}{z+\xi} \right|^{\alpha}$ has also this property by Lemma 1.9. More generally, let $G$ be non-vanishing such that $\log G$ is a Bloch function. It is well-known that then there exists a constant $C > 0$ such that $|\log G(z) - \log G(w)| \leq C q_\alpha(z, w)$ for all $z, w \in \mathbb{D}$. It follows that the weight $\omega$ defined by $\omega(z) = |G(z)|^\gamma$ is invariant for all $\gamma \in \mathbb{R}$. In particular, if $G$ is a non-vanishing univalent function and $0 < \gamma < 1$, then $\log G$ is a Bloch function by the Koebe $\frac{1}{4}$-theorem [38], and hence $\omega(z) = |G(z)|^\gamma$ is invariant and further, the polynomials are dense in $A^p_\omega$ by Lemma 1.9.
Finally, let us consider the class of weights that appears in a paper by Abkar [1]. We will see that also these weights are invariant. To do this, we need to introduce the following concepts. A function \( u \) defined on \( D \) is said to be superbiharmonic if \( \Delta^2 u \geq 0 \), where \( \Delta \) stands for the Laplace operator

\[
\Delta = \Delta_z = \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

in the complex plane \( \mathbb{C} \). The superbiharmonic weights play an essential role in the study of invariant subspaces of the Bergman space \( A^p \). In particular, [1, Theorem 2.6], that is stated as Theorem A below, is a key ingredient in the proof of the fact that any invariant subspace \( M \) of \( A^2 \), that is induced by the zero-set of a function in \( A^2 \), is generated by its extremal function \( \varphi_M \), see [1, Theorem 3.1] and [13].

**Theorem A.** Let \( \omega \) be a superbiharmonic weight such that

\[
\lim_{r \to 1^-} \int_T \omega(r \zeta) \, dm(\zeta) = 0.
\]

Then (1.34) is satisfied for all \( f \in A^p_\omega \). In particular, the polynomials are dense in \( A^p_\omega \).

The proof of Theorem A relies on showing that these type of weights \( \omega \) satisfy (1.35). For a similar result, see [2, Theorem 3.5]. Next, let

\[
\Gamma(u, \zeta) = |u - \zeta|^2 \log \left| \frac{u - \zeta}{1 - u \bar{\zeta}} \right| + (1 - |u|^2)(1 - |z|^2), \quad (u, \zeta) \in \mathbb{D} \times \mathbb{D},
\]

be the biharmonic Green function for the operator \( \Delta^2 \) in \( \mathbb{D} \), and let

\[
H(u, \zeta) = \frac{(1 - |\zeta|^2)^2}{|1 - \zeta u|^2}, \quad (u, \zeta) \in \mathbb{T} \times \mathbb{D},
\]

be the harmonic compensator.

**Lemma 1.11.** Every superbiharmonic weight that satisfies (1.37) is invariant.

**Proof.** The proof relies on the representation formula obtained in [3, Corollary 3.7]. It states that every superbiharmonic weight \( \omega \) that satisfies (1.37) can be represented in the form

\[
\omega(\zeta) = \int_\mathbb{D} \Gamma(u, \zeta) \Delta^2(u) \, dA(u) + \int_T H(u, \zeta) \, d\mu(u), \quad \zeta \in \mathbb{D},
\]

where \( \mu \) is a uniquely determined positive Borel measure on \( \mathbb{T} \). Therefore it suffices to prove the relations

\[
\Gamma(u, \zeta) \approx \Gamma(u, z), \quad z \in \Delta(\zeta, r), \quad u \in \mathbb{D},
\]

and

\[
H(u, \zeta) \approx H(u, z), \quad z \in \Delta(\zeta, r), \quad u \in \mathbb{T},
\]

where the constants of comparison may depend only on \( r \). The definition (1.39) and standard estimates yield (1.41). Moreover, by [1, Lemma 2.2(d)] or [13, Lemma 2.3(a)], we have

\[
\frac{1}{2} \frac{(1 - |\zeta|^2)^2(1 - |u|^2)^2}{|1 - \zeta u|^2} \leq \Gamma(u, \zeta) \leq \frac{(1 - |\zeta|^2)^2(1 - |u|^2)^2}{|1 - \bar{\zeta} u|^2}, \quad (u, \zeta) \in \mathbb{D} \times \mathbb{D},
\]

which easily gives (1.40), and finishes the proof. \( \square \)
CHAPTER 2

Description of $q$-Carleson Measures for $A^p_\omega$

In this chapter we give a complete description of $q$-Carleson measures for $A^p_\omega$ when $0 < p \leq q < \infty$ and $\omega \in I \cup R$. For a given Banach space (or a complete metric space) $X$ of analytic functions on $D$, a positive Borel measure $\mu$ on $D$ is called a $q$-Carleson measure for $X$ if the identity operator $I_d : X \to L^q(\mu)$ is bounded. It is known that a characterization of $q$-Carleson measures for a space $X \subset H(D)$ can be an effective tool, for example, in the study of several questions related to different operators acting on $X$. In this study we are particularly interested in the integral operator

$$T_\omega(f)(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad g \in H(D).$$

We will explain how the decay or growth of a radial weight $\omega$ affects to the study of $q$-Carleson measures for the weighted Bergman space $A^p_\omega$. The influence of $\omega$ appears naturally in the characterizations as well as in the techniques used to obtain them. We will draw special attention to rapidly increasing radial weights, because, to the best of our knowledge, this situation is not well understood yet.

Hastings, Luecking, Oleinik and Pavlov [37, 56, 58, 63], among others, have characterized $q$-Carleson measures for $A^p_\alpha$. In a recent paper Constantin [24] gives an extension of these classical results to the case when $\omega(z)(1-|z|)\eta$ belongs to the class $B^p_\alpha(\eta)$ of Bekollé-Bonami weights. Recall that these weights are not necessarily radial. A characterization of $q$-Carleson measures for $A^p_\omega$, $\omega \in R$, follows by [24, Theorems 3.1 and 3.2] because every regular weight $\omega$ satisfies $\omega(z)(1-|z|)^2 \in B^p_\alpha(\eta)$ for some $p_0 > 1$ and $\eta > -1$ by Lemma 1.4(i). If $\omega$ is a rapidly decreasing radial weight, such as those in (1.6), then different techniques from those that work for $\omega \in R$ must be used to describe $q$-Carleson measures for $A^p_\omega$ [64].

If $\omega \in I$, then the weight $\omega(z)(1-|z|)^2 \eta$ is not a Bekollé-Bonami weight for any $p_0 > 1$ and $\eta > -1$. This is a consequence of Theorem 4.1, Proposition 5.1(D) and [8, Theorem 4.1]. Therefore the above-mentioned results by Constantin [24] do not yield a characterization of $q$-Carleson measures for $A^p_\omega$ when $\omega \in I$. As far as we know these measures have not been characterized yet.

From now on and throughout we will write $\|T\|_{(X,Y)}$ for the norm of an operator $T : X \to Y$, and if no confusion arises with regards to $X$ and $Y$, we will simply write $\|T\|$. The following theorem is the main result of this chapter.

**Theorem 2.1.** Let $0 < p \leq q < \infty$ and $\omega \in I \cup R$, and let $\mu$ be a positive Borel measure on $D$. Then the following assertions hold:
(i) \( \mu \) is a \( q \)-Carleson measure for \( A^p_\omega \) if and only if

\[
\sup_{I \subset T} \frac{\mu(S(I))}{(\omega(S(I)))^{\frac{1}{q}}} < \infty.
\]

Moreover, if \( \mu \) is a \( q \)-Carleson measure for \( A^p_\omega \), then the identity operator \( I_d : A^p_\omega \to L^q(\mu) \) satisfies

\[
\|I_d\|_{(A^p_\omega, L^q(\mu))} \approx \sup_{I \subset T} \frac{\mu(S(I))}{(\omega(S(I)))^{\frac{1}{q}}}.
\]

(ii) The identity operator \( I_d : A^p_\omega \to L^q(\mu) \) is compact if and only if

\[
\lim_{|I| \to 0} \frac{\mu(S(I))}{(\omega(S(I)))^{\frac{1}{q}}} = 0.
\]

Recall that \( I_d : A^p_\omega \to L^q(\mu) \) is compact if it transforms bounded sets of \( A^p_\omega \) to relatively compact sets of \( L^q(\mu) \). Equivalently, \( I_d \) is compact if and only if for every bounded sequence \( \{f_n\} \) in \( A^p_\omega \) some subsequence of \( \{f_n\} \) converges in \( L^q(\mu) \).

One important feature of the proof of Theorem 2.1 is that it relies completely on Carleson squares and do not make any use of techniques related to (pseudo)hyperbolic discs as it is habitual when classical weighted Bergman spaces are considered. Therefore the reasoning gives an alternative proof of the known characterization of \( q \)-Carleson measures for the classical weighted Bergman space \( A^p_\omega \) [29, 84]. Indeed, it is known that, for any \( \omega \in \mathcal{R} \) and \( \beta \in (0, 1) \), there exists a constant \( C = C(\beta, \omega) > 0 \) such that

\[
C^{-1} \omega(D(a, \beta(1 - |a|))) \leq \omega(S(a)) \leq C\omega(D(a, \beta(1 - |a|)))
\]

for all \( a \in \mathbb{D} \), and so \( q \)-Carleson measures for \( \omega \in \mathcal{R} \) can be characterized either in terms of Carleson squares or pseudohyperbolic discs. However, this is no longer true when \( \omega \in \mathcal{I} \). The proof of Theorem 2.1 relies heavily on the properties of the maximal function

\[
M_\omega(\varphi)(z) = \sup_{I \subset T} \frac{1}{\omega(S(I))} \int_{S(I)} |\varphi(\xi)| \omega(\xi) dA(\xi), \quad z \in \mathbb{D},
\]

introduced by Hörmander [45]. Here we must require \( \varphi \in L^1_\omega \) and that \( \varphi(re^{i\theta}) \) is \( 2\pi \)-periodic with respect to \( \theta \) for all \( r \in (0, 1) \). It appears that \( M_\omega(\varphi) \) plays a similar role on \( A^p_\omega \) than the Hardy-Littlewood maximal function on the Hardy space \( H^p \). As a by-product of the proof of Theorem 2.1, we obtain the following result which is of independent interest.

**Corollary 2.2.** Let \( 0 < p \leq q < \infty \) and \( 0 < \alpha < \infty \) such that \( p\alpha > 1 \). Let \( \omega \in \mathcal{I} \cup \mathcal{R} \), and let \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then \( [M_\omega(\cdot)^\frac{1}{\alpha}]^\alpha : L^p_\omega \to L^q(\mu) \) is bounded if and only if \( \mu \) satisfies (2.1). Moreover,

\[
\|{[M_\omega(\cdot)^\frac{1}{\alpha}]^\alpha}\|_{(L^p_\omega, L^q(\mu))} \approx \sup_{I \subset T} \frac{\mu(S(I))}{(\omega(S(I)))^{\frac{1}{q}}}.
\]

The proof of Theorem 2.1 is presented in the next two sections. In Section 2.2.3 we give a counterexample that shows that Theorem 2.1(i) does not remain valid for \( \omega \in \mathcal{I} \) if Carleson squares are replaced by pseudohyperbolic discs.
2.1. Weighted maximal function \( M_\omega \)

We begin with constructing a family of test functions that allows us to show the necessity of conditions (2.1) and (2.2). To do so, we will need the following lemma.

Lemma 2.3. \( i \) If \( \omega \in \mathcal{R} \), then there exists \( \gamma_0 = \gamma_0(\omega) \) such that

\[
\int_D \frac{\omega(z)}{|1-\overline{a}z|^{\gamma+1}} dA(z) \asymp \frac{\int_0^1 \omega(r) dr}{(1-|a|)^\gamma}, \quad a \in \mathbb{D},
\]

for all \( \gamma > \gamma_0 \).

\( ii \) If \( \omega \in \mathcal{I} \), then

\[
\int_D \frac{\omega(z)}{|1-\overline{a}z|^{\gamma+1}} dA(z) \asymp \frac{\int_0^1 \omega(r) dr}{(1-|a|)^\gamma}, \quad a \in \mathbb{D},
\]

for all \( \gamma > 0 \).

Proof. (i) Let \( \omega \in \mathcal{R} \) and let \( \gamma_0 = \beta(\omega) > 0 \) be the constant in Lemma 1.1(i). Then

\[
\int_D \frac{\omega(z)}{|1-\overline{a}z|^{\gamma+1}} dA(z) \asymp \int_0^1 \frac{\omega(r)r}{(1-|a|r)^\gamma} \lesssim \frac{\omega(a)\psi_\omega(|a|)}{(1-|a|)^\gamma-1},
\]

and

\[
\int_D \frac{\omega(z)}{|1-\overline{a}z|^{\gamma+1}} dA(z) \asymp \left( \int_0^{|a|} + \int_{|a|}^1 \right) \frac{\omega(r)r}{(1-|a|r)^\gamma} \lesssim \frac{\omega(a)}{(1-|a|)^\gamma-1}.
\]

by Lemma 1.2. These inequalities give (2.3). Part (ii) can be proved in a similar manner. \( \square \)

Let now \( \gamma = \gamma(\omega) > 0 \) be the constant in Lemma 2.3, and define \( F_{a,p}(z) = \left( \frac{1-|a|^2}{1-|z|^2} \right)^{\frac{\gamma+1}{\gamma}} \). Lemma 2.3 immediately yields the following result.

Lemma 2.4. If \( \omega \in \mathcal{I} \cup \mathcal{R} \), then for each \( a \in \mathbb{D} \) and \( 0 < p < \infty \) there exists a function \( F_{a,p} \in \mathcal{H}(\mathbb{D}) \) such that

\[
|F_{a,p}(z)| \asymp 1, \quad z \in S(a),
\]

and

\[
\|F_{a,p}\|_{A_2^w}^p \asymp \omega(S(a)).
\]

To prove the sufficiency of conditions (2.1) and (2.2) in Theorem 2.1 ideas from [20, 28, 31, 45] are used. The following lemma plays an important role in the proof, but it is also of independent interest. It will be also used in Chapter 6 when the norms of reproducing kernels in the Hilbert space \( \mathcal{A}_2^w \) are estimated, see Lemma 6.2.

Lemma 2.5. Let \( 0 < \alpha < \infty \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \). Then there exists a constant \( C = C(\alpha, \omega) > 0 \) such that

\[
|f(z)|^\alpha \leq CM_\omega(|f|^{-}\alpha)(z), \quad z \in \mathbb{D},
\]

for all \( f \in \mathcal{H}(\mathbb{D}) \).
2. DESCRIPTION OF $q$-CARLESON MEASURES FOR $A^p_0$

PROOF. Let $0 < \alpha < \infty$ and $f \in \mathcal{H}(\mathbb{D})$. Let first $\omega \in \mathcal{R}$. Let $a \in \mathbb{D}$, and assume without loss of generality that $|a| > 1/2$. Set $a^* = \frac{3|a|-1}{2}e^{a \arg a}$ so that $D(a, \frac{1}{2}(1-|a|)) \subset S(a^*)$. This inclusion together with the subharmonicity property of $|f|^\alpha$, (1.4) and (1.2) give

\[
|f(a)|^\alpha \lesssim \frac{1}{\omega(a)(1-|a|)^2} \int_{D(a, \frac{1}{2}(1-|a|))} |f(z)|^\alpha \omega(z) \, dA(z)
\]

\[
\lesssim \frac{1}{\omega(a^*)(1-|a^*|)^2} \int_{S(a^*)} |f(z)|^\alpha \omega(z) \, dA(z)
\]

\[
\lesssim \frac{1}{\omega(S(a^*))} \int_{S(a^*)} |f(z)|^\alpha \omega(z) \, dA(z) \leq M_\omega(|f|^\alpha)(a),
\]

which is the desired inequality for $\omega \in \mathcal{R}$.

Let now $\omega \in \mathcal{I}$. It again suffices to prove the assertion for the points $re^{i\theta} \in \mathbb{D}$ with $r > \frac{1}{2}$. If $r < \rho < 1$, then

\[
|f(re^{i\theta})|^\alpha \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\rho^2}{1-r^2} |f(re^{i(t+\theta)})|^\alpha \, dt
\]

\[
= \int_{-\pi}^{\pi} P\left(\frac{r}{\rho}, t\right) |f(re^{i(t+\theta)})|^\alpha \, dt,
\]

where

\[
P(r, t) = \frac{1}{2\pi} \frac{1-r^2}{1-r e^{it} \bar{\rho}}, \quad 0 < r < 1,
\]

is the Poisson kernel. Set $t_n = 2^{n-1}(1-r)$ and $J_n = [-t_n, t_n]$ for $n = 0, 1, \ldots, N+1$, where $N$ is the largest natural number such that $t_N < \frac{1}{2}$. Further, set $G_0 = J_0$, $G_n = J_n \setminus J_{n-1}$ for $n = 1, \ldots, N$, and $G_{N+1} = [-\pi, \pi] \setminus J_N$. Then

\[
|f(re^{i\theta})|^\alpha \leq \sum_{n=0}^{N+1} \int_{G_n} P\left(\frac{r}{\rho}, t\right) |f(re^{i(t+\theta)})|^\alpha \, dt
\]

\[
\leq \sum_{n=0}^{N+1} P\left(\frac{r}{\rho}, t_{n-1}\right) \int_{G_n} |f(re^{i(t+\theta)})|^\alpha \, dt
\]

\[
\lesssim \frac{1}{1-\rho^2} \sum_{n=0}^{N+1} 4^{-n} \int_{G_n} |f(re^{i(t+\theta)})|^\alpha \, dt,
\]

and therefore

\[
|f(re^{i\theta})|^\alpha (1-r) \int_r^{1} \omega(\rho) \rho \, d\rho \leq 2 \int_{r}^{1} |f(re^{i\theta})|^\alpha (\rho - r) \omega(\rho) \rho \, d\rho
\]

\[
\lesssim \sum_{n=0}^{N+1} 4^{-n} \int_{G_n} \left|f\left(re^{i(t+\theta)}\right)\right|^\alpha dt \omega(\rho) \rho^2 \, d\rho.
\]
It follows that

\[
|f(re^{i\theta})|^\alpha \lesssim \sum_{n=0}^{N} 2^{-n} \int_{-t_n}^{t_n} \left[ f \left( pe^{i(t+\theta)} \right) \right]^\alpha \, dt \omega(\rho)\rho \, dp \\
+ 2^{-N} \int_{r}^{1} \int_{-\pi}^{\pi} |f(\rho e^{it})|^\alpha \, dt \omega(\rho)\rho \, dp \\
\lesssim \sum_{n=0}^{N} 2^{-n} \int_{-t_n}^{t_n} \left[ f \left( pe^{i(t+\theta)} \right) \right]^\alpha \, dt \omega(\rho)\rho \, dp \\
+ 2^{-N} \int_{r}^{1} \int_{-\pi}^{\pi} |f(\rho e^{it})|^\alpha \, dt \omega(\rho)\rho \, dp,
\]

where the last step is a consequence of the inequalities $0 < 1 - t_{n+1} \leq r$. Denoting the interval centered at $e^{i\theta}$ and of the same length as $J_n$ by $J_n(\theta)$, and applying Lemma 1.1, with $\beta \in (0, 1)$ fixed, to the denominators, we obtain

\[
|f(re^{i\theta})|^\alpha \lesssim \sum_{n=0}^{N} 2^{-n(1-\beta)} \int_{S(J_n(\theta))} |f(z)|^\alpha \omega(z) \, dA(z) \\
/ \omega(S(J_n(\theta))) \\
+ 2^{-N(1-\beta)} \int_{D} |f(z)|^\alpha \omega(z) \, dA(z) \\
/ \omega(D) \\
\lesssim \left( \sum_{n=0}^{\infty} 2^{-n(1-\beta)} \right) M_\omega(|f|^\alpha)(re^{i\theta}) \lesssim M_\omega(|f|^\alpha)(re^{i\theta}),
\]

which finishes the proof for $\omega \in \mathcal{I}$.

\[\square\]

### 2.2. Proof of the main result

#### 2.2.1. Boundedness

Let $0 < p \leq q < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$, and assume first that $\mu$ is a $q$-Carleson measure for $A_p^q$. Consider the test functions $F_{a,p}$ defined just before Lemma 2.4. Then the assumption together with relations (2.5) and (2.6) yield

\[
\mu(S(a)) \lesssim \int_{S(a)} |F_{a,p}(z)|^q \, d\mu(z) \lesssim \int_{D} |F_{a,p}(z)|^q \, d\mu(z) \lesssim \|F_{a,p}\|_{A_p^q}^q \lesssim \omega(S(a))^{\frac{q}{2}}
\]

for all $a \in \mathbb{D}$, and thus $\mu$ satisfies (2.1).

Conversely, let $\mu$ be a positive Borel measure on $\mathbb{D}$ such that (2.1) is satisfied. We begin with proving that there exists a constant $K = K(p, q, \omega) > 0$ such that the $L_{\omega}^p$-weak type inequality

\[
(2.8) \quad \mu(E_s) \leq K s^{-\frac{q}{p}} \|\varphi\|_{L_{\omega}^q}^{\frac{q}{2}}, \quad E_s = \{z \in \mathbb{D} : M_\omega(\varphi)(z) > s\},
\]

is valid for all $\varphi \in L_{\omega}^1$ and $0 < s < \infty$.

If $E_s = \emptyset$, then (2.8) is clearly satisfied. If $E_s \neq \emptyset$, then recall that $I_z = \{e^{i\theta} : |\arg(z e^{-i\theta})| < (1 - |z|)/2\}$ and $S(z) = S(I_z)$, and define for each $\varepsilon > 0$ the sets

\[
A_s^\varepsilon = \left\{ z \in \mathbb{D} : \int_{S(I_z)} |\varphi(\xi)| \omega(\xi) \, dA(\xi) > s (\varepsilon + \omega(S(z))) \right\}
\]
and
\[ B^\varepsilon_s = \{ z \in \mathbb{D} : I_z \subset I_u \text{ for some } u \in A^\varepsilon_s \}. \]
The sets \( B^\varepsilon_s \) expand as \( \varepsilon \to 0^+ \), and
\[ E_s = \{ z \in \mathbb{D} : M_\omega(\varphi)(z) > s \} = \bigcup_{\varepsilon > 0} B^\varepsilon_s, \]
so
\[ (2.9) \quad \mu(E_s) = \lim_{\varepsilon \to 0^+} \mu(B^\varepsilon_s). \]
We notice that for each \( \varepsilon > 0 \) and \( s > 0 \) there are finitely many points \( z_n \in A^\varepsilon_s \) such that the arcs \( I_{z_n} \) are disjoint. Namely, if there were infinitely many points \( z_n \in A^\varepsilon_s \) with this property, then the definition of \( A^\varepsilon_s \) would yield
\[ (2.10) \quad s \sum_n [\varepsilon + \omega(S(z))] \leq \sum_n \int_{S(I_n)} |\varphi(\xi)| \omega(\xi) dA(\xi) \leq \|\varphi\|_{L^q}, \]
and therefore
\[ \infty = s \sum_n \varepsilon \leq \|\varphi\|_{L^q}, \]
which is impossible because \( \varphi \in L^1_\omega \).

We now use Covering lemma [28, p. 161] to find \( z_1, \ldots, z_m \in A^\varepsilon_s \) such that the arcs \( I_{z_n} \) are disjoint and
\[ A^\varepsilon_s \subset \bigcup_{n=1}^m \{ z : I_z \subset I_{z_n} \}, \]
where \( I_z \) is the arc centered at the same point as \( I_z \) and of length \( 5|I_z| \). It follows easily that
\[ (2.11) \quad B^\varepsilon_s \subset \bigcup_{n=1}^m \{ z : I_z \subset I_{z_n} \}. \]
But now the assumption (2.1) and Lemma 1.1 give
\[ \mu(\{ z : I_z \subset J_{z_n} \}) = \mu(\{ z : S(z) \subset S(J_{z_n}) \}) \leq \mu(S(J_{z_n})) \]
\[ \lesssim (\omega(S(J_{z_n})))^{\frac{1}{q}} \lesssim (\omega(S(z_n)))^{\frac{1}{q}}, \quad n = 1, \ldots, m. \]
This combined with (2.11) and (2.10) yield
\[ \mu(B^\varepsilon_s) \lesssim \sum_{n=1}^m (\omega(S(z_n)))^{\frac{1}{q}} \leq \left( \sum_{n=1}^m \omega(S(z_n)) \right)^{\frac{1}{q}} \leq s^{\frac{1}{q}} \|\varphi\|_{L^q}, \]
which together with (2.9) gives (2.8) for some \( K = K(p, q, \omega) \).

We will now use Lemma 2.5 and (2.8) to show that \( \mu \) is a \( q \)-Carleson measure for \( A^p \). To do this, fix \( \alpha > \frac{1}{p} \) and let \( f \in A^p_\omega \). For \( s > 0 \), let \( |f|^{\frac{1}{p}} = \psi_{\frac{1}{p}, s} + \chi_{\frac{1}{p}, s} \),
where
\[ \psi_{\frac{1}{p}, s}(z) = \begin{cases} |f(z)|^{\frac{1}{p}}, & \text{if } |f(z)|^{\frac{1}{p}} > s/(2K) \\ 0, & \text{otherwise} \end{cases} \]
and \( K \) is the constant in (2.8), chosen such that \( K \geq 1 \). Since \( p > \frac{1}{\alpha} \), the function \( \psi_{\frac{1}{p}, s} \) belongs to \( L^1_\omega \) for all \( s > 0 \). Moreover,
\[ M_\omega(|f|^{\frac{1}{p}}) \leq M_\omega(\psi_{\frac{1}{p}, s}) + M_\omega(\chi_{0, s}) \leq M_\omega(\psi_{\frac{1}{p}, s}) + \frac{s}{2K}, \]
and therefore
\[ \begin{align*}
\left\{ z \in \mathbb{D} : M_\omega(|f|^{\frac{1}{\alpha}})(z) > s \right\} & \subset \left\{ z \in \mathbb{D} : M_\omega(\psi^{\frac{1}{\alpha}, \alpha})(z) > s/2 \right\}.
\end{align*} \]

Using Lemma 2.5, the inclusion (2.12), (2.8) and Minkowski’s inequality in continuous form (Fubini in the case \( q = p \)), we finally deduce
\[
\int_{\mathbb{D}} |f(z)|^q d\mu(z) \lesssim \int_{\mathbb{D}} \left(M_\omega(|f|^{\frac{1}{\alpha}})(z)\right)^{q\alpha} d\mu(z)
= q\alpha \int_0^\infty s^{q\alpha-1} \mu \left( \left\{ z \in \mathbb{D} : M_\omega(|f|^{\frac{1}{\alpha}})(z) > s \right\} \right) ds
\leq q\alpha \int_0^\infty s^{q\alpha-1} \mu \left( \left\{ z \in \mathbb{D} : M_\omega(\psi^{\frac{1}{\alpha}, \alpha})(z) > s/2 \right\} \right) ds
\lesssim \int_0^\infty s^{q\alpha-1-rac{2}{p}} \|\psi^{\frac{1}{\alpha}, s}\|_{L_1^\omega} ds
= \int_0^\infty s^{q\alpha-1-rac{2}{p}} \left( \int \{ z : |f(z)|^{\frac{1}{\alpha}} > \frac{s}{2} \} \right) \left( \int_0^{2K|f(z)|^{\frac{1}{\alpha}}} s^{q\alpha-1-rac{2}{p}} ds \right) dA(z)
\leq \left( \int_{\mathbb{D}} |f(z)|^{p_\omega}(z) dA(z) \right)^{\frac{q}{p}}.
\]

Therefore \( \mu \) is a \( q \)-Carleson measure for \( A^p_\omega \), and the proof of Theorem 2.1(i) is complete.

We note that the proof of Theorem 2.1(i) also yields Corollary 2.2.

### 2.2.2. Compactness

The proof is based on a well-known method that has been used in several times in the existing literature, see, for example, [64]. Let \( 0 < p \leq q < \infty \) and \( \omega \in I \cup R \), and assume first that \( I_d : A^p_\omega \to L^q(\mu) \) is compact. For each \( a \in \mathbb{D} \), consider the function
\[
f_{a,p}(z) = \frac{(1 - |a|)^{\frac{q+1}{p}}}{(1 - |a|)^{\frac{q+1}{p}} \omega(S(a))^{\frac{1}{p}}},
\]
where \( \gamma > 0 \) is chosen large enough such that \( \sup_{a \in \mathbb{D}} \|f_{a,p}\|_{A^p_\omega} < \infty \) by the proof of Lemma 2.4. Therefore the closure of set \( \{f_{a,p} : a \in \mathbb{D}\} \) is compact in \( L^q(\mu) \) by the assumption, and hence standard arguments yield
\[
\lim_{r \to 1^-} \int_{\mathbb{D} \setminus D(0, r)} |f_{a,p}(z)|^q d\mu(z) = 0
\]
uniformly in \( a \). Moreover, the proof of Lemma 1.1 gives
\[
\lim_{|a| \to 1^-} \int_{|a|}^{(1 - |a|)^{\gamma}} \frac{1}{\omega(s) s} ds = 0,
\]
if $\gamma > 0$ is again large enough. So, if $\gamma$ is fixed appropriately, then

$$\lim_{|a| \to 1^-} f_{a,p}(z) = \lim_{|a| \to 1^-} \left( \frac{(1 - |a|)^{\gamma \pi}}{(1 - |a|)^{\gamma I}} \right)^{\frac{1}{p}} = 0$$

uniformly on compact subsets of $\mathbb{D}$. By combining this with (2.14) we obtain

$$\lim_{|a| \to 1^-} \|f_{a,p}\|_{L^q(\mu)} = 0,$$

and consequently,

$$0 = \lim_{|a| \to 1^-} \|f_{a,p}\|_{L^q(\mu)} \geq \lim_{|a| \to 1^-} \int_{S(a)} |f_{a,p}(z)|^q \, d\mu(z) \geq \lim_{|a| \to 1^-} \frac{\mu(S(a))}{(\omega(S(a)))^q},$$

which proves (2.2).

Conversely, assume that $\mu$ satisfies (2.2), and set

$$d\mu_r(z) = \chi_{\{\tau \leq |z| < 1\}}(z) \, d\mu(z).$$

Then Theorem 2.1(i) implies

$$\|h\|_{L^q(\mu_r)} \leq K_{\mu_r} \|h\|_{A^p_\mu}, \quad h \in A^p_\omega,$$

where

$$\lim_{r \to 1^-} K_{\mu_r} = 0.$$ 

Let $\{f_n\}$ be a bounded sequence in $A^p_\mu$. Then $\{f_n\}$ is uniformly bounded on compact sets of $\mathbb{D}$ by Lemma 2.5, and hence $\{f_n\}$ constitutes a normal family by Montel’s theorem. Therefore we may extract a subsequence $\{f_{n_k}\}$ that converges uniformly on compact sets of $\mathbb{D}$ to some $f \in \mathcal{H}(\mathbb{D})$. Fatou’s lemma shows that $f \in A^p_\omega$. Let now $\varepsilon > 0$. By (2.15) there exists $r_0 \in (0, 1)$ such that $K_{\mu_r} \leq \varepsilon$ for all $r \geq r_0$. Moreover, by the uniform convergence on compact sets, we may choose $n_0 \in \mathbb{N}$ such that $|f_{n_k}(z) - f(z)|^q < \varepsilon$ for all $n_k \geq n_0$ and $z \in \overline{D}(0, r_0)$. It follows that

$$\|f_{n_k} - f\|_{L^q(\mu)} \leq \varepsilon \mu(\overline{D}(0, r_0)) + \|f_{n_k} - f\|_{L^q(\mu_{r_0})} \leq \varepsilon \mu(\mathbb{D}) + K_{\mu_{r_0}} \|f_{n_k} - f\|_{A^p_{\mu_{r_0}}} \leq \varepsilon (\mu(\mathbb{D}) + C), \quad n_k \geq n_0,$$

for some constant $C > 0$, and so $I_d : A^p_\mu \to L^q(\mu)$ is compact. 

**2.2.3. Counterexample.** Theorem 3.1 in [24], Theorem 2.1(i) and Lemma 1.4(i) imply that condition (2.1) is equivalent to

$$\sup_{a \in \mathbb{D}} \frac{\mu(D(a, \beta(1 - |a|)))}{\omega(D(a, \beta(1 - |a|)))}^{\frac{q}{p}} < \infty, \quad \beta \in (0, 1),$$

for all $q \geq p$ and $\omega \in \mathcal{R}$. However, this is no longer true if $\omega \in \overline{\mathcal{I}}$. Indeed, one can show that (2.16) is a sufficient condition for $\mu$ to be a $q$-Carleson measure for $A^p_\omega$, but it turns out that it is not necessary.

**Proposition 2.6.** Let $0 < p \leq q < \infty$ and $\omega \in \overline{\mathcal{I}}$. Then there exists a $q$-Carleson measure for $A^p_\omega$ which does not satisfy (2.16).

**Proof.** Let us consider the measure

$$d\mu(z) = (1 - |z|)^{\frac{p}{q} - 1} \left( \int_{|z|}^{1} \omega(s) \, ds \right)^{\frac{q}{p}} \chi_{[0, 1]}(z) d|z|, \quad \alpha \in (1, \infty),$$
which is supported on \([0, 1) \subset \mathbb{D}\). It is clear that
\[
\mu(S(|a|)) \leq \left( \int_{|a|}^1 \omega(s) \, ds \right) ^{\frac{q}{p}} \int_{|a|}^1 (1 - s)^{\frac{q}{p} - 1} \, ds \asymp (\omega(S(a)))^{\frac{q}{p}}, \quad |a| \in [0, 1).
\]
Moreover, bearing in mind the proof of Lemma 1.1, for each \(\beta > 0\), there exists \(r_0 = r_0(\beta)\) such that the function \(h_{1/\beta}(r) = \frac{\int_0^r \omega(s) \, ds}{(1 - r)^\beta}\) is increasing on \([r_0, 1)\). Therefore \(h_{1/\beta}(r)\) is essentially increasing on \([0, 1)\), and hence
\[
\mu(S(|a|)) \asymp \left( \int_{|a|}^1 \omega(s) \, ds \right) ^{\frac{q}{p}} \int_{|a|}^1 (1 - s)^{\frac{q}{p}(1 + \beta) - 1} \, ds \asymp (\omega(S(a)))^{\frac{q}{p}}.
\]

Theorem 2.1 now shows that \(\mu\) is a \(q\)-Carleson measure for \(A^p_{\omega}\).

It remains to show that (2.16) fails. If \(|a|\) is sufficiently close to 1, then, by Lemma 1.6, Lemma 1.7 and (1.2) for \(\omega^*\), we obtain
\[
\mu(D(|a|, \beta(1 - |a|))) \asymp \int_{|a| - \beta(1 - |a|)}^{(\omega^*(s))^{\frac{q}{p}}} \frac{(\omega^*(s))^{\frac{q}{p}}}{1 - s} \, ds \asymp (\omega^*(a))^{\frac{q}{p}} \int_{|a| - \beta(1 - |a|)}^{\omega(S(a)) (1 - |a|)} \frac{ds}{1 - s} \asymp (\omega(S(a)))^{\frac{q}{p}}.
\]

Moreover, since \(\omega \in \mathcal{I}\) by the assumption, (1.2) gives
\[
\omega(D(|a|, \beta(1 - |a|))) \asymp (1 - |a|)^2 \omega(a).
\]
Therefore
\[
\lim_{|a| \to 1^-} \frac{\mu(D(|a|, \beta(1 - |a|)))}{\omega(D(|a|, \beta(1 - |a|)))^{\frac{q}{p}}} \asymp \lim_{|a| \to 1^-} \left( \frac{\psi_{\omega}(a)}{1 - |a|} \right)^{\frac{q}{p}} = \infty,
\]
because \(\omega \in \mathcal{I}\), and consequently (2.16) does not hold. \(\square\)
CHAPTER 3

Factorization and Zeros of Functions in $A^p_\omega$

The main purpose of this chapter is to establish the following factorization theorem for functions in the weighted Bergman space $A^p_\omega$, under the assumption that $\omega$ is invariant and the polynomials are dense in $A^p_\omega$. Recall that a weight $\omega : \mathbb{D} \to (0, \infty)$ is invariant if $\omega(z) \approx \omega(a)$ in each pseudohyperbolic disc $\Delta(a, r)$ of a fixed radius $r \in (0, 1)$. The invariant weights and their relation to those $\omega$ for which the polynomials are dense in $A^p_\omega$ were discussed in Section 1.5.

**Theorem 3.1.** Let $0 < p < \infty$ and $\omega \in \text{Inv}$ such that the polynomials are dense in $A^p_\omega$. Let $f \in A^p_\omega$, and let $0 < p_1, p_2 < \infty$ such that $p^{-1} = p_1^{-1} + p_2^{-1}$. Then there exist $f_1 \in A^{p_1}_{\omega}$ and $f_2 \in A^{p_2}_{\omega}$ such that $f = f_1 \cdot f_2$ and

$$\|f_1\|_{A^{p_1}_{\omega}} \cdot \|f_2\|_{A^{p_2}_{\omega}} \leq \frac{p}{p_1} \|f_1\|_{A^{p_1}_{\omega}} + \frac{p}{p_2} \|f_2\|_{A^{p_2}_{\omega}} \leq C \|f\|_{A^p_\omega}$$

for some constant $C = C(p_1, p_2, \omega) > 0$.

If $\omega$ is radial, then polynomials are dense in $A^p_\omega$, and hence in this case the assumption on $\omega$ in Theorem 3.1 reduces to the requirement (1.2).

Theorem 3.1 is closely related to the zero distribution of functions in $A^p_\omega$. Therefore, and also for completeness, we put some attention to this matter. Our results follow the line of those due to Horowitz [46, 48, 49]. Roughly speaking we will study basic properties of unions, subsets and the dependence on $p$ of the zero sets of functions in $A^p_\omega$. In contrast to the factorization result, here the density of polynomials need not to be assumed. A good number of the results established are obtained as by-products of some key lemmas used in the proof of Theorem 3.1 and sharp conditions on some products defined on the moduli of the zeros. However, we do not venture into generalizing the theory, developed among others by Korenblum [52], Hedenmalm [40] and Seip [75, 76], and based on the use of densities defined in terms of partial Blaschke sums, Stolz star domains and Beurling-Carleson characteristic of the corresponding boundary set, which might be needed in order to obtain more complete results. Since the angular distribution of zeros plays a role in a description of the zero sets of functions in the classical weighted Bergman space $A^p_\alpha$, it is natural to expect that the same happens also in $A^p_\omega$, when $\omega \in \text{Inv} \cup \mathcal{R}$. In this chapter we will also briefly discuss the zero distribution of functions in the Bergman-Nevanlinna classes.

3.1. Factorization of functions in $A^p_\omega$

Most of the results and proofs in this section are inspired by those of Horowitz [47], and Horowitz and Schnaps [50]. It is worth noticing that in [50] the factorization of functions in $A^p_\omega$ is considered when $\omega$ belongs to a certain subclass of decreasing radial weights. Consequently, it does not cover the class $\text{Inv}$, and in particular it does
not contain the set $\tilde{T} \cup R$. One part of our contribution consists of detailed analysis on the constant $C = C(p_1, p_2, \omega) > 0$ appearing in (3.1), see Corollary 3.4 below. Later on, in Chapter 4, this corollary shows its importance when the bounded integral operators $T_\varphi : A^p_\omega \to A^q_\omega$ are characterized.

For $-1 < \alpha < \infty$, the Bergman-Nevanlinna class $BN_\alpha$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that
\[
\int_{\mathbb{D}} \log^+ |f(z)|(1 - |z|^2)^\alpha \, dA(z) < \infty.
\]
If $\{z_k\}$ are the zeros of $f \in BN_\alpha$, then
\[
\sum_k (1 - |z_k|^2)^{2+\alpha} < \infty,
\]
(3.2)
\[
n(r, f) = o\left(\frac{1}{(1 - r)^{2+\alpha}}\right), \quad r \to 1^-,
\]
\[
N(r, f) = o\left(\frac{1}{(1 - r)^{1+\alpha}}\right), \quad r \to 1^-,
\]
where $n(r, f)$ denotes the number of zeros of $f$ in $D(0, r)$, counted according to multiplicity, and
\[
N(r, f) = \int_0^r \frac{n(s) - n(0, f)}{s} \, ds + n(0, f) \log r
\]
is the integrated counting function. If no confusion arises with regards to $f$, we will simply write $n(r)$ and $N(r)$ instead of $n(r, f)$ and $N(r, f)$. The well-known facts (3.2) are consequences of Jensen’s formula, which states that
\[
\log |f(0)| + \sum_{k=1}^n \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta, \quad 0 < r < 1,
\]
where $\{z_k\}$ are the zeros of $f$ in the disc $D(0, r)$, repeated according to multiplicity and ordered by increasing moduli. See Lemma 3.17 for a more general result from which (3.2) follows by choosing $\omega(r) = (1 - r^2)^\alpha$.

We begin with showing that the class of invariant weights is in a sense a natural setting for the study of factorization of functions in $A^p_\omega$.

**Lemma 3.2.** If $0 < p < \infty$ and $\omega \in \mathcal{I}_w$, then $A^p_\omega \subset BN_0$.

**Proof.** Let $f \in A^p_\omega$, where $0 < p < \infty$ and $\omega \in \mathcal{I}_w$. Let $u \in \mathbb{D}$ be fixed. Then Lemma 1.8 yields
\[
\int_{\mathbb{D}} \log^+ |f(z)| \, dA(z) \leq \frac{1}{p} \int_{\mathbb{D}} \log^+ (|f(z)|^p \omega(z)) \, dA(z) + \frac{1}{p} \int_{\mathbb{D}} \log^+ \frac{1}{\omega(z)} \, dA(z)
\]
\[
\leq \frac{1}{p} \|f\|_{A^p_\omega}^p + \frac{1}{p} \int_{\mathbb{D}} \log^+ C(\varphi_u(z)) \, dA(z) + \frac{1}{p} \log^+ \frac{1}{\omega(u)}
\]
\[
\leq \frac{1}{p} \|f\|_{A^p_\omega}^p + \frac{1}{p} \left(1 + \frac{|u|}{1 - |u|}\right)^2 \int_{\mathbb{D}} \log^+ C(w) \, dA(w)
\]
\[
+ \frac{1}{p} \log^+ \frac{1}{\omega(u)} < \infty,
\]
and the inclusion $A^p_\omega \subset BN_0$ follows. \qed

The next result plays an important role in the proof of Theorem 3.1.
Lemma 3.3. Let $0 < p < q < \infty$ and $\omega \in \text{Inv}$. Let $\{z_k\}$ be the zero set of $f \in A^p_\omega$, and let
\[
g(z) = |f(z)|^p \prod_k \frac{1 - \frac{p}{q} + \frac{p}{q} |z_k|^q}{|z_k|^p}.
\]
Then there exists a constant $C = C(p, q, \omega) > 0$ such that
\[
\|g\|_{L^q_\omega} \leq C\|f\|^p_{A^p_\omega}.
\]
Moreover, the constant $C$ has the following properties:

(i) If $0 < p < q \leq 2$, then $C = C(\omega)$, that is, $C$ is independent of $p$ and $q$.

(ii) If $2 < q < \infty$ and $\frac{q}{p} \geq 1 + \epsilon > 1$, then $C = C_1 q e^{C_1 q}$, where $C_1 = C_1(\epsilon, \omega)$.

Proof. Let us start with considering $g(0)$. To do this, assume $f(0) \neq 0$. Lemma 3.2 yields $f \in B\omega_0$, and hence $\sum_k (1 - |z_k|)^2 < \infty$. Moreover, Bernouilli-l’Hôpital theorem shows that
\[
1 - \frac{1 - \frac{p}{q} + \frac{p}{q} |z_k|^q}{r^p} \approx (1 - r)^2, \quad r \to 1^-,
\]
for all $n > 1$, and hence the product
\[
\prod_k \frac{1 - \frac{p}{q} + \frac{p}{q} |z_k|^q}{|z_k|^p}
\]
converges. Now, an integration by parts gives
\[
\sum_k \log \left( \frac{\frac{p}{q} - 1}{|z_k|^p} |z_k|^q \right) = \int_0^1 \log \left( \frac{\frac{p}{q} - 1}{r^p} \frac{1 - \frac{p}{q} + \frac{p}{q} r^q}{r^p} \right) dn(r)
\]
\[
= \int_0^1 \frac{(\frac{q}{p} - 1)(1 - r^q)}{r - \frac{q}{p} - 1 + r^q} \frac{pn(r)}{r} dr,
\]
where the last equality follows by (3.5) and (3.2) with $\alpha = 0$. Another integration by parts and Jensen’s formula (3.3) show that the last integral in (3.6) equals to
\[
- \int_0^1 pN(r) du(r) = \int_D \log |f(z)|^p d\sigma(z) - \log |f(0)|^p,
\]
where
\[
d\sigma(z) = -u'(|z|) \frac{dA(z)}{2|z|}, \quad u(r) = \frac{(\frac{q}{p} - 1)(1 - r^q)}{\frac{q}{p} - 1 + r^q},
\]
is a positive measure of unit mass on $D$. By combining (3.6) and (3.7), we deduce
\[
\log(g(0)) = \log \left( |f(0)|^p \prod_k \left( \frac{1 - \frac{p}{q} + \frac{p}{q} |z_k|^q}{|z_k|^p} \right) \right)
\]
\[
= \log |f(0)|^p + \sum_k \log \left( \frac{1 - \frac{p}{q} + \frac{p}{q} |z_k|^q}{|z_k|^p} \right) = \int_D \log |f(z)|^p d\sigma(z).
\]
Replacing now $f$ by $f \circ \varphi_\zeta$, we obtain
\[
\log(g(\zeta)) = \int_D \log |f(\varphi_\zeta(z))|^p d\sigma(z)
\]
for $\zeta$ outside of zeros of $f$. Once this identity has been established we will prove (i).
Assume that $0 < q \leq 2$. We claim that there exists a unique point $x = x(p, q) \in (0, 1)$ such that $-u'(x) = 2x$, $2r \leq -u'(r)$ on $(0, x]$, and $2r \geq -u'(r)$ on $[x, 1)$. We first observe that

$$-u'(r) = \frac{q}{p} \left( \frac{q}{p} - 1 \right) \frac{qr^{q-1}}{(\frac{q}{p} - 1 + r^q)^2} = 2r$$

$$\Leftrightarrow 2r^{q+2} + 4r^2 \left( \frac{q}{p} - 1 \right) + 2r^{2-q} \left( \frac{q}{p} - 1 \right)^2 = \frac{q^2}{p} \left( \frac{q}{p} - 1 \right).$$

Now the left hand side is increasing, it vanishes at the origin, and at one it attains the value $2 \left( \frac{q}{p} \right)^2$, which is strictly larger than the constant on the right hand side because $q \leq 2$. The existence of the point $x \in (0, 1)$ with the desired properties follows.

By dividing the radial integral in (3.8) into $(0, x)$ and $(x, 1)$, using the fact that $\int_0^{2\pi} \log |f(\varphi_\zeta(re^{i\theta}))|^p d\theta$ is increasing on $(0, 1)$, and bearing in mind that $-u'(r)dr$ and $2rdr$ are probability measures on $(0, 1)$, we deduce

$$\int_D \log |f(\varphi_\zeta(z))|^p d\sigma(z) \leq \int_D \log |f(\varphi_\zeta(z))|^p dA(z).$$

By combining (3.8) and (3.9), we obtain

$$\log(g(\zeta)\omega(\zeta)) \leq \int_D \log |f(\varphi_\zeta(z))|^p \omega(\zeta) d\alpha(z)$$

$$\leq \int_D \log \left( |f(\varphi_\zeta(z))|^p \omega(\zeta) \frac{1 - |z|^2}{C(z)} \right) d\alpha(z) + C_1,$$

where $C$ is the function from Lemma 1.8, and hence

$$C_1 = C_1(\omega) = \int_D \log \left( \frac{C(z)}{1 - |z|^2} \right) d\alpha(z) < \infty$$

by (1.31). Now Jensen’s inequality together with (1.30) yields

$$g(\zeta)\omega(\zeta) \leq e^{C_1} \int_D |f(\varphi_\zeta(z))|^p \omega(\zeta) \frac{1 - |z|^2}{C(z)} d\alpha(z)$$

$$\leq e^{C_1} \int_D |f(\varphi_\zeta(z))|^p \omega(\varphi_\zeta(z))(1 - |z|^2) d\alpha(z)$$

for $\zeta$ outside of zeros of $f$. By integrating this inequality, changing a variable, and using Fubini’s theorem and [84, Lemma 3.10] we deduce

$$\|g\|_{L^p} \leq e^{C_1} \int_D |f(u)|^p \omega(u) \left( \int_B |\varphi_\zeta(u)|^2 (1 - |\varphi_\zeta(u)|^2) d\alpha(u) \right) d\alpha(u)$$

$$\leq C_2 \|f\|^p_{A^p_\zeta},$$

where $C_2 = C_2(\omega) > 0$ is a constant. This finishes the proof of (i).

Assume now that $q \geq 2$. In this case

$$-u'(r) = \frac{q}{p} \left( \frac{q}{p} - 1 \right) \frac{qr^{q-1}}{(\frac{q}{p} - 1 + r^q)^2} \leq \frac{q^2}{q-p} r^{q-1} \leq \frac{q^2}{q-p} r, \quad 0 \leq r \leq 1.$$

$$\Rightarrow 2r^{q+2} + 4r^2 \left( \frac{q}{p} - 1 \right) + 2r^{2-q} \left( \frac{q}{p} - 1 \right)^2 = \frac{q^2}{p} \left( \frac{q}{p} - 1 \right).$$
The identity (3.8) and (3.14) give
\[
\log(g(\zeta)\omega(\zeta)) \leq \int_{D} \log \left( |f(\varphi_\zeta(z))|^p \omega(\zeta) \frac{(1-|z|^2)}{C(z)} \right) d\sigma(z) \\
+ \int_{D} \log \left( \frac{C(z)}{(1-|z|^2)} \right) d\sigma(z) \\
\leq \int_{D} \log \left( |f(\varphi_\zeta(z))|^p \omega(\zeta) \frac{(1-|z|^2)}{C(z)} \right) d\sigma(z) + C_1(q,p,\omega),
\]
where \( C \) is the function from Lemma 1.8, and hence
\[
C_1(q,p,\omega) = \frac{q^2}{2(q-p)} \int_{D} \log \left( \frac{C(z)}{1-|z|^2} \right) dA(z) < \infty
\]
by (1.31). By arguing now similarly as in (3.12) and (3.13), and using (3.14), we obtain \( \|g\|_{L^p} \leq C_2\|f\|_{A^p_\omega} \), where \( C_2 = C_2(p,q,\omega) = \frac{q^2}{2(q-p)}e^{C_1(p,q,\omega)} > 0 \) is a constant. Consequently, joining this with (i), we deduce (3.4) for all \( 0 < p < q < \infty \).

Finally we will prove (ii). Assume that \( 2 < q < \infty \) and \( \frac{q}{p} \geq 1+\epsilon > 1 \). A direct calculation shows that
\[
(3.15) \quad -u'(r) = \frac{q}{p} \left( \frac{q}{p} - 1 \right) \frac{q^{r-1} - 1}{\left( \frac{q}{p} - 1 + r^q \right)^2} \leq \frac{1 + \epsilon}{\epsilon} qr, \quad 0 \leq r \leq 1.
\]
The identity (3.8) and (3.15) give
\[
(3.16) \quad \log(g(\zeta)\omega(\zeta)) \leq \int_{D} \log \left( |f(\varphi_\zeta(z))|^p \omega(\zeta) \frac{(1-|z|^2)}{C(z)} \right) d\sigma(z) + C_1,
\]
where
\[
C_1 = C_1(p,q,\omega) = q \frac{1+\epsilon}{\epsilon} \int_{D} \log \left( \frac{C(z)}{1-|z|^2} \right) dA(z).
\]
Therefore \( C_1 = C_2C_3q \), where \( C_2 = C_2(\epsilon) = \frac{1+\epsilon}{\epsilon} \) and \( C_3 = C_3(\omega) < \infty \) by (1.31). Jensen’s inequality together with (3.16) and (1.30) yield
\[
g(\zeta)\omega(\zeta) \leq e^{C_2C_3q}C_2q \int_{D} |f(\varphi_\zeta(z))|^p \omega(\zeta) \frac{(1-|z|^2)}{C(z)} dA(z) \\
\leq e^{C_2C_3q}C_2q \int_{D} |f(\varphi_\zeta(z))|^p \omega(\zeta)(1-|z|^2) dA(z).
\]
By arguing similarly as in (3.13) we finally obtain \( \|g\|_{L^p} \leq C_4\|f\|_{A^p_\omega} \), where \( C_4 = e^{C_2C_3q}C_2C_5q \) and \( C_5 > 0 \) is a constant. This finishes the proof.

Now we are in position to prove the announced factorization for functions in \( A^p_\omega \).

**Proof of Theorem 3.1.** Let \( 0 < p < \infty \) and \( \omega \in \mathcal{I}nv \) such that the polynomials are dense in \( A^p_\omega \), and let \( f \in A^p_\omega \). Assume first that \( f \) has finitely many zeros only. Such functions are of the form \( f = gB \), where \( g \in A^p_\omega \) has no zeros and \( B \) is a finite Blaschke product. Let \( z_1, \ldots, z_m \) be the zeros of \( f \) so that \( B = \prod_{k=1}^{m} B_k \), where \( B_k = \frac{z_k}{|z_k|}\varphi_{z_k} \). Write \( B = B^{(1)} \cdot B^{(2)} \), where the factors \( B^{(1)} \) and \( B^{(2)} \) are random subproducts of \( B_0, B_1, \ldots, B_m \), where \( B_0 \equiv 1 \). Setting \( f_j = \left( \frac{f}{B_j} \right)^{\frac{p}{q}} B^{(j)} \), we have \( f = f_1 \cdot f_2 \). We now choose \( B^{(j)} \) probabilistically. For a given \( j \in \{1, 2\} \),
the factor $B^{(j)}$ will contain each $B_k$ with the probability $p/p_j$. The obtained $m$ random variables are independent, so the expected value of $|f_j(z)|^{p_j}$ is

$\begin{align*}
E(|f_j(z)|^{p_j}) &= \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left( 1 - \frac{p}{p_j} + \frac{p}{p_j} |\varphi_{z_k}(z)|^{p_j} \right) \\
(3.17) &= |f(z)|^p \prod_{k=1}^m \left( 1 - \frac{p}{p_j} + \frac{p}{p_j} |\varphi_{z_k}(z)|^{p_j} \right) |\varphi_{z_k}(z)|^{p_j}
\end{align*}$

for all $z \in \mathbb{D}$ and $j \in \{1, 2\}$. Now, bearing in mind (3.17) and Lemma 3.3, we find a constant $C_1 = C_1(p, p_1, \omega) > 0$ such that

$\begin{align*}
\|E(f_1^{p_1})\|_{L^\omega_\mathbb{D}} &= \int_\mathbb{D} \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left( 1 - \frac{p}{p_1} + \frac{p}{p_1} |\varphi_{z_k}(z)|^{p_1} \right) |\varphi_{z_k}(z)|^{p_1} \omega(z) dA(z) \\
&= \int_\mathbb{D} \left| f(z) \prod_{k=1}^m \left( 1 - \frac{p_2}{p_2} + \frac{p_2}{p_2} |\varphi_{z_k}(z)|^{p_2} \right) |\varphi_{z_k}(z)|^{p_2} \right| \omega(z) dA(z) \leq C_1 \|f\|_{A^\omega p}^p.
\end{align*}$

Analogously, by (3.17) and Lemma 3.3 there exists a constant $C_2 = C_2(p, p_2, \omega) > 0$ such that

$\begin{align*}
\|E(f_2^{p_2})\|_{L^\omega_\mathbb{D}} &= \int_\mathbb{D} \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left( 1 - \frac{p_2}{p_2} + \frac{p_2}{p_2} |\varphi_{z_k}(z)|^{p_2} \right) |\varphi_{z_k}(z)|^{p_2} \omega(z) dA(z) \leq C_2 \|f\|_{A^\omega p}^p.
\end{align*}$

By combining the two previous inequalities, we obtain

$\begin{align*}
(3.18) \quad \left\| E \left( \frac{p}{p_1} f_1^{p_1} \right) \right\|_{L^\omega_\mathbb{D}} + \left\| E \left( \frac{p}{p_2} f_2^{p_2} \right) \right\|_{L^\omega_\mathbb{D}} \leq \left( \frac{p}{p_1} C_1 + \frac{p}{p_2} C_2 \right) \|f\|_{A^\omega p}^p.
\end{align*}$

On the other hand,

$\begin{align*}
\left\| E \left( \frac{p}{p_1} f_1^{p_1} \right) \right\|_{L^\omega_\mathbb{D}} + \left\| E \left( \frac{p}{p_2} f_2^{p_2} \right) \right\|_{L^\omega_\mathbb{D}} &= \frac{p}{p_1} \int_\mathbb{D} \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left( \frac{p}{p_1} + \left( 1 - \frac{p}{p_1} \right) |\varphi_{z_k}(z)|^{p_1} \right) |\varphi_{z_k}(z)|^{p_1} \omega(z) dA(z) \\
&+ \frac{p}{p_2} \int_\mathbb{D} \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left( \frac{p}{p_2} + \left( 1 - \frac{p}{p_2} \right) |\varphi_{z_k}(z)|^{p_2} \right) |\varphi_{z_k}(z)|^{p_2} \omega(z) dA(z) \\
(3.19) &= \int_\mathbb{D} I(z) \omega(z) dA(z),
\end{align*}$

where

$\begin{align*}
I(z) = \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left( \frac{p}{p_1} + \left( 1 - \frac{p}{p_1} \right) |\varphi_{z_k}(z)|^{p_1} \right) \\
+ \frac{p}{p_2} \prod_{k=1}^m \left( \frac{p}{p_2} + \left( 1 - \frac{p}{p_2} \right) |\varphi_{z_k}(z)|^{p_2} \right).
\end{align*}$
It is clear that the $m$ zeros of $f$ must be distributed to the factors $f_1$ and $f_2$, so if $f_1$ has $n$ zeros, then $f_2$ has the remaining $(m - n)$ zeros. Therefore

$$I(z) = \sum_{f_1, f_2 = f} \left( \left(1 - \frac{p}{p_2}\right)^n \left(\frac{p}{p_2}\right)^{m-n} \left[\frac{p}{p_1} |f_1(z)|^{p_1} + \frac{p}{p_2} |f_2(z)|^{p_2}\right] \right).$$

This sum consists of $2^m$ addends, $f_1$ contains $\left(\frac{n}{p_1}\right)$ and $n$ zeros of $f$, and $f_2$ contains $\left(\frac{m-n}{p_2}\right)$ and the remaining $(m - n)$ zeros of $f$, and thus $f = f_1 \cdot f_2$.

Further, for a fixed $n$, (3.20) shows that there must exist a concrete factorization $f = f_1 \cdot f_2$.

By joining (3.18), (3.19) and (3.20), we deduce

$$\sum_{f_1, f_2 = f} \left(1 - \frac{p}{p_2}\right)^n \left(\frac{p}{p_2}\right)^{m-n} \left[\frac{p}{p_1} ||f_1||_{A^p_{p_1}} + \frac{p}{p_2} ||f_2||_{A^p_{p_2}}\right] \leq \left(\frac{p}{p_1} C_1 + \frac{p}{p_2} C_2\right) ||f||_{A^p_{\infty}}^p.$$

This together with (3.21) shows that there must exist a concrete factorization $f = f_1 \cdot f_2$ such that

$$\frac{p}{p_1} ||f_1||_{A^p_{p_1}} + \frac{p}{p_2} ||f_2||_{A^p_{p_2}} \leq C(p_1, p_2, \omega)||f||_{A^p_{\infty}}^p.$$

By combining this with the inequality

$$x^\alpha \cdot y^\beta \leq ax + \beta y, \quad x, y \geq 0, \quad \alpha + \beta = 1,$$

we finally obtain (3.1) under the hypotheses that $f$ has finitely many zeros only.

To deal with the general case, we first prove that every norm-bounded family in $A^p_{\infty}$ is a normal family of analytic functions. If $f \in A^p_{\infty}$, then

$$||f||_{A^p_{\infty}}^p \geq \int_{D(0, \frac{1+r}{2}) \setminus D(0, \rho)} |f(z)|^p \omega(z) dA(z) \geq M_p^p(\rho, f) \left(\min_{|z| \leq \frac{1+r}{2}} \omega(z)\right), \quad 0 \leq \rho < 1,$$

from which the well-known relation $M_{\infty}(r, f) \leq M_p(\frac{1+r}{2}, f)(1-r)^{-1/p}$ yields

$$M_{\infty}(r, f) \leq \frac{||f||_{A^p_{\infty}}^p}{(1-r) \left(\min_{|z| \leq \frac{1+r}{2}} \omega(z)\right)}, \quad 0 \leq r < 1.$$

Therefore every norm-bounded family in $A^p_{\infty}$ is a normal family of analytic functions by Montel’s theorem.

Finally, assume that $f \in A^p_{\infty}$ has infinitely many zeros. Since polynomials are dense in $A^p_{\infty}$ by the assumption, we can choose a sequence $f_i$ of functions with finitely many zeros that converges to $f$ in norm, and then, by the previous argument, we can factorize each $f_i = f_{i,1} \cdot f_{i,2}$ as earlier. Now, since every norm-bounded family in $A^p_{\infty}$ is a normal family of analytic functions, by passing to subsequences of $\{f_{i,j}\}$
3.1

We will mimic the proof of Theorem 3.1 and replace by any assumption that ensures the existence of a dense family of functions with finitely many zeros only.

It is worth noticing that the hypothesis on the density of polynomials in Theorem 3.1 can be replaced by any assumption that ensures the existence of a dense family of linear operators, see Proposition 4.8. Usually this kind of properties are established by using interpolation theorems similar to [83, Theorem 2.34] or [84, Theorem 4.29]. Indeed, it would be interesting to know whether or not an extension of [84, Theorem 4.29] remains valid for \( A_p^0 \) if either \( \omega \in I \cup R \) or \( \omega \in I \cup R \).

Corollary 3.4. Let \( 0 < p < 2 \) and \( \omega \in Inv \) such that the polynomials are dense in \( A_p^0 \). Let \( 0 < p_1 \leq 2 < p_2 < \infty \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( p_2 \geq 2p \). If \( f \in A_p^0 \), then there exist \( f_1 \in A_{p_1}^0 \) and \( f_2 \in A_{p_2}^0 \) such that \( f = f_1 \cdot f_2 \) and

\[
\|f_1\|_{A_{p_1}^0} \cdot \|f_2\|_{A_{p_2}^0} \leq C\|f\|_{A_p^0}
\]

for some constant \( C = C(p_1, \omega) > 0 \).

Proof. We will mimic the proof of Theorem 3.1, but we will have to pay special attention to the constants coming from Lemma 3.3. Therefore we will keep the notation used in the proof of Theorem 3.1, and will only explain the steps where the proof significantly differs from that of Theorem 3.1.

It suffices to establish the desired factorization for those \( f \in A_p^0 \) having finitely many zeros only. Following the proof of Theorem 3.1, we write \( f = gB \), where \( B = B^{(1)} \cdot B^{(2)} \), and \( f_j = \left( \frac{1}{B} \right)^{\frac{1}{p_j}} B^{(j)} \), so that \( f = f_1 \cdot f_2 \). Now, bearing in mind (3.17), and the cases (i) and (ii) of Lemma 3.3, we find constants \( C_1 = C_1(\omega) > 0 \) and \( C_2 = C_2(\omega) > 0 \) such that

\[
\|E(f_1^{p_1})\|_{L_2^p} = \int_D \left( |f(z)|^p \prod_{k=1}^m \frac{P_k}{p_2} + \left( 1 - \frac{P_k}{p_2} \right) |\varphi_{z_k}(z)|^{p_1} \right) \omega(z) dA(z) \leq C_1 \|f\|_{A_p^0}^p
\]

and

\[
\|E(f_2^{p_2})\|_{L_2^p} = \int_D \left( |f(z)|^p \prod_{k=1}^m \frac{1 - P_k}{p_2} |\varphi_{z_k}(z)|^{p_2} \right) \omega(z) dA(z) 
\leq C_2 p_2 e^{C_2 p_2} \|f\|_{A_p^0}^p = M_2 \|f\|_{A_p^0}^p,
\]

where we may choose \( C_2 \) sufficiently large so that \( M_2 = C_2 p_2 e^{C_2 p_2} \geq 2 \). These inequalities yield

\[
\|E\left(1 - \frac{1}{M_2}\right) f_1^{p_1}\|_{L_2^p} + \|E\left(\frac{1}{M_2} f_2^{p_2}\right)\|_{L_2^p} \leq \left( C_1 \left(1 - \frac{1}{M_2}\right) + 1 \right) \|f\|_{A_p^0}^p,
\]

with respect to \( l \) if necessary, we have \( f_{i,j} \to f_j \), where the functions \( f_j \) form the desired bounded factorization \( f = f_1 \cdot f_2 \), satisfying (3.1). This finishes the proof of Theorem 3.1. \( \square \)
where
\[
\left\| E \left( \left( 1 - \frac{1}{M_2} \right) f_1^{p_1} \right) \right\|_{L^1_\omega} + \left\| E \left( \frac{1}{M_2} f_2^{p_2} \right) \right\|_{L^1_\omega} \\
= \left( 1 - \frac{1}{M_2} \right) \int_D \left| \frac{f(z)}{B(z)} \right|^{p_1} \prod_{k=1}^m \left( \frac{p}{p_2} + \left( 1 - \frac{p}{p_2} \right) |\varphi_{z_k}(z)|^{p_1} \right) \omega(z) dA(z) \\
+ \frac{1}{M_2} \int_D \left| \frac{f(z)}{B(z)} \right|^{p_2} \prod_{k=1}^m \left( \left( 1 - \frac{p}{p_2} \right) + \frac{p}{p_2} |\varphi_{z_k}(z)|^{p_2} \right) \omega(z) dA(z).
\]
Therefore
\[
\sum_{f_1, f_2 \neq f} \left( 1 - \frac{p}{p_2} \right)^n \left( \frac{p}{p_2} \right)^{m-n} \left[ \left( 1 - \frac{1}{M_2} \right) \| f_1 \|_{A^p_{C_1}} + \frac{1}{M_2} \| f_2 \|_{A^p_{C_2}} \right]
\leq \left( C_1 \left( 1 - \frac{1}{M_2} \right) + 1 \right) \| f \|_{A^p_{\omega}},
\]
which together with (3.21) implies that there exists a concrete factorization \( f = f_1 \cdot f_2 \) such that
\[
\left( 1 - \frac{1}{M_2} \right) \| f_1 \|_{A^p_{C_1}} + \frac{1}{M_2} \| f_2 \|_{A^p_{C_2}} \leq \left( C_1 \left( 1 - \frac{1}{M_2} \right) + 1 \right) \| f \|_{A^p_{\omega}}.
\]
It follows that
\[
\| f_1 \|_{A^p_{\omega}} \leq \left( 2 \left( C_1 \left( 1 - \frac{1}{M_2} \right) + 1 \right) \right)^{\frac{1}{p_1}} \| f \|_{A^p_{\omega}} \leq K_1 \| f \|_{A^p_{\omega}},
\]
where \( K_1 = K_1(p_1, \omega) = (2(C_1 + 1))^{\frac{1}{p_1}} \), and
\[
\| f_2 \|_{A^p_{\omega}} \leq (M_2 + C_1 (M_2 - 1))^{\frac{1}{p_2}} \| f \|_{A^p_{\omega}} \leq K_2 \| f \|_{A^p_{\omega}} < \infty,
\]
where
\[
K_2 = K_2(\omega) = \sup_{p_2 > 2} (M_2 + C_1 (M_2 - 1))^{\frac{1}{p_2}} < \infty.
\]
By multiplying these inequalities we obtain
\[
\| f_1 \|_{A^p_{\omega}} \| f_2 \|_{A^p_{\omega}} \leq K_1 \cdot K_2 \| f \|_{A^p_{\omega}}.
\]
This establishes the assertion with \( C(p_1, \omega) = K_1(p_1, \omega)K_2(\omega) \). \( \square \)

### 3.2. Zeros of functions in \( A^p_{\omega} \)

For a given space \( X \) of analytic functions in \( \mathbb{D} \), a sequence \( \{ z_k \} \) is called an \( X \)-zero set, if there exists a function \( f \) in \( X \) such that \( f \) vanishes precisely on the points \( z_k \) and nowhere else. As far as we know, it is still an open problem to find a complete description of zero sets of functions in the Bergman spaces \( A^p = A^p_0 \), but the gap between the known necessary and sufficient conditions is very small. We refer to [29, Chapter 4] and [41, Chapter 4] as well as [52, 59, 75, 76] for more information on this topic.

We begin with proving that a subset of an \( A^p_0 \)-zero set is also an \( A^p_0 \)-zero set if \( \omega \) is invariant.
Let 0 < p < \infty and \omega \in \mathcal{I}_{nv}. Let \{z_k\} be an arbitrary subset of the zero set of f \in \mathcal{A}_\omega^p, and let

\[ H(z) = \prod_k B_k(z)(2 - B_k(z)), \quad B_k = \frac{z_k}{|z_k|} \varphi_{z_k}, \]

with the convention z_k/|z_k| = 1 if z_k = 0. Then there exists a constant C = C(\omega) > 0 such that \( \|f/H\|_{\mathcal{A}_\omega^p}^p \leq C\|f\|_{\mathcal{A}_\omega^p}^p \). In particular, each subset of an \( \mathcal{A}_\omega^p \)-zero set is an \( \mathcal{A}_\omega^p \)-zero set.

Since

\[ \frac{|f(z)|}{\prod_k |\varphi_{z_k}(z)|(2 - |\varphi_{z_k}(z)|)} \geq \frac{|f(z)|}{H(z)}, \]

the assertion in Theorem 3.5 follows by the following lemma.

**Lemma 3.6.** Let 0 < p < \infty and \omega \in \mathcal{I}_{nv}. Let \{z_k\} be an arbitrary subset of the zero set of f \in \mathcal{A}_\omega^p, and set

\[ \hat{f}(z) = \prod_k |\varphi_{z_k}(z)|(2 - |\varphi_{z_k}(z)|). \]

Then there exists a constant C = C(\omega) > 0 such that

\[ \|\hat{f}\|_{L_\omega^p}^p \leq C\|f\|_{\mathcal{A}_\omega^p}^p. \]

**Proof.** Assume f(0) \neq 0. By arguing as in (3.6) and (3.7), and using Lemma 3.2 and (3.2), we obtain

\[ \sum_k \log \frac{1}{|z_k|(2 - |z_k|)} = \int_0^1 \log \frac{1}{r(2 - r)} \, dn(r) = \int_0^1 \frac{2(1 - r)}{2 - r} \, n(r) \, dr \]

\[ = \int_{\mathbb{D}} \log |f(z)| \frac{dA(z)}{(2 - |z|)^2|z|} - \log |f(0)| \]

\[ \leq \int_{\mathbb{D}} \log |f(z)| \, dA(z) - \log |f(0)|, \]

where the last inequality can be established as (3.9). Replacing f by f \circ \varphi_\zeta, we obtain

\[ \log \hat{f}(\zeta) = \log \frac{|f(\zeta)|}{\prod_k |\varphi_{z_k}(\zeta)|(2 - |\varphi_{z_k}(\zeta)|)} \leq \int_{\mathbb{D}} \log |f(\varphi_\zeta(z))| \, dA(z) \]

for all \zeta outside of zeros of f. By proceeding as in (3.10) and (3.12), we deduce

\[ \|\hat{f}\|_{L_\omega^p}^p \leq e^{C_1} \int_{\mathbb{D}} |f(u)|^{p\omega(u)} \left( \int_{\mathbb{D}} |\varphi_\zeta(u)|^2(2 - |\varphi_\zeta(u)|^2) \, dA(z) \right) \omega(u), \]

where \( C_1 \) is given by (3.11) and is thus bounded by Lemma 1.8. The assertion follows.

Theorem 3.5 gives a partial answer to a question posed by Aleman and Sundberg in [12, p. 10]. They asked:

If \omega is a radial continuous and decreasing weight such that \omega(r) < 1 for all 0 \leq r < 1, does the condition

\[ (3.26) \limsup_{r \to 1^-} \frac{\log \omega(r)}{\log(1 - r)} < \infty \]

guarantee that any subset of an \( \mathcal{A}_\omega^p \)-zero set is necessarily an \( \mathcal{A}_\omega^p \)-zero set?
The assumption (3.26) on $\omega$ is equivalent to saying that $\omega$ is continuous and decreasing, and there exists $C = C(\omega) > 0$ such that $(1 - r)^C \leq \omega(r) < 1$ for all $0 \leq r < 1$. By Theorem 3.5 and the definition of invariant weights the answer is affirmative if $\omega$ is a radial continuous weight satisfying (1.2). In this case $\psi(\omega(r)) \geq (1 - r)$ and $\omega(r) \geq (1 - r)^{\alpha}$ for some $\alpha > -1$, so (3.26) is satisfied. However, there are radial continuous weights $\omega$ that satisfy (3.26), and for which (1.2) fails. For example, consider $\omega$ whose graph lies between those of $\omega_1(r) = (1 - r)/2$ and $\omega_2 \equiv 1/2$, and that is constructed in the following manner. Let $r_k = 1 - e^{-\varepsilon^k}$ and $s_k = \frac{r_k + 1}{1 + \varepsilon}$ for all $k \in \mathbb{N}$. Then $0 < r_k < s_k < r_{k+1} < 1$ and
\begin{equation}
(3.27) \quad \frac{1 - r_k}{3} < 1 - s_k < 1 - r_k, \quad k \in \mathbb{N}.
\end{equation}
Define the decreasing continuous function $\omega$ by
\begin{equation}
(3.28) \quad \omega(r) = \begin{cases} 
\frac{1 - r}{3}, & r \in [0, s_1], \\
\frac{1 - s_k}{2} + \frac{s_k - s_{k+1}}{2(1 + r_k)} (r - r_{k+1}), & r \in [s_k, r_{k+1}], \quad k \in \mathbb{N},
\end{cases}
\end{equation}
Then (3.27) yields $1 - r_k \approx 1 - s_k$ and
\[ \frac{\omega(r_k)}{\omega(s_k)} = \frac{1 - s_{k+1} - 1}{s_k} \geq \frac{1 - r_{k+1}}{3(1 - r_k)} \to \infty, \quad k \to \infty, \]
and hence $\omega$ does not satisfy (1.2). Since clearly $\omega \not\in \mathcal{I}$, we also deduce $\omega \not\in \mathcal{I} \cup \mathcal{R} \cup \mathcal{I}_{nv}$.

The next two results on zero sets of functions in $A^p_{\mathcal{I}}$ are readily obtained by [47, p. 208–209] with Lemma 3.6 in hand, and therefore the details are omitted. These results say, in particular, that if $\{z_k\}$ is an $A^p_{\mathcal{I}}$-zero set and $\{w_k\}$ are sufficiently close to $\{z_k\}$, then $\{w_k\}$ is also an $A^p_{\mathcal{I}}$-zero set.

**Theorem 3.7.** Let $0 < p < \infty$ and $\omega \in \mathcal{I}_{nv}$. Let $\{z_k\}$ be an arbitrary subset of the zero set of $f \in A^p_{\mathcal{I}}$. If $\{w_k\}$ is a sequence such that
\begin{equation}
(3.29) \quad \sum_k \left| \frac{w_k - z_k}{1 - \overline{z_k} w_k} \right| < \infty,
\end{equation}
then the function $g$, defined by
\[ g(z) = f(z) \prod_k \frac{B_{w_k}(z)}{B_{z_k}(z)}, \quad z \in \mathbb{D}, \]
belongs to $A^p_{\mathcal{I}}$.

**Corollary 3.8.** Let $0 < p < \infty$ and $\omega \in \mathcal{I}_{nv}$. Let $\{z_k\}$ be the zero set of $f \in A^p_{\mathcal{I}}$. If $\{w_k\}$ satisfies (3.29), then $\{w_k\}$ is an $A^p_{\mathcal{I}}$-zero set.

The rest of the results in this section concern radial weights. The first of them will be used to show that $A^p_{\mathcal{I}}$-zero sets depend on $p$.

**Theorem 3.9.** Let $0 < p < \infty$ and let $\omega$ be a radial weight. Let $f \in A^p_{\mathcal{I}}$, $f(0) \neq 0$, and let $\{z_k\}$ be its zero sequence repeated according to multiplicity and ordered by increasing moduli. Then
\begin{equation}
(3.30) \quad \prod_{k=1}^n \frac{1}{|z_k|} = o \left( \left( \int_{1 - \frac{1}{n}}^1 \omega(r) \frac{1}{r} \right)^{-\frac{1}{p}} \right), \quad n \to \infty.
\end{equation}
Proof. Let \( f \in A^p_\omega \) and \( f(0) \neq 0 \). By multiplying Jensen’s formula (3.3) by \( p \), and applying the arithmetic-geometric mean inequality, we obtain
\[
|f(0)|^p \prod_{k=1}^n \frac{r^p}{|z_k|^p} \leq M^p_p(r, f)
\]
for all \( 0 < r < 1 \) and \( n \in \mathbb{N} \). Moreover,
\[
\lim_{r \to 1^-} M^p_p(r, f) \int_r^1 \omega(s) \, ds \leq \lim_{r \to 1^-} \int_r^1 M^p_p(s, f) \omega(s) \, ds = 0,
\]
so taking \( r = 1 - \frac{1}{n} \) in (3.31), we deduce
\[
\prod_{k=1}^n \frac{1}{|z_k|} \leq M_p \left( 1 - \frac{1}{n} \right) f = o \left( \left( \int_{1 - \frac{1}{n}}^1 \omega(r) \, dr \right)^{-\frac{1}{2}} \right), \quad n \to \infty,
\]
as desired. \( \square \)

The next result improves and generalizes [46, Theorem 4.6]. Moreover, its proof shows that condition (3.30) is a sharp necessary condition for \( \{z_k\} \) to be an \( A^p_\omega \)-zero set.

**Theorem 3.10.** Let \( 0 < q < \infty \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \). Then there exists \( f \in \cap_{p<q} A^p_\omega \) such that its zero sequence \( \{z_k\} \), repeated according to multiplicity and ordered by increasing moduli, does not satisfy (3.30) with \( p = q \). In particular, there is a \( \cap_{p<q} A^p_\omega \)-zero set which is not an \( A^p_\omega \)-zero set.

**Proof.** The proof uses ideas from [32, Theorem 3], see also [48, 49]. Define
\[
f(z) = \prod_{k=1}^\infty F_k(z), \quad z \in \mathbb{D},
\]
where
\[
F_k(z) = \frac{1 + a_k z^{2^k}}{1 + a_k^{-1} z^{2^k}}, \quad z \in \mathbb{D}, \quad k \in \mathbb{N},
\]
and
\[
a_k = \left( \frac{\int_{1-2^{k+1}}^1 \omega(s) \, ds}{\int_{1-2^k}^1 \omega(s) \, ds} \right)^{1/q}, \quad k \in \mathbb{N}.
\]
By Lemma 1.1 there exists a constant \( C_1 = C_1(q, \omega) > 0 \) such that
\[
1 < a_k \leq C_1 < \infty, \quad k \in \mathbb{N}.
\]
Therefore \( \limsup_{k \to \infty} (a_k - a_k^{-1}) 2^{-k} \leq \limsup_{k \to \infty} a_k^{2^{-k}} = 1 \), and hence the product in (3.33) defines an analytic function in \( \mathbb{D} \). The zero set of \( f \) is the union of the zero sets of the functions \( F_k \), so \( f \) has exactly \( 2^k \) simple zeros on the circle \( \{z : |z| = a_k^{-2^{-k}}\} \) for each \( k \in \mathbb{N} \). Let \( \{z_j\}_{j=1}^\infty \) be the sequence of zeros of \( f \) ordered by increasing moduli, and denote \( N_n = 2 + 2^2 + \cdots + 2^n \). Then \( 2^n \leq N_n \leq 2^{n+1} \), and hence
\[
\prod_{k=1}^{N_n} \frac{1}{|z_k|} \geq \prod_{k=1}^n a_k = \left( \frac{\int_{1-2^{n+1}}^1 \omega(s) \, ds}{\int_{1-2^{n+1}}^1 \omega(s) \, ds} \right)^{1/q} \geq \left( \frac{\int_{1-\frac{1}{n}}^1 \omega(s) \, ds}{\int_{1-\frac{1}{n}}^1 \omega(s) \, ds} \right)^{1/q}.
\]
It follows that \( \{z_j\}_{j=1}^{\infty} \) does not satisfy (3.30), and thus \( \{z_j\}_{j=1}^{\infty} \) is not an \( A_\omega^p \)-zero set by Theorem 3.9.

We turn to prove that the function \( f \) defined in (3.33) belongs to \( A_\omega^p \) for all \( p \in (0, q) \). Set \( r_n = e^{-2^{-n}} \) for \( n \in \mathbb{N} \), and observe that

\[
|f(z)| = \prod_{k=1}^{n} \frac{a_k}{1 + |z|^{2^k}} \prod_{j=1}^{\infty} \left| \frac{1 + a_n z^{2^{n+j}}}{1 + a_{n+j} z^{2^{n+j}}} \right|.
\]  

(3.35)

The function \( h_1(x) = \frac{x + a}{1 + a x} \) is increasing on \((0, 1)\) for each \( a \in [0, 1) \), and therefore

\[
\left| \frac{1 + a_n z^{2^{n+j}}}{1 + a_{n+j} z^{2^{n+j}}} \right| = a^{n+j} \left| \frac{1 + a^{-1} z^{2^{n+j}}}{1 + a_{n+j} z^{2^{n+j}}} \right| 
\leq a^{n+j} \frac{1 + a_{n+j} \left( \frac{1}{z} \right)^{2^j}}{1 + a_{n+j} \left( \frac{1}{z} \right)^{2^j}}, \quad |z| \leq r_n, \quad j, n \in \mathbb{N}
\]  

(3.36)

Since \( h_2(x) = \frac{1 + a_n x}{1 + a x} \) is increasing on \((0, \infty)\) for each \( a \in (0, \infty) \), (3.34) and (3.36) yield

\[
\prod_{j=1}^{\infty} \left| \frac{1 + a_n z^{2^{n+j}}}{1 + a_{n+j} z^{2^{n+j}}} \right| \leq \prod_{j=1}^{\infty} \left( 1 + C_1 \left( \frac{1}{z} \right)^{2^j} \right) = C_2 < \infty,
\]  

(3.37)

whenever \( |z| \leq r_n \) and \( n \in \mathbb{N} \). So, by using (3.35), (3.37), Lemma 1.1 and the inequality \( e^{-x} \geq 1 - x, \ x \geq 0 \), we obtain

\[
|f(z)| \leq C_2 \prod_{k=1}^{n} a_k \lesssim \left( \frac{1}{\int_{1 - 2^{-n+1}}^{1} \omega(s) \, ds} \right)^{1/q} \lesssim \left( \frac{1}{\int_{1 - 2^{-n}}^{1} \omega(s) \, ds} \right)^{1/q}
\leq \left( \frac{1}{\int_{r_n}^{1} \omega(s) \, ds} \right)^{1/q}, \quad |z| \leq r_n, \quad n \in \mathbb{N}
\]  

(3.38)

Let now \( |z| \geq 1/\sqrt{e} \) be given and fix \( n \in \mathbb{N} \) such that \( r_n \leq |z| < r_{n+1} \). Then (3.38), the inequality \( 1 - x \leq e^{-x} \leq 1 - \frac{x}{2}, \ x \in [0, 1] \), and Lemma 1.1 give

\[
|f(z)| \leq M_\infty(r_{n+1}, f) \lesssim \left( \frac{1}{\int_{r_{n+1}}^{1} \omega(s) \, ds} \right)^{1/q}
\leq \left( \frac{1}{\int_{1 - 2^{-(n+2)}}^{1} \omega(s) \, ds} \right)^{1/q} \lesssim \left( \frac{1}{\int_{1 - 2^{-n}}^{1} \omega(s) \, ds} \right)^{1/q}
\leq \left( \frac{1}{\int_{r_n}^{1} \omega(s) \, ds} \right)^{1/q},
\]

and hence

\[
M_\infty(r, f) \lesssim \left( \frac{1}{\int_{r}^{1} \omega(s) \, ds} \right)^{1/q}, \quad 0 < r < 1.
\]
This and the identity $\psi_\omega(r) = \frac{1}{1 - r} \psi_\omega(r)$ of Lemma 1.4(iii), with $\alpha = p/q < 1$ and $r = 0$, yield

$$
\|f\|_{A^p_\omega}^p \lesssim \int_0^1 \frac{\omega(r) \, dr}{\left( \int_r^1 \omega(s) \, ds \right)^{p/q}} = \int_0^1 \tilde{\omega}(r) \, dr = \frac{q}{q - p} \left( \int_0^1 \omega(s) \, ds \right)^{\frac{q - p}{q}} < \infty.
$$

This finishes the proof. \qed

The proof of Theorem 3.10 remains valid for any radial weight satisfying (1.14) for some $\beta > 0$. By observing the proof of Lemma 1.1, this is the case, in particular, if $\psi_\omega(r) \gtrsim (1 - r)$ for all $r$ close to one.

The proof of Theorem 3.10 implies that the union of two $A^p_\omega$-zero sets is not necessarily an $A^p_\omega$-zero set if $\omega \in \mathcal{I} \cup \mathcal{R}$. More precisely, by arguing similarly as in the proof of [46, Theorem 5.1] we obtain the following result.

**Corollary 3.11.** Let $0 < p < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$. Then the union of two $A^p_\omega$-zero sets is an $A^{p/2}_{\omega^2}$-zero set. However, there are two $\cap_{p<q} A^p_\omega$-zero sets such that their union is not an $A^{p/2}_{\omega^2}$-zero set.

**Proof.** The first assertion is an immediate consequence of the Cauchy-Schwarz inequality. To see the second one, use the proof of Theorem 3.10 to find $f \in \cap_{p<q} A^p_\omega$, $f(0) \neq 0$, such that its zero sequence $\{z_k\}$ satisfies

$$
(3.39) \quad \prod_{k=1}^n \frac{1}{|z_k|} \gtrsim \left( \int_{1-\frac{1}{n}}^1 \omega(r) \, dr \right)^{\frac{1}{q}}.
$$

Choose $\theta$ such that $\{e^{i\theta} z_k\} \cap \{z_k\} = \emptyset$, and let $\{w_k\}$ be the union $\{e^{i\theta} z_k\} \cup \{z_k\}$ organized by increasing moduli. Then (3.39) and Lemma 1.1 yield

$$
\frac{2n}{k=1} \frac{1}{|w_k|} = \left( \frac{1}{\sum_{k=1}^n |z_k|} \right)^2 \gtrsim \left( \int_{1-\frac{1}{n}}^1 \omega(r) \, dr \right)^{\frac{2}{q}} \gtrsim \left( \int_{1-\frac{1}{n}}^1 \omega(r) \, dr \right)^{\frac{2}{q}}.
$$

Therefore $\{w_k\}$ is not an $A^{p/2}_{\omega^2}$-zero set by Theorem 3.9. \qed

The next result can be obtained by bearing in mind Lemma 1.1 and using arguments similar to those in [41, p. 101–103]. One shows that if $f \in A^p_\omega$, then (3.40) is satisfied, and this together with Lemma 1.1 implies (3.41).

**Proposition 3.12.** Let $0 < p < \infty$, $\omega \in \mathcal{I} \cup \mathcal{R}$ and $f \in A^p_\omega$. Then

$$
(3.40) \quad N(r) \lesssim \left( \log \frac{1}{\int_0^1 \omega(s) \, ds} \right)^{p/q}, \quad r \to 1^-,
$$

and

$$
(3.41) \quad n(r) \lesssim \left( \frac{1}{1 - r} \log \frac{1}{\int_0^1 \omega(s) \, ds} \right)^{p/q}, \quad r \to 1^-.
$$

It is well known that if $\omega$ is any radial weight, then there is an $A^p_\omega$-zero set which does not satisfy the Blaschke condition [29, p. 94]. However, if $\{z_k\}$ is an $A^p_\omega$-zero set, then

$$
(3.42) \quad \sum_k (1 - |z_k|) \left( \log \frac{1}{1 - |z_k|} \right)^{-p/q} < \infty.
$$
3.2. ZEROS OF FUNCTIONS IN $A^p_\omega$

for all $\varepsilon > 0$ by [29, p. 95]. This together with the observations (ii) and (iii) to Lemma 1.1 show that the same is true for $A^p_\omega$-zero sets if $\omega \in I \cup R$. We will improve this last statement. To do so we let $\omega$ be a radial weight such that $\int_0^1 \omega(r) \, dr < 1$, and consider the increasing differentiable function

\[
(3.43) \quad h(r) = \log \frac{1}{\int_0^r \omega(s) \, ds}, \quad 0 \leq r < 1,
\]

that satisfies $h'(r) = 1/\psi_\omega(r)$ and $h(r) > 0$ for all $0 \leq r < 1$.

**Lemma 3.13.** Let $f \in H(D)$ such that its ordered sequence of zeros $\{z_k\}$ satisfies $N(r) \asymp h(r)$, as $r \to 1^-$. Further, let $\omega \in I \cup R$ with $\int_0^1 \omega(r) \, dr < 1$, and let $\tau : [0, \infty) \to [0, \infty)$ be an increasing function such that $x \lesssim \tau(x)$ and $\tau'(x) \lesssim \tau(x) + 1$. Then

\[
(3.44) \quad \sum \frac{1 - |z_k|}{\tau \left( \log \frac{1}{\int_{|z_k|}^1 \omega(s) \, ds} \right)} < \infty
\]

if and only if

\[
(3.45) \quad \int_{R_0}^\infty \frac{\tau'(x)}{\tau^2(x)} \, dx < \infty
\]

for some $R_0 \in (0, \infty)$.

**Proof.** Let $f, \{z_k\}, \omega$ and $\tau$ be as in the statement, and let $h$ be the function given in (3.43). Without loss of generality we may assume that $f(0) \neq 0$. Since $N(r) \asymp h(r)$, we have $n(r) \lesssim h(r)/(1-r)$ by the paragraph just before Proposition 3.12. Therefore (1.3) for $\omega \in R$ and (1.7) for $\omega \in I$, and two integrations by parts yield

\[
\sum_{|z_k| \geq \frac{1}{2}} \tau \left( \log \frac{1}{\int_{|z_k|}^1 \omega(s) \, ds} \right) \leq \int_{\frac{1}{2}}^1 \left( 1 - \frac{1}{\tau(h(r))} \right) n(r) \, dr
\]

\[
\leq 1 + \int_{\frac{1}{2}}^1 \frac{1}{\tau(h(r))} \left( 1 + \frac{\tau'(h(r))h'(r)(1-r)}{\tau(h(r))} \right) n(r) \, dr
\]

\[
\leq 1 + \int_{\frac{1}{2}}^1 \frac{1}{\tau(h(r))} \frac{n(r)}{r} \, dr
\]

\[
\leq 1 + \int_{\frac{1}{2}}^1 \frac{\tau'(h(r))h'(r)}{\tau^2(h(r))} N(r) \, dr \leq 1 + \int_{h(\frac{1}{2})}^\infty \frac{\tau'(x)}{\tau^2(x)} \, dx,
\]

which shows that (3.45) implies (3.44).
A reasoning similar to that above gives

\[
\sum_k \frac{1 - |z_k|}{\tau \left( \log \frac{1}{\int_{|z_k|} \omega(s) \, ds} \right)} \geq \int_{r_0}^{1} \frac{1 - r}{\tau(h(r))} \, dn(r)
\]

\[
\geq -1 + \int_{r_0}^{1} \frac{1}{\tau(h(r))} \left( 1 + \frac{\tau'(h(r))h'(r)(1 - r)}{\tau(h(r))} \right) n(r) \, dr
\]

\[
\geq -1 + r_0 \int_{r_0}^{1} \frac{n(r)}{r} \, dr
\]

\[
\geq -1 + \int_{r_0}^{1} \frac{\tau'(h(r))h'(r)}{\tau^2(h(r))} N(r) \, dr \geq -1 + \int_{h(r_0)}^{\infty} \frac{\tau'(x)}{\tau^2(x)} x \, dx
\]

for all sufficiently large \( r_0 \), and thus (3.44) implies (3.45).

It is worth noticing that (3.45) is equivalent to

(3.46) \[
\int_{R_0}^{\infty} \frac{dx}{\tau(x)} < \infty.
\]

This can be seen by integrating the identity

\[
\frac{\tau'(x)}{\tau^2(x)} x + \frac{d}{dx} \left( \frac{x}{\tau(x)} \right) = \frac{1}{\tau(x)}
\]

from \( R_0 \) to \( R \), and then letting \( R \to \infty \).

**Theorem 3.14.** Let \( 0 < p < \infty \) and \( \omega \in \mathbb{I} \cup \mathbb{R} \) such that \( \int_{0}^{1} \omega(r) \, dr < 1 \). Let \( \tau : [0, \infty) \to [0, \infty) \) be an increasing function such that \( x \leq \tau(x) \) and \( \tau'(x) \leq \tau(x) + 1 \).

(i) If \( f \in A_p^\omega \), then its ordered sequence of zeros \( \{z_k\} \) satisfies (3.44) for each \( \tau \) that obeys (3.46).

(ii) There exists \( f \in A_p^\omega \) such that its ordered sequence of zeros \( \{z_k\} \) satisfies

(3.47) \[
\sum_k \frac{1 - |z_k|}{\tau \left( \log \frac{1}{\int_{|z_k|} \omega(s) \, ds} \right)} = \infty
\]

for each \( \tau \) for which (3.46) fails for some \( R_0 \in (0, \infty) \).

**Proof.** The assertion (i) follows by the first part of the proof of Proposition 3.12, Lemma 3.13 and the discussion related to (3.46). To prove (ii) it suffices to find \( f \in A_p^\omega \) such that \( N(r) \geq h(r) \), where \( h \) is the function defined in (3.43). To do so, we will use arguments similar to those in the proof of [32, Theorem 6]. Take \( \alpha > p \), and let \( \{r_n\} \) be the increasing sequence defined by

(3.48) \[
\int_{r_n}^{1} \omega(r) \, dr = \frac{1}{2^n \alpha}.
\]

Denote \( M_n = E \left( \frac{1}{r_n} \right) \), where \( E(x) \) is the integer such that \( E(x) \leq x < E(x) + 1 \).

By the proof of Lemma 1.1 there exists \( \beta = \beta(\omega) > 0 \) such that (1.14) with \( C = 1 \)
holds for all $r$ sufficiently close to 1. Hence there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

\[
\frac{M_{n+1}}{M_n} \geq \frac{1 - r_n}{1 - r_{n+1}} - (1 - r_n) \geq \left( \frac{\int_{r_n}^1 \omega(r) \, dr}{\int_{r_{n+1}}^1 \omega(r) \, dr} \right)^{\frac{1}{\beta}} - (1 - r_n)
\]

\[
\geq 2^{\frac{1}{2\pi}} - (1 - r_n) \geq \lambda, \quad n \geq n_0.
\]

Therefore the analytic function

\[
g(z) = \sum_{n=n_0}^{\infty} 2^n z^{M_n}, \quad z \in \mathbb{D},
\]

is a lacunary series, and hence $M_p(r, g) \simeq M_2(r, g)$ for all $0 < p < \infty$ [85].

Next, we claim that there exists $r_0 \in (0, 1)$ such that

\[
M_2(r, g) \simeq \left( \frac{1}{r} \int_{r}^1 \omega(r) \, dr \right)^{-\frac{1}{\alpha}}, \quad r \geq r_0,
\]

which together Lemma 1.4(iii) for $\frac{p}{q} < 1$ implies $g \in A^p_q$.

Since $g$ is a lacunary series, arguing as in the proof of [32, Theorem 6], we deduce

\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| \, d\theta \gtrsim h(r), \quad r \in [r_1, 1),
\]

for some $r_1 \in (0, 1)$. This together with [61, Theorem on p. 276] implies that there exists $a \in \mathbb{C}$ such that

\[
N(r, g - a) \gtrsim h(r), \quad r_2 \leq r < 1,
\]

for some $r_2 \in [r_1, 1)$. Therefore $f = g - a \in A^p_q$ has the desired properties. Finally, we will prove (3.50). We begin with proving (3.50) for $r = r_N$, where $N \geq n_0$. To do this, note first that

\[
\sum_{n=n_0}^{N} 2^{2n} r_N^{\alpha} \left( \frac{1}{r_n} \right)^{1/\alpha} \leq \sum_{n=0}^{N} 2^{2n} \leq 4 \sum_{n=0}^{N} 2^{2n} = \frac{4}{3} \left( \int_{r_N}^1 \omega(r) \, dr \right)^{-\frac{2}{\pi}}.
\]

To deal with the remainder of the sum, we observe that (3.49) implies

\[
\frac{1 - r_n}{1 - r_{n+j}} \geq \left( \frac{\int_{r_n}^1 \omega(r) \, dr}{\int_{r_{n+j}}^1 \omega(r) \, dr} \right)^{1/\beta} = 2^{\frac{1}{2\pi}}, \quad n \geq n_0.
\]

This and the inequality $1 - r \leq \log \frac{1}{r}$ give

\[
\sum_{n=N+1}^{\infty} 2^{2n} r_N^{\alpha} \left( \frac{1}{r_n} \right)^{1/\alpha} \leq 2^{2N} \sum_{j=1}^{\infty} 2^{2j} e^{-C \frac{r_n}{r_{n+j}}} \leq 2^{2N} \sum_{j=1}^{\infty} 2^{2j} e^{-C \frac{r_n}{r_{n+j}}} = C(\beta, \alpha, \omega) \left( \int_{r_N}^1 \omega(s) \, ds \right)^{-\frac{2}{\pi}}.
\]

Since $\beta = \beta(\omega)$, this together with (3.53) yields

\[
M_2^2(r_N, g) \lesssim \left( \int_{r_N}^1 \omega(r) \, dr \right)^{-\frac{2}{\pi}}, \quad N \geq n_0.
\]
Let now \( r \in [r_{n_0}, 1) \) be given and fix \( N \geq n_0 \) such that \( r_N \leq r < r_{N+1} \). Then, by (3.54), there exists \( C = C(\alpha, \omega) \) such that

\[
M_2^2(r, g) \leq M_2^2(r_{N+1}, g) \leq C 2^{2N} \left( \int_r^{1} \omega(s) \, ds \right)^{-\frac{2}{\omega}}.
\]

Further,

\[
M_2^2(r_N, g) \geq \sum_{n=n_0}^{N} 2^{2n} r_N^{-2E\left(\frac{1}{\omega}\right)} \geq C 2^{2N} \left( \int_{r_N}^{1} \omega(r) \, dr \right)^{-\frac{2}{\omega}}
\]

for all \( N \geq n_0 \). Now for a given \( r \in [r_{n_0}, 1) \), choose \( N \in \mathbb{N} \) such that \( r_N \leq r < r_{N+1} \).

It follows that

\[
M_2^2(r, g) \geq M_2^2(r_{N}, g) \geq C 2^{2N+2} \geq C \left( \int_{r_{N+1}}^{1} \omega(r) \, dr \right)^{-\frac{2}{\omega}} = \left( \int_{r}^{1} \omega(r) \, dr \right)^{-\frac{2}{\omega}},
\]

which gives (3.50).

The proof of Theorem 3.14(ii) has a consequence which is of independent interest. To state it, we recall two things. First, the proximity function of an analytic (or meromorphic) function \( g \) in \( D \) is defined as

\[
m(r, g) = \frac{1}{2\pi} \int_0^{2\pi} \log |g(re^{i\theta})| \, d\theta, \quad 0 < r < 1.
\]

This is the function appearing in (3.51). Second, each convex function \( \Phi \) can be written as \( \Phi(r) = \int_0^r \phi(t) \, dt \), where \( \phi \) is continuous increasing unbounded function, see, for example, [85, p. 24].

**Theorem 3.15.** Let \( \Phi(r) = \int_0^r \phi(t) \, dt \) be an increasing unbounded convex function on \((0, 1)\) such that either \( \phi(r)(1-r) \asymp 1 \) or \( \lim_{r \to 1^-} \phi(r)(1-r) = 0 \). Then there exists \( g \in \mathcal{H}(D) \) such that

\[
\log M_\infty(r, g) \asymp m(r, g) \asymp \Phi(r), \quad r \to 1^-.
\]

**Proof.** For a given \( \omega \in I \cup R \), the function \( g \in \mathcal{H}(D) \) constructed in the proof of Theorem 3.14 satisfies

\[
m(r, g) \asymp \log \frac{1}{\int_r^1 \omega(s) \, ds}, \quad r \to 1^-.
\]

Namely, by (3.51) the growth of \( m(r, g) \) has the correct lower bound, and the same upper bound follows by Jensen’s inequality and (3.50). Moreover, (3.50) and a minor modification in its proof show that

\[
M_\infty(r, g) \asymp \frac{1}{\left( \int_r^1 \omega(s) \, ds \right)^{\frac{1}{\omega}}}, \quad r \to 1^-.
\]

If we now take

\[
\omega(r) = \phi(r) \exp \left( - \int_0^r \phi(t) \, dt \right),
\]

then \( \psi_\omega(r) = (\phi(r))^{-1} \) and

\[
\log \frac{1}{\int_r^1 \omega(s) \, ds} = \int_0^r \phi(t) \, dt = \Phi(r).
\]
Therefore \( \omega \in I \cup R \) by the assumptions, and \( g \) has the desired properties. \( \square \)

Theorem 3.15 shows that we can find \( g \in \mathcal{H}(\mathbb{D}) \) such that \( m(r, g) \) grows very slowly. For example, by choosing \( \omega \in I \) appropriately, there exists \( g \in \mathcal{H}(\mathbb{D}) \) such that
\[
m(r, g) \asymp \log \log \frac{1}{1-r}, \quad r \to 1^-.\]

If the function \( \Phi \) in Theorem 3.15 exceeds \( \log \frac{1}{1-r} \) in growth, then the existence of \( g \) such that \( m(r, g) \asymp \Phi(r) \) was proved by Shea [78, Theorem 1], see also related results by Clunie [23] and Linden [55].

Typical examples of functions \( \tau \) satisfying (3.46) are
\[
\tau_{N, \varepsilon}(x) = x \prod_{n=1}^{N-1} \log \exp \frac{1}{1-|z_k|} (\log \log \exp \frac{1}{1-|z_k|})^{1-|z_k|}, \quad N \in \mathbb{N}, \quad \varepsilon > 0.
\]
To see a concrete example, consider \( \omega = v_\alpha \) and \( \tau = \tau_{1, \varepsilon} \), where \( 1 < \alpha < \infty \) and \( \varepsilon > 0 \). If \( f \in A^p_{v_\alpha} \) and \( \{z_k\} \) is its ordered sequence of zeros, then
\[
\sum_k \log \exp \frac{1}{1-|z_k|} (\log \log \exp \frac{1}{1-|z_k|})^{1+\varepsilon} < \infty
\]
for all \( \varepsilon > 0 \). The functions \( \tau_{N, 0} \) do not satisfy (3.46), and therefore there exists \( f \in A^p_{v_\alpha} \) such that its ordered sequence of zeros \( \{z_k\} \) satisfies
\[
\sum_k \log \exp \frac{1}{1-|z_k|} (\log \log \exp \frac{1}{1-|z_k|})^{1+\varepsilon} = \infty.
\]
Whenever \( \omega \in I \cup R \), Theorem 3.14 is the best we can say about the zero distribution of functions in \( A^p_{\omega} \) in terms of conditions depending on their moduli only.

### 3.3. Zeros of functions in the Bergman-Nevanlinna class \( BN_\omega \)

We now turn back to consider the condition (3.42) for further reference. It is not hard to find a space of analytic functions on \( \mathbb{D} \) for which the zero sets are characterized by this neat condition. To give the precise statement, we say that \( f \in H(\mathbb{D}) \) belongs to the Bergman-Nevanlinna class \( BN_\omega \), if
\[
\|f\|_{BN_\omega} = \int_\mathbb{D} \log^+ |f(z)| \omega(z) dA(z) < \infty.
\]

**Proposition 3.16.** Let \( \omega \in I \cup R \). Then \( \{z_k\} \) is a \( BN_\omega \)-zero set if and only if
\[
\sum_k \omega^*(z_k) < \infty.
\]

**Proof.** If \( f \in BN_\omega \) and \( \{z_k\} \) is its zero set, then by integrating Jensen's formula we obtain 2 \( \sum_k \omega^*(z_k) \leq \|f\|_{BN_\omega} \). Conversely, let \( \{z_k\} \) be a sequence such that (3.55) is satisfied. By following the construction in [41, p. 131–132] and using Lemma 2.3, we find \( f \in BN_\omega \) such that it vanishes at the points \( z_k \) and nowhere else. We omit the details. \( \square \)

Lemma 3.17 shows that the condition (3.55) is equivalent to the convergence of certain integrals involving the counting functions. We will use this result in Section 7.2 when studying linear differential equations with solutions in Bergman-Nevanlinna classes.
Lemma 3.17. Let \( \omega \) be a radial weight and denote \( \tilde{\omega}(r) = \int_{1}^{r} \omega(s) \, ds \). Let \( f \in \mathcal{H}(\mathbb{D}) \) and let \( \{ z_k \} \) be its zero sequence. Then the following conditions are equivalent:

1. \( \sum_k \omega^*(z_k) < \infty \);
2. \( \int_{0}^{1} N(r, f) \omega(r) \, dr < \infty \);
3. \( \int_{0}^{1} n(r, f) \tilde{\omega}(r) \, dr < \infty \).

Proof. Let \( \rho \in (\rho_0, 1) \), where \( \rho_0 \in (0, 1) \) is fixed. An integration by parts gives

\[
\int_{\rho_0}^{\rho} N(r, f) \omega(r) \, dr = -N(\rho, f) \tilde{\omega}(\rho) + N(\rho_0, f) \tilde{\omega}(\rho_0) + \int_{\rho_0}^{\rho} \frac{n(r, f)}{r} \tilde{\omega}(r) \, dr.
\]

(3.56)

If (2) is satisfied, then \( N(\rho, f) \tilde{\omega}(\rho) \to 0 \), as \( \rho \to 1^- \). Therefore (3.56) shows that (2) and (3) are equivalent. Moreover,

\[
\int_{\rho_0}^{\rho} n(r, f) \tilde{\omega}(r) \, dr = -n(\rho, f) \int_{\rho}^{1} \tilde{\omega}(s) \, ds + n(\rho_0, f) \int_{\rho_0}^{1} \tilde{\omega}(s) \, ds + \sum_{\rho_0 < |z_k| < \rho} \int_{|z_k|}^{1} \tilde{\omega}(s) \, ds,
\]

(3.57)

where

\[
\int_{|z_k|}^{1} \tilde{\omega}(s) \, ds = \int_{|z_k|}^{1} \omega(t)(t - |z_k|) \, dt \asymp \omega^*(z_k), \quad |z_k| \to 1^-.
\]

We deduce from (3.57) that also (1) and (3) are equivalent. \( \square \)

By Proposition 3.16 the condition (3.42) characterizes zero sets in \( BN^{\alpha_2}_{\epsilon} \). Proposition 3.16 further implies that for each increasing differentiable unbounded function \( \lambda : [0, 1) \to (0, \infty) \) such that \( \omega_\lambda = \lambda' / \lambda^2 \in \mathcal{I} \cup \mathcal{R} \), the condition

\[
\sum_k \frac{1 - |z_k|}{\lambda(z_k)} < \infty
\]

characterizes the zero sets in \( BN_{\omega_\lambda} \). This because \( \omega_\lambda^*(r) \asymp \frac{1 - r}{\lambda(r)} \) by Lemma 1.6.
CHAPTER 4

Integral Operators and Equivalent Norms

One of our main objectives is to characterize those symbols $g \in \mathcal{H}(D)$ such that the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad z \in D,$$

is bounded or compact from $A_p^\omega$ to $A_q^\omega$, when $\omega$ is a rapidly increasing weight. Pommerenke was probably one of the first authors to consider the operator $T_g$. He used it in [68] to study the space BMOA, which consists of those functions in the Hardy space $H^1$ that have bounded mean oscillation on the boundary $T$ [15, 33]. The space BMOA can be equipped with several different equivalent norms [33]. We will use the one given by

$$\|g\|^2_{\text{BMOA}} = \sup_{a \in D} \frac{\int_{S(a)} |g'(z)|^2 (1 - |z|^2) dA(z)}{1 - |a|} + |g(0)|^2.$$

The operator $T_g$ has received different names in the literature: the Pommerenke operator, a Volterra type operator (the choice $g(z) = z$ gives the usual Volterra operator), the generalized Cesàro operator (the usual Cesàro operator is obtained when $g(z) = -\log(1-z)$), a Riemann-Stieltjes type operator, or simply an integral operator. The operator $T_g$ began to be extensively studied after the appearance of the works by Aleman and Siskakis [10, 11], see also [6, 81] for two surveys on the topic. If $\omega$ is either regular or rapidly decreasing, then a description of those $g \in \mathcal{H}(D)$ for which $T_g : A_p^\omega \to A_q^\omega$ is bounded or compact can be found in [8, 11, 64, 65]. In particular, for these weights the boundedness of $T_g : A_p^\omega \to A_p^\omega$ is equivalent to a growth condition on $M_\infty^\omega(r,g')$. It follows from Lemma 1.4(i) and [8, Theorem 4.1] that if $\omega$ is regular, then $T_g : A_p^\omega \to A_p^\omega$ is bounded if and only if $g$ is a Bloch function. Recall that the Bloch space $B$ [14] consists of those $f \in \mathcal{H}(D)$ such that

$$\|f\|_B = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) + |f(0)| < \infty.$$

We will see soon that if $\omega$ is rapidly increasing, then the boundedness of $T_g : A_p^\omega \to A_p^\omega$ can not be characterized by a simple condition depending only on $M_\infty^\omega(r,g')$. The above-mentioned results on the boundedness and compactness of $T_g : A_p^\omega \to A_q^\omega$ are usually discovered by the aid of a Littlewood-Paley type formula that allows one to get rid of the integral appearing from the definition of $T_g$. However, this kind of approach does not work for $A_p^\omega$ when $\omega \in I$, because such a Littlewood-Paley type formula for $A_p^\omega$, with $\omega \in I$ and $p \neq 2$, does not exist by Proposition 4.3. Therefore we are forced to search for alternative norms in $A_p^\omega$ in terms of the first derivative. This leads us to employ ideas that are closely related to the theory of Hardy spaces [7, 6, 10]. It is also worth noticing that if $0 < p \leq 1$ and $\omega \in I$, then the Riesz projection is not necessarily bounded on $L_p^\omega$. 
We say that \( g \in H(\mathbb{D}) \) belongs to \( C^{q,p}(\omega^*) \), \( 0 < p, q < \infty \), if the measure \( |g'(z)|^2 \omega^*(z) \, dA(z) \) is a \( q \)-Carleson measure for \( A^p_\omega \). Moreover, \( g \in C^{q,p}(\omega^*) \) if the identity operator \( I_g : A^p_\omega \to L^q(|g'|^2 \omega^* dA) \) is compact. If \( q \geq p \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), then Theorem 2.1 shows that these spaces only depend on the quotient \( \omega / \varphi \). Consequently, for \( q \geq p \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), we simply write \( C^{q,p}(\omega^*) \) instead of \( C^{q,p}(\omega^*) \). Thus, if \( \alpha \geq 1 \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), then \( C^{\alpha}(\omega^*) \) consists of those \( g \in H(\mathbb{D}) \) such that

\[
\|g\|_{C^{\alpha}(\omega^*)}^2 = |g(0)|^2 + \sup_{I \subset \mathbb{T}} \int_{S(I)} |g'(z)|^2 \omega^*(z) \, dA(z) \leq \frac{1}{\alpha^2} < \infty.
\]

An analogue of this identity is valid for the little space \( C^{q,p}_0(\omega^*) \). The above characterization of the Banach space \( (C^{\alpha}(\omega^*), \| \cdot \|_{C^{\alpha}(\omega^*)}) \) has all the flavor of known characterizations of \( BMOA \) [33], the Lipschitz class \( \Lambda_\alpha \) [7] or the Bloch space \( B \) [82] when \( \alpha \) is chosen appropriately. In fact, Proposition 5.1 shows that \( C^1(\omega^*) = B \) for all \( \omega \in \mathcal{R} \), and

\[
\text{BMOA} \subset C^1(\omega^*) \subset B, \quad \omega \in \mathcal{I},
\]

where both embeddings can be strict at the same time. Unlike \( B \), the space \( C^1(\omega^*) \) can not be described by a simple growth condition on the maximum modulus of \( g' \) if \( \omega \in \mathcal{I} \). This follows by Proposition 5.1 and the fact that \( \log(1 - z) \in A^p_\omega \) for all \( \omega \in \mathcal{I} \). However, if \( \alpha > 1 \) and \( \omega \in \mathcal{I} \), then \( g \in C^{\alpha}(\omega^*) \) if and only if

\[
M_\infty(r, g') \leq \frac{(\omega^*(r))^{\frac{q-1}{q}}}{1-r}, \quad 0 < r < 1,
\]

by Proposition 4.7.

The spaces \( \text{BMOA} \) and \( \mathcal{B} \) are conformally invariant. This property has been used, among other things, in describing those symbols \( g \in H(\mathbb{D}) \) for which \( T_g \) is bounded on \( H^p \) or \( A^p_\omega \). However, the space \( C^1(\omega^*) \) is not necessarily conformally invariant by Proposition 5.6, and therefore different techniques must be employed in the case of \( A^p_\omega \) with \( \omega \in \mathcal{I} \).

The next result gives a complete characterization of when \( T_g : A^p_\omega \to A^q_\omega \) is bounded, provided \( \omega \in \mathcal{I} \cup \mathcal{R} \).

**Theorem 4.1.** Let \( 0 < p, q < \infty \), \( \omega \in \mathcal{I} \cup \mathcal{R} \) and \( g \in H(\mathbb{D}) \).

(i) The following conditions are equivalent:

(a) \( T_g : A^p_\omega \to A^p_\omega \) is bounded;

(b) \( g \in C^1(\omega^*) \).

(ii) If \( 0 < p < q \) and \( \frac{1}{p} - \frac{1}{q} < 1 \), then the following conditions are equivalent:

(a) \( T_g : A^p_\omega \to A^q_\omega \) is bounded;

(b) \( M_\infty(r, g') \leq \frac{(\omega^*(r))^{\frac{q-1}{q}}}{1-r}, \quad r \to 1^-; \)

(c) \( g \in C^2(\omega^*) \).

(iii) If \( \frac{1}{p} - \frac{1}{q} \geq 1 \), then \( T_g : A^p_\omega \to A^q_\omega \) is bounded if and only if \( g \) is constant.

(iv) If \( 0 < q < p < \infty \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), then the following conditions are equivalent:

(a) \( T_g : A^p_\omega \to A^q_\omega \) is bounded;

(b) \( g \in A^q_\omega \), where \( \frac{1}{q} = \frac{1}{p} - \frac{1}{2} \).
4.1. Equivalent norms on $A^p_\omega$

It is worth noticing that the regularity assumption (1.2), that all the weights in $\mathcal{I} \cup \mathcal{R}$ satisfy, is only used in the proof of Theorem 4.1 when showing that (aiv)⇒(biv), the rest of the proof is valid under the hypothesis $\omega \in \mathcal{I} \cup \mathcal{R}$.

A description of the symbols $g \in \mathcal{H}(\mathbb{D})$ such that $T_g : A^p_\omega \to A^q_\omega$ is compact is given in Theorem 4.9.

In Section 4.3 we will show how the techniques developed on the way to the proof of Theorem 4.1 can be applied to the case when $T_g$ acts on the Hardy spaces. In particular, we will give a new proof for the fact that $T_g : H^p \to H^p$ is bounded if and only if $g \in \text{BMOA}$.

### 4.1. Equivalent norms on $A^p_\omega$

For a large class of radial weights, which includes any differentiable decreasing weight and all the standard ones, the most appropriate way to obtain a useful norm involving the first derivative is to establish a kind of Littlewood-Paley type formula [67]. However, if $\omega \in \mathcal{I}$ and $p \neq 2$, this is not possible in general by Proposition 4.3 below. Consequently, we will equip the space $A^p_\omega$ with other norms that are inherited from different equivalent $H^p$ norms. The one that we will obtain by using square functions via the classical Fefferman-Stein estimate [30] appears to be the most useful for our purposes. The square function that arises naturally is obtained by integrating over the lens type region $\Gamma(u)$, see (1.25) for the definition, and also Picture 1.

**Theorem 4.2.** Let $0 < p < \infty$, $n \in \mathbb{N}$ and $f \in \mathcal{H}(\mathbb{D})$, and let $\omega$ be a radial weight. Then

\[
\|f\|_{A^p_\omega}^p = p^2 \int_D |f(z)|^{p-2} |f'(z)|^2 \omega^*(z) dA(z) + \omega(\mathbb{D})|f(0)|^p,
\]

and

\[
\|f\|_{A^p_\omega}^p \leq \int_D \left( \int_{\Gamma(u)} |f^{(n)}(z)|^2 \left( 1 - \left| \frac{z}{u} \right| \right)^{2n-2} dA(z) \right)^{\frac{p}{2}} \omega(u) dA(u)
+ \sum_{j=0}^{n-1} |f^{(j)}(0)|^p,
\]

where the constants of comparison depend only on $p$, $n$ and $\omega$. In particular,

\[
\|f\|_{A^p_\omega}^2 = 4\|f'\|^2_{A^2_\omega} + \omega(\mathbb{D})|f(0)|^2.
\]

**Proof.** Hardy-Stein-Spencer identity [31]

\[
\|f\|_{H^p}^p = \frac{p^2}{2} \int_D |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z) + |f(0)|^p,
\]
applied to \( f_r(z) = f(rz) \), and Fubini’s theorem yield

\[
\|f\|_{A_p^c}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) = 2 \int_0^1 \|f_r\|_{H^p} \omega(r)r \, dr
\]

\[
= p^2 \int_0^1 \left( \int_{\mathbb{D}} |f(rz)|^{p-2} |f'(rz)|^2 r^2 \log \frac{r}{z} \, dA(z) \right) \omega(r) \, dr
\]

\[
+ \omega(\mathbb{D}) |f(0)|^p
\]

\[
= p^2 \int_0^1 \left( \int_0^r \int_0^{2\pi} |f(se^{i\theta})|^{p-2} |f'(se^{i\theta})|^2 d\theta \log \frac{r}{s} \, ds \right) \omega(r) \, dr
\]

\[
+ \omega(\mathbb{D}) |f(0)|^p
\]

\[
= p^2 \int_0^1 \left( \int_0^r \int_0^{2\pi} |f(se^{i\theta})|^{p-2} |f'(se^{i\theta})|^2 d\theta \left( \int_s^1 \log \frac{r}{s} \omega(r) \, dr \right) \right) \, ds
\]

\[
+ \omega(\mathbb{D}) |f(0)|^p
\]

\[
= p^2 \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \omega^*(z) dA(z) + \omega(\mathbb{D}) |f(0)|^p,
\]

which proves (4.3). In particular, the identity (4.5) is valid.

Analogously, the extension of the Littlewood-Paley identity to any \( H^p \), proved by Ahern and Bruna [3], states that

\[
(4.6) \quad \|f\|_{H^p} \approx \int_{\mathbb{T}} \left( \int_{\Gamma_c} |f^{(n)}(z)|^2 (1 - |z|)^{2n-2} dA(z) \right)^{p/2} d\theta + \sum_{j=0}^{n-1} |f^{(j)}(0)|,
\]

see also the earlier results in [30] and [85, Vol II. Chapter 14]. This and the change of variable \( rz = \xi \) give

\[
\|f\|_{A_p^c}^p = 2 \int_0^1 \|f_r\|_{H^p} \omega(r)r \, dr
\]

\[
= \int_0^1 \left( \int_{\mathbb{T}} \left( \int_{\Gamma_c(\xi)} |f^{(n)}(rz)|^2 r^{2n} (1 - |z|)^{2n-2} dA(z) \right)^{p/2} d\xi \right) \omega(r) \, dr
\]

\[
+ \sum_{j=0}^{n-1} |f^{(j)}(0)|^p
\]

\[
= \int_0^1 \int_{\mathbb{T}} \left( \int_{\Gamma_c(\xi)} |f^{(n)}(\xi)|^2 r^{2n-2} \left( 1 - \left| \frac{\xi}{r} \right| \right)^{2n-2} dA(\xi) \right)^{p/2} |d\xi| \omega(r) \, dr
\]

\[
+ \sum_{j=0}^{n-1} |f^{(j)}(0)|^p
\]

\[
= \int_{\mathbb{S}} \left( \int_{\Gamma_u} |f^{(n)}(\xi)|^2 \left( 1 - \left| \frac{\xi}{u} \right| \right)^{2n-2} dA(\xi) \right)^{p/2} \omega(u) \, dA(u)
\]

\[
+ \sum_{j=0}^{n-1} |f^{(j)}(0)|^p,
\]

and thus also (4.4) is valid. \( \square \)
Theorem 4.2 is of very general nature because \( \omega \) is only assumed to be radial. In particular, the Littlewood-Paley formula (4.5) is valid for all radial weights \( \omega \). The next result shows that no asymptotic Littlewood-Paley formula can be found for \( \Lambda_p^\omega \) when \( \omega \in \mathcal{I} \) and \( p \neq 2 \).

**Proposition 4.3.** Let \( p \neq 2 \). Then there exists \( \omega \in \mathcal{I} \) such that, for any function \( \varphi : [0, 1) \to (0, \infty) \), the relation

\[
(4.7) \quad \|f\|_{\Lambda_p^\omega}^p = \int_\mathbb{D} |f'(z)|^p \varphi(|z|) \omega(z) \, dA(z) + |f(0)|^p
\]

can not be valid for all \( f \in \mathcal{H}(\mathbb{D}) \).

**Proof.** Let first \( p > 2 \) and consider the weight \( v_\alpha \), where \( \alpha \) is fixed such that \( 2 < 2(\alpha - 1) \leq p \). Assume on the contrary to the assertion that (4.7) is satisfied for all \( f \in \mathcal{H}(\mathbb{D}) \). Applying this relation to the function \( h_\alpha(z) = z^n \), we obtain

\[
(4.8) \quad \int_0^1 r^{np} v_\alpha(r) \, dr \asymp n^p \int_0^1 r^{(n-1)p} \varphi(r)^p v_\alpha(r) \, dr, \quad n \in \mathbb{N}.
\]

Consider now the lacunary series \( h(z) = \sum_{k=0}^{\infty} z^{\alpha_k} \). It is easy to see that

\[
M_p(r, h) \asymp \left( \log \frac{1}{1-r} \right)^{1/2}, \quad M_p(r, h') \asymp \frac{1}{1-r}, \quad 0 \leq r < 1.
\]

By combining the relations (4.8), (4.9) and

\[
\left( \frac{1}{1-r^p} \right)^p \asymp \sum_{n=1}^{\infty} n^{p-1} r^{(n-1)p}, \quad \log \frac{1}{1-r^p} \asymp \sum_{n=1}^{\infty} n^{-1} r^{np}, \quad 0 \leq r < 1,
\]

we obtain

\[
\int_{\mathbb{D}} |h'(z)|^p \varphi(z)^p v_\alpha(z) \, dA(z) \asymp \int_0^1 \left( \frac{1}{1-r^p} \right)^p \varphi(r)^p v_\alpha(r) \, dr
\]
\[
\asymp \int_0^1 \left( \sum_{n=1}^{\infty} n^{p-1} r^{(n-1)p} \right) \varphi(r)^p v_\alpha(r) \, dr
\]
\[
\asymp \sum_{n=1}^{\infty} n^{p-1} \int_0^1 r^{(n-1)p} \varphi(r)^p v_\alpha(r) \, dr
\]
\[
\asymp \sum_{n=1}^{\infty} n^{-1} \int_0^1 r^{np} v_\alpha(r) \, dr
\]
\[
\asymp \int_0^1 \left( \sum_{n=1}^{\infty} n^{-1} r^{np} \right) v_\alpha(r) \, dr
\]
\[
\asymp \int_0^1 \log \frac{1}{1-r^p} v_\alpha(r) \, dr,
\]

where the last integral is convergent because \( \alpha > 2 \). However,

\[
\|h\|_{\Lambda_p^v_\alpha}^p \asymp \int_0^1 \left( \log \frac{1}{1-r} \right)^{p/2} v_\alpha(r) \, dr = \infty,
\]

since \( p \geq 2(\alpha - 1) \), and therefore (4.7) fails for \( h \in \mathcal{H}(\mathbb{D}) \). This is the desired contradiction.
If $0 < p < 2$, we again consider $v_\alpha$, where $\alpha$ is chosen such that $p < 2(\alpha - 1) \leq 2$, and use an analogous reasoning to that above to prove the assertion. Details are omitted.

Let $f \in H(D)$, and define the non-tangential maximal function of $f$ in the (punctured) unit disc by

$$N(f)(u) = \sup_{z \in \Gamma(u)} |f(z)|, \quad u \in \mathbb{D} \setminus \{0\}.$$  

The following equivalent norm will be used in the proof of Theorem 4.1.

**Lemma 4.4.** Let $0 < p < \infty$ and let $\omega$ be a radial weight. Then there exists a constant $C > 0$ such that

$$\|f\|_{A^p_\omega} \leq \|N(f)\|_{L^p_\omega} \leq C\|f\|_{A^p_\omega}$$

for all $f \in H(\mathbb{D})$.

**Proof.** It follows from [31, Theorem 3.1 on p. 57] that there exists a constant $C > 0$ such that the classical non-tangential maximal function

$$f^*(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{T},$$

satisfies

(4.10) $$\|f^*\|_{L^p(\mathbb{T})} \leq C\|f\|_{H^p}$$

for all $0 < p < \infty$ and $f \in H(\mathbb{D})$. Therefore

$$\|f\|_{A^p_\omega} \leq \|N(f)\|_{L^p_\omega} = \int_D \omega(u) dA(u)$$

$$= \int_0^1 \omega(r) r \int_\mathbb{T} |f(r\zeta)|^p |d\zeta| dr$$

$$\leq C \int_0^1 \omega(r) r \int_\mathbb{T} |f(r\zeta)|^p |d\zeta| dr = C\|f\|_{A^p_\omega},$$

and the assertion is proved. □

With these preparations we are now in position to prove our main results on the boundedness and compactness of the integral operator $T_g : A^p_\omega \to A^q_\omega$ with $\omega \in \mathcal{I} \cup \mathcal{R}$.

### 4.2. Integral operator $T_g$ on the weighted Bergman space $A^p_\omega$

We begin with the following lemma which says, in particular, that the symbol $g \in H(\mathbb{D})$ must belong to the Bloch space $\mathcal{B}$ whenever the operator $T_g$ is bounded on $A^p_\omega$ and $\omega \in \mathcal{I} \cup \mathcal{R}$.

**Lemma 4.5.** Let $0 < p, q < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$.

(i) If $T_g : A^p_\omega \to A^q_\omega$ is bounded, then

(4.11) $$M_\infty(r, g') \lesssim \left(\frac{\omega^*(r)}{1 - r}\right)^{\frac{1}{p} - \frac{1}{q}}, \quad 0 < r < 1.$$
(ii) If $T_g : A^p_\omega \to A^q_\omega$ is compact, then

$$M_\infty(r, g') = o \left( \left( \frac{\omega^*(r))^{\frac{1}{p} + \frac{1}{q}}}{1 - r} \right)^{\frac{1}{1 - r}} \right), \quad r \to 1^-.$$  

Proof. (i) Let $0 < p, q < \infty$ and $\omega \in I \cup R$, and assume that $T_g : A^p_\omega \to A^q_\omega$ is bounded. Consider the functions

$$f_{a,p}(z) = \frac{(1 - |a|)^{\frac{1}{p} + \frac{1}{q}}}{(1 - az)^{\frac{1}{p} + \frac{1}{q}} \omega(S(a))^{\frac{1}{q}}}, \quad a \in \mathbb{D},$$

defined in (2.13). By choosing $\gamma > 0$ large enough and arguing as in the proof of Theorem 2.1(ii), we deduce $\sup_{a \in \mathbb{D}} \|f_{a,p}\|_{A^p_\omega} < \infty$ and also $f_{a,p} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$. Since

$$\|h\|^q_{A^p_\omega} \geq \int_{\mathbb{D} \setminus D(0,r)} |h(z)|^q \omega(z) dA(z) \geq M_q^r(r,h) \int_r^1 \omega(s) ds, \quad r \geq \frac{1}{2},$$

for all $h \in A^p_\omega$, we obtain

$$M_q^r(r, T_g(f_{a,p})) \leq \frac{\|T_g(f_{a,p})\|^q_{A^p_\omega}}{\int_r^1 \omega(s) ds} \leq \frac{\|T_g\|^q_{(A^p_\omega, A^q_\omega)}}{\int_r^1 \omega(s) ds}.\quad \frac{1}{f_r \omega(s) ds}, \quad r \geq \frac{1}{2}$$

for all $a \in \mathbb{D}$. This together with the well-known relations $M_\infty(r, f) \leq M_q(r, f)(1 - r)^{-1/q}$ and $M_q(r, f') \leq M_q(r, f)/(1 - r)$, $\rho = (1 + r)/2$, Lemma 1.1 and Lemma 1.6 yield

$$|g'(a)| \leq (\omega^*(a))^{\frac{1}{q}} |T_g(f_{a,p})'(a)| \leq (\omega^*(a))^{\frac{1}{q}} \frac{M_q \left( \frac{1 + |a|}{2} \right)}{(1 - |a|)^{\frac{1}{q}}}, \quad |a| \geq \frac{1}{2}$$

The assertion follows from this inequality.

(ii) We will need the following standard lemma whose proof will be omitted.

Lemma 4.6. Let $0 < p, q < \infty$, and let $\omega$ be a radial weight. Then $T_g : A^p_\omega \to A^q_\omega$ is compact if and only if $\lim_{n \to \infty} \|T_g(f_n)\|_{A^q_\omega} = 0$ for all sequences $\{f_n\} \subset A^p_\omega$ such that $\sup_n \|f_n\|_{A^p_\omega} < \infty$ and $\lim_{n \to \infty} f_n(z) = 0$ uniformly on compact subsets of $\mathbb{D}$.

Assume now that $T_g : A^p_\omega \to A^q_\omega$ is compact, and consider again the functions $f_{a,p}$. Recall that $\sup_{a \in \mathbb{D}} \|f_{a,p}\|_{A^p_\omega} < \infty$ and also $f_{a,p} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$. Therefore $\lim_{|a| \to 1^-} \|T_g(f_{a,p})\|_{A^q_\omega} = 0$ by Lemma 4.6. A reasoning similar to that in the proof of (i) yields the assertion. □

Proof of Theorem 4.1. The proof is split into several parts. We will first prove (iii) and (ii) because they are less involved than (i). Part (iv) will be proved last.

(iii) If $0 < p < q$ and $\frac{1}{p} - \frac{1}{q} \geq 1$, then (4.11) implies $g' \equiv 0$, and so $g$ is a constant. The converse is trivial.
(ii) If \( 0 < p < q \) and \( \frac{1}{p} - \frac{1}{q} < 1 \), then Lemma 4.5 gives the implication (a(ii))⇒(b(ii)). The equivalence (b(ii))⇔(c(ii)) follows from the next more general result.

**Proposition 4.7.** Let \( 0 < \alpha < \infty \), \( \omega \in \mathcal{I} \cup \mathcal{R} \) and \( g \in \mathcal{H}(\mathbb{D}) \). Then the following assertions hold:

(i) \( g \in C^{2\alpha+1}(\omega^*) \) if and only if

\[
M_{\alpha}(r,g') \lesssim \frac{(\omega^*(r))^\alpha}{1-r}, \quad 0 < r < 1;
\]

(ii) \( g \in C_0^{2\alpha+1}(\omega^*) \) if and only if

\[
M_{\alpha}(r,g') = o\left(\frac{(\omega^*(r))^\alpha}{1-r}\right), \quad r \to 1^-.
\]

**Proof.** (i) Let \( 0 < \alpha < \infty \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), and assume first that \( g \in C^{2\alpha+1}(\omega^*) \). We observe that \( \omega^* \in \mathcal{R} \) by Lemma 1.7. Let \( z \in \mathbb{D} \), and assume without loss of generality that \( |z| > 1/2 \). Set \( z^* = \frac{3|z|-1}{2}e^{i\text{arg}z} \) so that \( D(z,\frac{1}{2}(1-|z|)) \subset S(z^*) \). This inclusion together with the subharmonicity property of \( |g'|^2 \) and the fact \( \omega^* \in \mathcal{R} \) gives

\[
|g'(z)|^2 \lesssim \frac{1}{(1-|z|^2)\omega^*(z)} \int_{D(z,\frac{1}{2}(1-|z|))} |g'(|\zeta|)|^2 \omega^*(\zeta) dA(\zeta)
\]

\[
\leq \frac{1}{(1-|z|^2)\omega^*(z)} \int_{S(z^*)} |g'(|\zeta|)|^2 \omega^*(\zeta) dA(\zeta).
\]

A standard reasoning involving (4.1), Lemma 1.1 and Lemma 1.6 now show that (4.13) is satisfied.

Conversely, (4.13) together with Lemma 1.6 yield

\[
\int_{S(\alpha)} |g'(z)|^2 \omega^*(z) dA(z) \lesssim (1 - |a|) \int_{|a|}^1 \left(\frac{\omega^*(r)}{1-r}\right)^{2\alpha+1} dr
\]

\[
\lesssim (1 - |a|) \int_{|a|}^1 \frac{\left(1-r\int_r^1 \omega(s) ds\right)^{2\alpha+1}}{(1-r)^2} dr
\]

\[
\lesssim (\omega(S(a)))^{2\alpha+1} \approx (\omega^*(a))^{2\alpha+1}, \quad |a| \geq \frac{1}{2},
\]

and it follows that \( g \in C_0^{2\alpha+1}(\omega^*) \).

The second assertion (ii) can be proved by appropriately modifying the reasoning above.

We now return to the proof of the case (ii) of Theorem 4.1. It remains to show that (a(ii)) is satisfied if either of the equivalent conditions (b(ii)) and (c(ii)) is valid. We first observe that, if \( q = 2 \), then (a(ii))⇔(c(ii)) by Theorem 4.2, the definition of \( C^{2/\alpha}(\omega^*) \) and (4.1). The cases \( q > 2 \) and \( q < 2 \) are now treated separately.

Let \( q > 2 \) and assume that \( g \in C^{2\left(\frac{1}{q} + \frac{1}{2}\right)+1}(\omega^*) \). In this case \( L_\omega^{q/2} \) can be identified with the dual of \( L_\omega^{q+q} \), that is, \( L_\omega^{q/2} = (L_\omega^{q+q})^* \). Therefore formula (4.4) in Theorem 4.2 shows that \( T_g : A^p_\omega \to A^q_\omega \) is bounded if and only if

\[
\left| \int_{\mathbb{D}} h(u) \left( \int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2 dA(z) \right) \omega(u) dA(u) \right| \lesssim ||h||_{L_\omega^{q/2}} ||f||_{A^p_\omega}^2
\]
for all \( h \in L^\frac{q}{q-2} \). To see this, we use first Fubini’s theorem, Lemma 1.6 and the
definition of the maximal function \( M_\omega(|h|) \) to obtain
\[
\left| \int_D h(u) \left( \int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right) \omega(u) \, dA(u) \right|
\leq \int_D \left| f(z) \right|^2 \left| g'(z) \right|^2 \left( \int_{\Gamma(z)} |h(u)| \omega(u) \, dA(u) \right) \, dA(z)
\leq \int_D \left| f(z) \right|^2 \left| g'(z) \right|^2 \omega(T(z)) \cdot \left( \frac{1}{\omega(S(z))} \int_{\Gamma(z)} |h(u)| \omega(u) \, dA(u) \right) \, dA(z)
\leq \int_D \left| f(z) \right|^2 M_\omega(|h|)(z) \left| g'(z) \right|^2 \omega(z) \, dA(z).
\]
But \( g \in \mathcal{C}^{2(\frac{1}{p} - \frac{1}{q}) + 1}(\omega^*) \) by the assumption, and \( 2(\frac{1}{p} - \frac{1}{q}) + 1 = (2 + p - \frac{2p}{q}) / p \), so we may estimate the last integral upwards by Hölder’s inequality and Corollary 2.2 to
\[
\left( \int_D \left| f(z) \right|^{2 + p - \frac{2p}{q}} \left| g'(z) \right|^2 \omega(z) \, dA(z) \right)^{\frac{q}{q-2}} \leq \left( \int_D (M_\omega(|h|)(z))^{1 + \frac{2p}{q(2-q)}} \left| g'(z) \right|^2 \omega(z) \, dA(z) \right)^{\frac{2-q}{2}} \leq \| f \|_{A^p} \| h \|_{L^\frac{q}{q-2}}.
\]
These estimates give the desired inequality for all \( h \in L^\frac{q}{q-2} \), and thus \( T_g : A^p \to A^q \) is bounded.

Let now \( 0 < q < 2 \), and assume again that \( g \in \mathcal{C}^{2(\frac{1}{p} - \frac{1}{q}) + 1}(\omega^*) \). We will use ideas from [25]. Then (4.4), Hölder’s inequality, Lemma 4.4, Fubini’s theorem, and Lemma 1.6 give
\[
\| T_g(f) \|_{A^q} \leq \int_D \left( \int_{\Gamma(u)} |f(z)|^2 \left| g'(z) \right|^2 \, dA(z) \right)^{\frac{q}{2}} \omega(u) \, dA(u)
\leq \int_D N(f)(u)^{\frac{q(2-q)}{2}} \cdot \left( \int_{\Gamma(u)} |f(z)|^{2 - \frac{2p}{q} + p} \left| g'(z) \right|^2 \, dA(z) \right)^{\frac{2-q}{2}} \omega(u) \, dA(u)
\leq \left( \int_D N(f)(u)^p \omega(u) \, dA(u) \right)^{\frac{2-q}{2}} \left( \int_D \int_{\Gamma(u)} |f(z)|^{2 - \frac{2p}{q} + p} \left| g'(z) \right|^2 \omega(u) \, dA(u) \right)^{\frac{2-q}{2}} \leq \| f \|_{A^p} \| g \|_{A^q} \| T_g \|_{A^q}.
\]
and thus \( T_g : A^p \to A^q \) is bounded. The proof of the case (ii) is now complete.

(i) This case requires more work than the previous ones mainly because \( \mathcal{C}^1(\omega^*) \) can not be characterized by a simple radial condition like \( \mathcal{C}^\alpha(\omega^*) \) for each \( \alpha > 1 \).
We first observe that, if $p = 2$, then (ai)$\Leftrightarrow$(bi) by Theorem 4.2, the definition of $C^1(\omega^*)$ and (4.1). Moreover, by following the two different cases of the proof of (cii)$\Rightarrow$(aii) we obtain (bi)$\Rightarrow$(ai) for $p \neq 2$. It remains to show that (ai)$\Rightarrow$(bi) for $p \neq 2$. This will be done in two pieces.

Let first $p > 2$, and assume that $T_g : A_p^\omega \rightarrow A_p^\omega$ is bounded. By (4.4) in Theorem 4.2 this is equivalent to

$$
\|T_g(f)\|_{A_p^\omega}^p \simeq \int_D \left( \int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) dA(u) \lesssim \|f\|_{A_p^\omega}^p
$$

for all $f \in A_p^\omega$. By using this together with Lemma 1.6, Fubini’s theorem, Hölder’s inequality and Lemma 4.4, we obtain

$$
\int_D |f(z)|^p |g'(z)|^2 \omega^*(z) dA(z) \simeq \int_D |f(z)|^p |g'(z)|^2 \omega(T(z)) dA(z)
$$

$$
= \int_D \int_{\Gamma(u)} |f(z)|^p |g'(z)|^2 dA(z) \omega(u) dA(u)
$$

$$
\leq \int_D N(f)(u)^{p-2} \int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2 dA(z) \omega(u) dA(u)
$$

$$
\leq \left( \int_D N(f)(u)^p \omega(u) dA(u) \right)^{\frac{p-2}{p}}
$$

$$
\cdot \left( \int_D \left( \int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) dA(u) \right)^{\frac{2}{p}} \lesssim \|f\|_{A_p^\omega}^p
$$

for all $f \in A_p^\omega$. Therefore $|g'(z)|^2 \omega^*(z) dA(z)$ is a $p$-Carleson measure for $A_p^\omega$, and thus $g \in C^1(\omega^*)$ by the definition. This implication can also be proved by using Theorem 2.1, Theorem 4.2 and the functions $F_{a,p}$ of Lemma 2.4.

Let now $0 < p < 2$, and assume that $T_g : A_p^\omega \rightarrow A_p^\omega$ is bounded. Then Lemma 4.5 and its proof imply $g \in B$ and

(4.14)$$
\|g\|_B \lesssim \|T_g\|. \tag{4.14}
$$

Choose $\gamma > 0$ large enough, and consider the functions $F_{a,p} = \left( \frac{1-|a|^2}{1-\frac{1}{\alpha}} \right)^{\frac{p-1}{p}}$ of Lemma 2.4. Let $1 < \alpha, \beta < \infty$ such that $\beta/\alpha = p/2 < 1$, and let $\alpha'$ and $\beta'$ be the conjugate indexes of $\alpha$ and $\beta$. Then Lemma 1.6, Fubini’s theorem, Hölder’s
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Inequality, (2.5) and (4.4) yield

\[
\int_{S(a)} |g'(z)|^2 \omega^*(z) \, dA(z) \\
\leq \int_{\mathbb{D}} \left( \int_{S(a) \cap \Gamma(u)} |g'(z)|^2 |F_{a,p}(z)|^2 \, dA(z) \right)^{\frac{\alpha}{\beta}} \omega(u) \, dA(u) \\
\leq \left( \int_{\mathbb{D}} \left( \int_{\Gamma(u) \cap S(a)} |g'(z)|^2 \, dA(z) \right)^{\frac{\alpha'}{\beta}} \omega(u) \, dA(u) \right)^{\frac{\beta}{\alpha'}} \\
\cdot \left( \int_{\mathbb{D}} \left( \int_{\Gamma(u) \cap S(a)} |g'(z)|^2 \, dA(z) \right)^{\frac{\alpha'}{\beta}} \omega(u) \, dA(u) \right)^{\frac{\beta}{\alpha'}} \\
\leq \|T_g (F_{a,p})\|_{A^p_\omega}^\frac{\alpha'}{\beta} \|S_g (\chi_{S(a)})\|_{L^\beta_{\omega}}^\frac{\beta}{\alpha'}, \quad a \in \mathbb{D},
\]

where

\[
S_g (\varphi)(u) = \int_{\Gamma(u)} \varphi(z)^2 |g'(z)|^2 \, dA(z), \quad u \in \mathbb{D} \setminus \{0\},
\]

for any bounded function $\varphi$ on $\mathbb{D}$. Since $\beta/\alpha = p/2 < 1$, we have $\frac{\alpha'}{\alpha} > 1$ with the conjugate exponent $\left( \frac{\alpha'}{\alpha} \right)' = \frac{\alpha (\alpha - 1)}{\alpha - \beta} > 1$. Therefore

\[
\|S_g (\chi_{S(a)})\|_{L^\beta_{\omega}}^\frac{\beta}{\alpha'} = \sup_{L^\beta_{\omega}} \frac{\|h\|_{L^\beta_{\omega}}}{\|h\|_{L^\beta_{\omega}}} \left| \int_{\mathbb{D}} h(u) S_g (\chi_{S(a)})(u) \omega(u) \, dA(u) \right|.
\]

By using Fubini’s theorem, Lemma 1.6, Hölder’s inequality and Corollary 2.2, we deduce

\[
\left| \int_{\mathbb{D}} h(u) S_g (\chi_{S(a)})(u) \omega(u) \, dA(u) \right| \\
\leq \int_{\mathbb{D}} |h(u)| \int_{\Gamma(u) \cap S(a)} |g'(z)|^2 \, dA(z) \omega(u) \, dA(u) \\
\leq \int_{S(a)} M_w(|h|)(z) |g'(z)|^2 \omega^*(z) \, dA(z) \\
\leq \left( \int_{S(a)} |g'(z)|^2 \omega^*(z) \, dA(z) \right)^{\frac{\alpha'}{\beta'}} \\
\cdot \left( \int_{\mathbb{D}} M_w(|h|)^{\frac{\alpha'}{\beta'}} |g'(z)|^2 \omega^*(z) \, dA(z) \right)^{\frac{\beta}{\alpha'}} \\
\leq \left( \int_{S(a)} |g'(z)|^2 \omega^*(z) \, dA(z) \right)^{\frac{\alpha'}{\beta'}} \\
\cdot \left( \sup_{u \in \mathbb{D}} \frac{\int_{S(a)} |g'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \right)^{1 - \frac{\alpha'}{\beta'}} \beta.
\]
By replacing \( g(z) \) by \( g_r(z) = g(rz) \), \( 0 < r < 1 \), and combining (4.15)–(4.17), we obtain

\[
\int_{S(a)} |g_r'(z)|^2 \omega^*(z) \, dA(z) \lesssim \|T_{g_r}(F_{a,p})\|_{A_p^*}^p \left( \int_{S(a)} |g_r'(z)|^2 \omega^*(z) \, dA(z) \right)^{\frac{p}{p'}} \cdot \left( \sup_{a \in \mathbb{D}} \frac{\int_{S(a)} |g_r'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \right)^{\frac{1}{p'}} (1 - \frac{r}{r'})^p.
\]

We now claim that there exists a constant \( C = C(\omega) > 0 \) such that

\[
(4.18) \quad \sup_{0 < r < 1} \|T_{g_r}(F_{a,p})\|_{A_p^*}^p \leq C \|T_g\|_{A_p^*}^p \omega(S(a)), \quad a \in \mathbb{D}.
\]

Taking this for granted for a moment, we deduce

\[
\left( \frac{\int_{S(a)} |g_r'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \right)^{\frac{1}{p'}} \lesssim \|T_g\|_{A_p^*}^p \left( \sup_{a \in \mathbb{D}} \frac{\int_{S(a)} |g_r'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \right)^{\frac{1}{p'}} (1 - \frac{r}{r'})^p
\]

for all \( 0 < r < 1 \) and \( a \in \mathbb{D} \). This yields

\[
\frac{\int_{S(a)} |g_r'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \lesssim \|T_g\|^2, \quad a \in \mathbb{D},
\]

and so

\[
\sup_{a \in \mathbb{D}} \frac{\int_{S(a)} |g(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \leq \sup_{a \in \mathbb{D}} \liminf_{r \to 1^-} \left( \frac{\int_{S(a)} |g_r'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \right) \lesssim \|T_g\|^2
\]

by Fatou’s lemma. Therefore \( g \in C^1(\omega^*) \) by Theorem 2.1.

It remains to prove (4.18). To do this, let \( a \in \mathbb{D} \). If \( |a| \leq r_0 \), where \( r_0 \in (0, 1) \) is fixed, then the inequality in (4.18) follows by Theorem 4.2, the change of variable \( rz = \zeta \), the fact

\[
(4.19) \quad \Gamma(ru) \subset \Gamma(u), \quad 0 < r < 1,
\]

and the assumption that \( T_g : A_p^\omega \to A_p^\omega \) is bounded. If \( a \in \mathbb{D} \) is close to the boundary, we consider two separate cases.

Let first \( \frac{1}{2} < |a| \leq \frac{1}{2r} \). Then

\[
|1 - \overline{az}| \leq \left| 1 - \frac{\overline{z}}{r} \right| + \frac{1-r}{2-r} \leq 2 \left| 1 - \frac{\overline{z}}{r} \right|, \quad |z| \leq r.
\]
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Therefore Theorem 4.2, (4.19) and (2.6) yield

\[
\|T_g(F_{a,p})\|_{A^p_\omega} \leq \int_\mathbb{D} \left( \int_{\Gamma(u)} r^2 |g'(rz)|^2 |F_{a,p}(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) du
\]

(4.20)

\[
\leq 2^{\gamma+1} \int_\mathbb{D} \left( \int_{\Gamma(u)} |g'(z)|^2 |F_{a,p}(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) du
\]

\[
\leq 2^{\gamma+1} \int_\mathbb{D} \left( \int_{\Gamma(u)} |g'(z)|^2 |F_{a,p}(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) du
\]

\[
\leq \|T_g(F_{a,p})\|_{A^p_\omega} \lesssim \|T_g\|_{A^p_\omega} \omega(S(a)),
\]

and hence

(4.21)

\[
\|T_g(F_{a,p})\|_{A^p_\omega} \lesssim \|T_g\|_{A^p_\omega} \omega(S(a)), \quad \frac{1}{2} < |a| \leq \frac{1}{2 - r}.
\]

Let now \(|a| > \max\{\frac{1}{2 - r}, \frac{1}{2}\}\). Then, by Theorem 4.2, (4.14) and Lemma 2.3, with \(\gamma + 2 - p\) in place of \(\gamma\), we deduce

\[
\|T_g(F_{a,p})\|_{A^p_\omega} \lesssim \int_\mathbb{D} \left( \int_{\Gamma(u)} r^2 |g'(rz)|^2 |F_{a,p}(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) du
\]

(4.22)

\[
\leq M_\infty (r, g')^p \int_\mathbb{D} \left( \int_{\Gamma(u)} |F_{a,p}(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) du
\]

\[
\lesssim M_\infty \left( 2 - \frac{1}{|a|} g' \right)^p (1 - |a|)^p \left\| \left( 1 - \frac{|a|^2}{1 - rz} \right)^{\frac{\gamma+1}{p} - 1} \right\|_{A^p_\omega}
\]

\[
\lesssim \|g\|_{B^p_\omega} \omega(S(a)) \lesssim \|T_g\|_{A^p_\omega} \omega(S(a))
\]

for \(\gamma > 0\) large enough. This together with (4.21) gives (4.18). The proof of (i) is now complete.
(iv) Let first \( g \in A^p_\omega \), where \( s = \frac{p}{p-q} \). Then Theorem 4.2, Hölder’s inequality and Lemma 4.4 yield
\[
\|T_g(f)\|_{A^q_\omega}^q \leq \int_D \left( \int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right)^{\frac{q}{2}} \omega(u) \, dA(u)
\leq \int_D \left( \int_{\Gamma(u)} |g'(z)|^2 \, dA(z) \right)^{\frac{q}{2}} \omega(u) \, dA(u)
\leq \left( \int_D \left( \int_{\Gamma(u)} |g'(z)|^2 \, dA(z) \right) \right)^{\frac{q}{2}} \omega(u) \, dA(u)
\leq C \frac{q}{2} C_2(p, q, \omega) \|g\|_{A^q_\omega}^q.
\]
Thus \( T_g : A^p_\omega \rightarrow A^q_\omega \) is bounded.

To prove the converse implication, we will use ideas from [7, p. 170–171], where \( T_g \) acting on Hardy spaces is studied. We begin with the following result whose proof relies on Corollary 3.4.

**Proposition 4.8.** Let \( 0 < q < p < \infty \) and \( \omega \in \mathcal{T} \cup \mathcal{R} \), and let \( T_g : A^p_\omega \rightarrow A^q_\omega \) be bounded. Then \( T_g : A^p_\omega \rightarrow A^q_\omega \) is bounded for any \( \hat{p} < p \) and \( \hat{q} < q \) with \( \frac{1}{p} - \frac{1}{\hat{p}} = \frac{1}{q} - \frac{1}{\hat{q}} \).

Further, if \( 0 < p \leq 2 \), then there exists \( C = C(p, q, \omega) > 0 \) such that
\[
\limsup_{\hat{p} \to p^-} \|T_g\|_{(A^p_\omega, A^q_\omega)} \leq C \|T_g\|_{(A^p_\omega, A^q_\omega)}.
\]

**Proof.** Theorem 3.1 shows that for any \( f \in A^p_\omega \), there exist \( f_1 \in A^p_\omega \) and \( f_2 \in A^{\hat{p}}_\omega \) such that
\[
f = f_1 f_2 \quad \text{and} \quad \|f_1\|_{A^p_\omega} \cdot \|f_2\|_{A^{\hat{p}}_\omega} \leq C_3 \|f\|_{A^p_\omega}
\]
for some constant \( C_3 = C_3(p, \hat{p}, \omega) > 0 \). We observe that \( T_g(f) = T_F(f_2) \), where \( F = T_g(f_1) \). Since \( T_g : A^p_\omega \rightarrow A^q_\omega \) is bounded,
\[
\|F\|_{A^q_\omega} = \|T_g(f_1)\|_{A^q_\omega} \leq \|T_g\|_{(A^p_\omega, A^q_\omega)} \|f_1\|_{A^p_\omega} < \infty,
\]
and hence \( F \in A^q_\omega \). Then (4.23) and the identity \( \frac{1}{q} = \frac{1}{\hat{q}} - \frac{1}{\hat{p}} \) yield
\[
\|T_g(f)\|_{A^q_\omega} = \|T_F(f_2)\|_{A^q_\omega} \leq C_1^{\frac{1}{q}} C_2^{\frac{1}{\hat{q}}} C_3 \|f_2\|_{A^{\hat{p}}_\omega} \|F\|_{A^q_\omega},
\]
where \( C_2 = C_2(q, \omega) > 0 \). This together with (4.25) and (4.26) gives
\[
\|T_g(f)\|_{A^q_\omega} \leq C_1^{\frac{1}{q}} C_2^{\frac{1}{\hat{q}}} C_3 \|T_g\|_{(A^p_\omega, A^q_\omega)} \|f_1\|_{A^p_\omega} \cdot \|f_2\|_{A^{\hat{p}}_\omega} \|F\|_{A^q_\omega},
\]
where \( \hat{C}_2 = \hat{C}_2(q, \omega) > 0 \). Therefore \( T_g : A^p_\omega \rightarrow A^q_\omega \) is bounded.

To prove (4.24), let \( 0 < p \leq 2 \) and let \( 0 < \hat{p} < \hat{q} \) be close enough to \( p \) such that
\[
\min \left\{ \frac{p}{p-\hat{p}}, \frac{\hat{p}p}{p-\hat{p}} \right\} > 2.
\]
If \( f \in A^p_\omega \), then Corollary 3.4 shows that (4.25) holds with \( C_3 = C_3(p, \omega) \). Therefore the reasoning in the previous paragraph and (4.27) give (4.24). \( \square \)

With this result in hand, we are ready to prove (aiv)\( \Rightarrow \) (biv). Let \( 0 < q < p < \infty \) and \( \omega \in I \cup R \), and let \( T_g : A^p_\omega \to A^p_\omega \) be bounded. Denote \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \). By the first part of Proposition 4.8, we may assume that \( p \leq 2 \). We may also assume, without loss of generality, that \( q(0) = 0 \). Define \( t^* = \sup \{ t : g \in A^t_\omega \} \). Since the constant function \( 1 \) belongs to \( A^p_\omega \), we have \( g = T_g(1) \in A^p_\omega \), and hence \( t^* \geq q = 0 \). Fix a positive integer \( m \) such that \( \frac{1}{r} = \frac{1}{q} < p \). For each \( t < t^* \), set \( \hat{p} = \hat{p}(t) = \frac{t}{m} \), and define \( \hat{q} = \hat{q}(t) \) by the equation \( \frac{1}{s} = \frac{1}{q} - \frac{1}{p} \). Then \( \hat{p} < p, \hat{q} < q \) and \( T_g : A^\hat{p}_\omega \to A^\hat{q}_\omega \) is bounded by Proposition 4.8. Since \( g^m = g^\hat{r} \in A^\hat{p}_\omega \), we have \( g^{m+1} = (m+1)T_g(g^m) \in A^\hat{q}_\omega \) and

\[
\|g^{m+1}\|_{A^\hat{q}_\omega} \leq (m+1)\|T_g\|_{(A^\hat{p}_\omega,A^\hat{q}_\omega)}\|g^m\|_{A^\hat{p}_\omega},
\]

that is,

\[
(4.28) \quad \|g\|_{A^{m+1}_\omega} \leq (m+1)\|T_g\|_{(A^\hat{p}_\omega,A^\hat{q}_\omega)}\|g\|_{A^\hat{p}_\omega}.
\]

Suppose first that for some \( t < t^* \), we have

\[
t \geq (m+1)\hat{q} = \left(\frac{t}{p} + 1\right)\hat{q} = \hat{q} + t\left(1 - \frac{1}{s}\right).
\]

Then \( s \leq t < t^* \), and the result follows from the definition of \( t^* \). It remains to consider the case in which \( t < (m+1)\hat{q} \) for all \( t < t^* \). By Hölder’s inequality, \( \|g\|_{A^\hat{q}_\omega} \leq C_1(m,\omega)\|g\|_{A^{m+1}_\omega} \). This and (4.28) yield

\[
(4.29) \quad \|g\|_{A^{m+1}_\omega} \leq C_2(m,\omega)\|T_g\|_{(A^\hat{p}_\omega,A^\hat{q}_\omega)},
\]

where \( C_2(m,\omega) = C_1(m,\omega)(m+1) \). Now, as \( t \) increases to \( t^* \), \( \hat{p} \) increases to \( \frac{t}{m} \) and \( \hat{q} \) increases to \( \frac{t}{t+ms} \), so by (4.29) and (4.24) we deduce

\[
\|g\|_{A^{m+1}_\omega} \leq \limsup_{t \to t^*} \|g\|_{A^{m+1}_\omega} \leq C_2(m,\omega) \|T_g\|_{(A^\hat{p}_\omega,A^\hat{q}_\omega)} \leq C(p,q,m,\omega)\|T_g\|_{(A^\hat{p}_\omega,A^\hat{q}_\omega)} < \infty.
\]

The definition of \( t^* \) implies \( \frac{m+1}{t^*+ms} \leq t^* \), and so \( t^* \geq s \). This finishes the proof. \( \square \)

We next characterize the symbols \( g \in \mathcal{H}(\mathbb{D}) \) such that \( T_g : A^p_\omega \to A^q_\omega \) is compact.

**Theorem 4.9.** Let \( 0 < p, q < \infty \), \( \omega \in I \cup R \) and \( g \in \mathcal{H}(\mathbb{D}) \).

(i) The following conditions are equivalent:

(a) \( T_g : A^p_\omega \to A^q_\omega \) is compact;
(b) \( g \in C^1_0(\omega^*) \).

(ii) If \( 0 < p < q \) and \( \frac{1}{p} - \frac{1}{q} < 1 \), then the following conditions are equivalent:

(a) \( T_g : A^p_\omega \to A^q_\omega \) is compact;
(b) \( M_{\infty}(r,g') = o\left(\frac{(\omega^*(r))^{\frac{1}{p} - \frac{1}{q}}}{1 - r}\right) \), \( r \to 1^- \);
(c) \( g \in C^2(\frac{1}{p} - \frac{1}{q} + 1)(\omega^*) \).

(iii) If \( 0 < q < p < \infty \) and \( \omega \in \tilde{I} \cup R \), then the following conditions are equivalent:

(a) \( T_g : A^p_\omega \to A^q_\omega \) is compact;
(biii) $T_g : A^p \to A^q$ is bounded;
(ciii) $g \in A^p_\omega$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.

PROOF. (ii) Appropriate modifications in the proofs of Theorem 2.1(ii) and Theorem 4.1(ii) together with Lemma 4.5(ii), Lemma 4.6 and Proposition 4.7(ii) give the assertion.

(i) By using Lemma 4.6, the proof of Theorem 2.1(ii) and arguing as in the proof of Theorem 4.1(i), with appropriate modifications, we obtain (bii)$\Rightarrow$(ai) for all $0 < p < \infty$, and the converse implication (ai)$\Rightarrow$(bii) for $p \geq 2$. To prove the remaining case, let $0 < p < 2$ and assume that $T_g : A^p \to A^q_\omega$ is compact. Recall that the functions $f_{a,p}$ defined in (2.13) satisfy $\|f_{a,p}\|_{A^p_\omega} \approx 1$ and $f_{a,p} \to 0$, as $|a| \to 1^-$, uniformly in compact subsets of $\mathbb{D}$. Therefore $\|T_g(f_{a,p})\|_{A^q_\omega} \to 0$, as $|a| \to 1^-$, by Lemma 4.6. Now, let $1 < \alpha, \beta < \infty$ such that $\beta/\alpha = p/2 < 1$. Arguing as in (4.15) we deduce

$$
\frac{1}{(\omega(S(a)))^{\frac{1}{2}}} \int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z) \lesssim \|T_g(f_{a,p})\|_{A^q_\omega} \|S_g(\chi_{S(a)} f_{a,p})\|_{A^p_\omega} \omega(S(a))^{\frac{1}{\beta}}
$$

for all $a \in \mathbb{D}$. Following the reasoning in the proof of Theorem 4.1(i) further, we obtain

$$
\frac{\int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z)}{(\omega(S(a)))^{\frac{1}{2}}} \lesssim \|T_g(f_{a,p})\|_{A^q_\omega} \left(\frac{\int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z)}{(\omega(S(a)))^{\frac{1}{2}}} \right)^{\frac{\beta}{\alpha}} \frac{1}{\beta},
$$

which is equivalent to

$$
\frac{\int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z)}{\omega(S(a))} \lesssim \|T_g(f_{a,p})\|_{A^q_\omega}^{p}.\]

Thus Theorem 2.1 implies $g \in C^1_b(\omega^*)$.

(iii) The equivalence (biii)$\Leftrightarrow$(ciii) is Theorem 4.1(iv), and (a iii)$\Rightarrow$(biii) is obvious. To prove the remaining implication (ciii)$\Rightarrow$(a iii), let $g \in A^s_\omega$, where $s = \frac{p\alpha}{p-q}$, and let $\{f_n\} \subset A^p_\omega$ such that $\sup_n \|f_n\|_{A^p_\omega} < \infty$ and $\lim_{n \to \infty} f_n(z) = 0$ uniformly on compact subsets of $\mathbb{D}$. Let $\varepsilon > 0$. By using (4.4) in Theorem 4.2, we can find $r_0 \in (0,1)$ such that

$$
\int_{|z| < 1} \left(\int_{T(u)} |g'(z)|^2 dA(z)\right)^{\frac{pq}{2(p-q)}} \omega(u) dA(u) < \varepsilon^{\frac{p}{p-q}}.
$$
Now, take \( n_0 \in \mathbb{N} \) such that \( \sup_{n \geq n_0} |f_n(z)| < \varepsilon^{1/q} \) for all \( z \in D(0, r_0) \). Then Theorem 4.2, Hölder’s inequality and Lemma 4.4 yield

\[
\|T_g(f_n)\|_{A^q} \leq \varepsilon \int_{D} \left( \int_{\Gamma(u)} |f_n(z)|^2 |g'(z)|^2 dA(z) \right)^{\frac{1}{2}} \omega(u) dA(u)
\]

\[
\leq \varepsilon \int_{|z| < r_0} \left( \int_{\Gamma(u)} |g'(z)|^2 dA(z) \right)^{\frac{1}{2}} \omega(u) dA(u)
\]

\[
+ \int_{r_0 \leq |z| < 1} \left( \int_{\Gamma(u)} |f_n(z)|^2 |g'(z)|^2 dA(z) \right)^{\frac{1}{2}} \omega(u) dA(u)
\]

\[
\leq \varepsilon \|g\|_{A^q}^q + \int_{r_0 \leq |z| < 1} (N(f_n)(u))^q \left( \int_{\Gamma(u)} |g'(z)|^2 dA(z) \right)^{\frac{1}{2}} \omega(u) dA(u)
\]

\[
\leq \varepsilon \|g\|_{A^q}^q + \varepsilon \left( \int_{r_0 \leq |z| < 1} (N(f_n)(u))^p \omega(u) dA(u) \right)^{\frac{1}{p}}
\]

\[
\lesssim \varepsilon \left( \|g\|_{A^q}^q + \left( \sup_n \|f_n\|_{A^q} \right)^q \right) \lesssim \varepsilon
\]

for all \( n \geq n_0 \). In the last step we used the fact \( g \in A^q_\phi \), which follows by Theorem 4.1(iv). Therefore, \( \lim_{n \to \infty} \|T_g(f_n)\|_{A^q} = 0 \), and so \( T_g : A^p_\phi \to A^q_\phi \) is compact by Lemma 4.6. This finishes the proof of Theorem 4.9. \( \square \)

It is worth noticing that Theorem 4.1(iii) is a consequence of the fact that condition (4.13) implies \( g' \equiv 0 \) for all \( \alpha \in [1, \infty) \). If \( \omega \) is regular, then this implication remains valid also for some \( \alpha < 1 \) by (1.16). However, no such conclusion can be made if \( \omega \) is rapidly increasing by the observation (ii) to Lemma 1.1.

If \( \omega \) is a regular weight, a description of those \( g \in \mathcal{H}(\mathbb{D}) \) such that \( T_g : A^p_\omega \to A^q_\omega \) is bounded follows by [8, Theorem 4.1] and Lemma 1.4(i), see also [11]. The reasoning in the proof of Theorem 4.1 gives a different way to establish these results, in particular, when \( q \leq p \). In the proofs found in the existing literature, the correct necessary conditions for \( \mu \) to be a \( q \)-Carleson measure for \( A^p_\omega \), with \( q < p \), are achieved by using Luecking’s approach based on Kinchine’s inequality [58]. In contrast to this, the corresponding part of the proof of Theorem 4.1(iv) relies on an argument inherited from [7] and an appropriate estimate for the constant in the reverse type Hölder’s inequality, which is obtained through factorization of functions in \( A^p_\omega \) in Corollary 3.4. If \( q = p \) and \( \omega \) is rapidly increasing, then the proof of Theorem 4.1 is much more involved than in the case when \( \omega \) is regular. This is due to the fact that \( C^1(\omega^*) \) is a proper subspace of the Bloch space \( \mathcal{B} \) by Proposition 5.1, it is not necessarily conformally invariant and it can not be characterized by a simple growth condition on \( M_\infty(r, g') \).

### 4.3. Integral operator \( T_g \) on the Hardy space \( H^p \)

The question of when \( T_g : H^p \to H^q \) is bounded was completely solved by Aleman, Cima and Siskakis [7, 10], see also [6, 26, 68, 81]. We next quote their result for further reference.

**Theorem B.** Let \( 0 < p, q < \infty \) and \( g \in \mathcal{H}(\mathbb{D}) \).
The following conditions are equivalent:

(i) The following conditions are equivalent:
   (a) $T_g : H^p \to H^p$ is bounded;
   (b) $g \in \text{BMOA}$.

(ii) If $0 < p < q$ and $\frac{1}{p} - \frac{1}{q} \leq 1$, then the following conditions are equivalent:
   (a) $T_g : H^p \to H^q$ is bounded;
   (b) $M_\infty(r, g') \lesssim \left( \frac{1}{1-r} \right)^{1 - \left( \frac{1}{p} - \frac{1}{q} \right)}$, $r \to 1$;
   (c) $g \in A\left( \frac{1}{p} - \frac{1}{q} \right)$;
   (d) The measure $d\mu_g(z) = |g'(z)|^2(1 - |z|^2)dA(z)$ satisfies

   $$\sup_{I \subset \mathbb{T}} \frac{\mu_g(S(I))}{|I|^{\frac{1}{p} - \frac{1}{q} + 1}} < \infty.$$ 

(iii) If $\frac{1}{p} - \frac{1}{q} > 1$, then $T_g : A^p_\omega \to A^q_\omega$ is bounded if and only if $g$ is constant.

(iv) If $0 < q < p < \infty$, then the following conditions are equivalent:
   (a) $T_g : H^p \to H^q$ is bounded;
   (b) $g \in H^p$, where $\frac{1}{p} = \frac{1}{q} - \frac{1}{p}$.

For $0 < \alpha \leq 1$, the Lipschitz space $\Lambda(\alpha)$ consists of those $g \in \mathcal{H} (\mathbb{D})$, having a non-tangential limit $g(e^{i\theta})$ almost everywhere, such that

$$\sup_{\theta \in [0, 2\pi], 0 < t < 1} \frac{|g(e^{i(\theta + t)}) - g(e^{i\theta})|}{t^\alpha} < \infty.$$ 

The proof of Theorem B in [7, 10] uses several well-known properties of BMOA and $H^p$ such as the conformal invariance of BMOA, Fefferman’s duality identity $(H^1)^* = \text{BMOA}$ [15, 33], a Riesz-Thorin type interpolation theorem for $H^p$ [85, 66], and the inner-outer factorization of $H^p$-functions. However, the proof of Theorem 4.1 does not rely on such properties for $C^1(\omega^*)$ and $A^p_\omega$. In fact, we will see in Chapter 5 that the space $C^1(\omega^*)$ is not necessarily conformally invariant if $\omega$ is rapidly increasing. The objective of this section is to offer an alternative proof for some cases of Theorem B by using the techniques developed on the way to the proof of Theorem 4.1. We will omit analogous steps and we will deepen only in the cases when the proof significantly differs from the original one.

We begin with recalling some definitions and known results needed. For $\beta > 0$, a positive Borel measure on $\mathbb{D}$ is a $\beta$-classical Carleson measure if

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|^\beta} < \infty.$$ 

If $\beta \geq 1$ and $0 < p < \infty$, then $H^p \subset L^{p\beta}(\mu)$ if and only if $\mu$ is a $\beta$-classical Carleson measure [28, Section 9.5]. For $0 < \alpha < \infty$ and a $2\pi$-periodic function $\varphi(\cdot) \in L^1(\mathbb{T})$, the Hardy-Littlewood maximal function is defined by

$$M(\varphi)(z) = \sup_{I : z \in S(I)} \frac{1}{|I|} \int_I |\varphi(\zeta)| \frac{|d\zeta|}{2\pi}, \quad z \in \mathbb{D}.$$ 

The following result can be obtained by carefully observing either [28, Section 9.5] or the proof of Theorem 2.1.

**Corollary 4.10.** Let $0 < p \leq q < \infty$ and $0 < \alpha < \infty$ such that $p\alpha > 1$. Let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then $[M((\cdot)^{\frac{\alpha}{p}})]^\alpha : L^p(\mathbb{T}) \to L^q(\mu)$ is bounded if and
only if \( \mu \) is a \( \frac{2}{p} \)-Carleson measure. Moreover,

\[
\| [M(\cdot)]^q \|_q \lesssim \sup_{I \subset T} \frac{\mu(S(I))}{|I|^{\frac{2}{p}}}
\]

**Proof of Theorem B.** We will prove in detail only the implication (ai)\( \Rightarrow \) (bi) and discuss (aiv)\( \Rightarrow \) (biv) because their proofs are significantly different from the original ones, and because some adjustments to the arguments in the proof of Theorem 4.1 should be made. After these proofs, we will indicate how the reader can modify the proof of Theorem 4.1 to obtain the remaining implications.

(ai)\( \Rightarrow \) (bi). If \( p = 2 \), then the equivalence (ai)\( \Leftrightarrow \) (bi) follows by using the fact that \( g \in \text{BMOA} \) if and only if \( |g'(z)|^2 (1 - |z|^2) \) is a \( 1 \)-classical Carleson measure \([31, 33]\).

Let \( p > 2 \), and assume that \( T_g : H^p \to H^p \) is bounded. Then (4.6), with \( n = 1 \), yields

\[
\| T_g(f) \|_{H^p}^p \asymp \int_T \left( \int_{\Gamma(\zeta)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, |d\zeta| \lesssim \| f \|_{H^p}^p
\]

for all \( f \in H^p \). This together with Fubini’s theorem, Hölder’s inequality and (4.10) give

\[
\int_D |f(z)|^p |g'(z)|^2 (1 - |z|^2) \, dA(z) \asymp \int_D |f(z)|^p |g'(z)|^2 |I_z| \, dA(z)
\]

\[
\asymp \int_T \int_{\Gamma(\zeta)} |f(z)|^p |g'(z)|^2 \, dA(z) \, |d\zeta|
\]

\[
\leq \int_T (f^*(\zeta))^{p-2} \left( \int_{\Gamma(\zeta)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right) \, |d\zeta|
\]

\[
\leq \left( \int_T (f^*(\zeta))^p \, |d\zeta| \right)^{\frac{p-2}{p}} \left( \int_D \left( \int_{\Gamma(\zeta)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, |d\zeta| \right)^{\frac{2}{p}} \lesssim \| f \|_{H^p}^p,
\]

and thus \( |g'(z)|^2 (1 - |z|^2) \, dA(z) \) is a \( p \)-Carleson measure for \( H^p \). Therefore \( g \in \text{BMOA} \) by \([28, \text{Theorem 9.4}]\).

Let now \( 0 < p < 2 \), and assume that \( T_g : H^p \to H^p \) is bounded. First, we will show that \( g \) is a Bloch function and

\[
\| g \|_B \lesssim \| T_g \|_{(H^p, H^p)}.
\]

For \( a \in \mathbb{D} \), let \( D_a = \{ z : |z - a| < \frac{1 - |a|^2}{2} \} \) and consider the functions \( F_{a,\mu}(z) = \left( \frac{1 - |a|^2}{1 - \mu z} \right)^{\frac{\gamma+1}{p}} \), where \( \gamma > 0 \) is fixed. Clearly, \( \| F_{a,\mu} \|_{H^p} \asymp (1 - |a|) \) and \( |F_{a,\mu}(z)| \asymp 1 \) for all \( z \in D_a \). Moreover, there exists \( r_0 \in (0, 1) \) such that

\[
|\{ \zeta \in \mathbb{T} : D_a \subset \Gamma(\zeta) \}| \asymp (1 - |a|), \quad |a| \geq r_0.
\]

Therefore

\[
(1 - |a|) \left( \int_{D_a} |g'(z)|^2 \, dA(z) \right)^{p/2} \lesssim \int_T \left( \int_{\Gamma(\zeta)} |F_{a,\mu}(z)|^2 |g'(z)|^2 \, dA(z) \right)^{p/2} \, |d\zeta|
\]

\[
\lesssim \| T_g \|_{(H^p, H^p)}^p \| F_{a,\mu} \|_{H^p}^p
\]

\[
\asymp \| T_g \|_{(H^p, H^p)}^p (1 - |a|), \quad |a| \geq r_0,
\]
from which the subharmonicity of $|g'|^2$ yields

$$|g'(a)|^2(1 - |a|^2)^2 \lesssim \int_{D_a} |g'(z)|^2 \, dA(z) \lesssim \|T_g\|_{(H^p, H^p)}^2 < \infty.$$ 

This implies $g \in \mathcal{B}$ and (4.30).

Let now $1 < \alpha, \beta < \infty$ such that $\beta/\alpha = p/2 < 1$, and let $\alpha'$ and $\beta'$ be the conjugate indexes of $\alpha$ and $\beta$. Then Fubini’s theorem, (2.5) and Hölder’s inequality yield

$$\int_{S(a)} |g'(z)|^2(1 - |z|^2) \, dA(z) \lesssim \left( \int_T \left( \int_{S(a) \cap \Gamma(\zeta)} |g'(z)|^2 |F_{a,p}(z)|^2 \, dA(z) \right)^{\frac{\beta}{2}} \, |d\zeta| \right)^{\frac{1}{\beta'}} \cdot \left( \int_T \left( \int_{\Gamma(\zeta) \cap S(a)} |g'(z)|^2 \, dA(z) \right)^{\frac{\alpha'}{2}} \, |d\zeta| \right)^{\frac{1}{\alpha'}} = \|T_g(F_{a,p})\|_{H^p} \|S_g(\chi_{S(a)})\|_{L^\frac{\alpha'}{\beta'}(T)} \cdot a \in \mathbb{D},$$

(4.31)

where

$$S_g(\varphi)(\zeta) = \int_{\Gamma(\zeta)} |\varphi(z)|^2 |g'(z)|^2 \, dA(z), \quad \zeta \in T,$$

for any bounded function $\varphi$ on $\mathbb{D}$. Now $\left( \frac{\beta'}{\alpha'} \right)' = \frac{\beta - 1}{\alpha - \beta} > 1$, and hence

$$\|S_g(\chi_{S(a)})\|_{L^\frac{\alpha'}{\beta'}(T)} = \|h\|_{L^\frac{\beta - 1}{\alpha - \beta}(T)} \leq \left| \int_T h(\zeta) S_g(\chi_{S(a)})(\zeta) \, |d\zeta| \right|.$$ 

(4.32)
Next, using Fubini’s theorem, Hölder’s inequality and Corollary 4.10, we deduce
\[
\left| \int_{\mathcal{T}} h(\zeta) S_g(\chi_S(\alpha))(\zeta) \, d\zeta \right| \\
\leq \int_{\mathcal{T}} \left| h(\zeta) \right| \int_{\Gamma(\zeta) \cap S(\alpha)} \left| g'(z) \right|^2 dA(z) \, d\zeta \\
\leq \int_{S(\alpha)} \left| g'(z) \right|^2 (1 - |z|^2) M(|h|)(z) \, dA(z) \\
\leq \left( \int_{S(\alpha)} \left| g'(z) \right|^2 (1 - |z|^2) \, dA(z) \right)^{\frac{\gamma'}{\gamma'}} \\
\cdot \left( \int_{\mathcal{D}} M(|h|) \left( \frac{\gamma'}{\gamma} \right)^\gamma \left| g'(z) \right|^2 (1 - |z|^2) \, dA(z) \right)^{1 - \frac{\gamma'}{\gamma}} \\
\leq \left( \int_{S(\alpha)} \left| g'(z) \right|^2 (1 - |z|^2) \, dA(z) \right)^{\frac{\gamma'}{\gamma'}} \\
\cdot \left( \sup_{a \in \mathcal{B}} \frac{\int_{S(\alpha)} \left| g'(z) \right|^2 (1 - |z|^2) \, dA(z)}{1 - |a|} \right)^{1 - \frac{\gamma'}{\gamma}} \| h \|_{L^\gamma(\mathcal{T})}^{\frac{\gamma'}{\gamma}}.
\]
(4.33)

By replacing \( g \) by \( g_r \) in (4.31)–(4.33), we obtain
\[
\int_{S(\alpha)} \left| g'_r(z) \right|^2 (1 - |z|^2) \, dA(z) \\
\leq \left\| T_g(F_{a,p}) \right\|_{H^p}^p \left( \int_{S(\alpha)} \left| g'_r(z) \right|^2 (1 - |z|^2) \, dA(z) \right)^{\frac{\gamma'}{\gamma'}} \\
\cdot \left( \sup_{a \in \mathcal{B}} \frac{\int_{S(\alpha)} \left| g'_r(z) \right|^2 (1 - |z|^2) \, dA(z)}{1 - |a|} \right)^{1 - \frac{\gamma'}{\gamma}}.
\]
(4.34)

By arguing as in the proof of (4.18) we find a constant \( C > 0 \) such that
\[
\sup_{0 < r < 1} \left\| T_g(F_{a,p}) \right\|_{H^p}^p \leq C\left\| T_q \right\|_{(H^p,H^q)}^p (1 - |a|), \quad a \in \mathcal{D}.
\]
This combined with (4.34) and Fatou’s lemma yield
\[
\sup_{a \in \mathcal{B}} \frac{\int_{S(\alpha)} \left| g'(z) \right|^2 (1 - |z|^2) \, dA(z)}{1 - |a|} \lesssim \left\| T_g \right\|_{(H^p,H^q)}^2,
\]
and so \( g \in \text{BMOA} \). Now the proof of (ai)⇒(bi) is complete.

(ai)⇒(biv). One of the key ingredients in the original proof of this implication is [7, Proposition p. 170] that is stated in weaker form as Proposition C below. Our contribution consists of indicating how this weaker result can be established by using the standard factorization of \( H^p \)-functions instead of appealing to the interpolation theory. The disadvantage of this method is that we will not obtain the sharp constant \( C = 1 \) that follows by interpolating.

**Proposition C.** Let \( 0 < q < p < \infty \) and let \( T_g : H^p \to H^q \) be bounded. Then \( T_g : H^p \to H^q \) is bounded for any \( \hat{p} < p \) and \( \hat{q} < q \) with \( \frac{1}{\hat{q}} - \frac{1}{p} = \frac{1}{q} - \frac{1}{\hat{p}} \). Further, if
0 < p < ∞, then there exists $C = C(q) > 0$ such that
\begin{equation}
\limsup_{\hat{p} \to p^{-}} \|Tg\|_{(H^{\hat{p}}, H^{p})} \leq C\|Tg\|_{(H^{p}, H^{\hat{p}})}.
\end{equation}

Indeed, if $f \in H^p$, then $f = Bg$, where $B$ is the Blaschke product whose zeros are those of $f$, and $g$ is a non-vanishing analytic function with $\|g\|_{H^{\hat{p}}} = \|f\|_{H^{\hat{p}}}$. So, if we take $f_1 = Bg^{\hat{p}} \in H^p$ and $f_2 = g^{\frac{\hat{p} - p}{p}} \in H^\frac{\hat{p} - p}{p}$, then
\[ f = f_1f_2 \quad \text{and} \quad \|f_1\|_{H^p} \cdot \|f_2\|_{H^\frac{\hat{p} - p}{p}} = \|f\|_{H^{\hat{p}}}.
\]
Consequently, (4.35) can be proved in the same way as (4.24).

The remaining parts of Theorem B can be proved by appropriately modifying the proof of Theorem 4.1. Basically one has to use the norm (4.6) in $H^p$ when the norm (4.4) in $A^p_\omega$ is used, and use the maximal function $f^*$ whenever $N(f)$ appears. Further, the maximal operator $M$ should be used instead of $M_\omega$ and the weight $1 - |z|$ should appear instead of $\omega^*(z)$. Furthermore, one will need to use \[28, Theorem 9.3\], \[28, Theorem 9.4\], \[28, Theorem 5.1\], and Corollary 4.10. With this guidance on the remaining implications we consider Theorem B proved.

We finish the section by two observations on the proof of Theorem B. First, the techniques can be also used to establish \[7, Corollary 1\]. Second, in the proof we used repeatedly the fact that for $h \in L^p(\mathbb{T})$, $1 < q \leq q$, and a $q/p$-classical Carleson measure $\mu$, we have
\[
\int_{\mathbb{D}} \left( \frac{1}{1 - |z|^2} \int_{\mathbb{T}} h(\zeta) \, dm(\zeta) \right) \, d\mu(z) \lesssim \int_{\mathbb{D}} M(h)(z) \, d\mu(z) \\
\lesssim \|M(h)\|_{L^q(\mu)} \lesssim \|h\|_{L^p(\mathbb{T})} < \infty.
\]
It is also worth noticing that here the maximal function $M(h)$ can be replaced by the Poisson integral $P(h)$ of $h$. 
CHAPTER 5

Non-conformally Invariant Space Induced by $T_g$ on $A^p_\omega$

The boundedness of the integral operator $T_g$ on the Hardy space $H^p$ and the classical weighted Bergman space $A^p_\omega$ are characterized by the conditions $g \in BMOA$ and $g \in B^{\alpha}$, respectively. Moreover, we saw in Chapter 4 that if $\omega$ is rapidly increasing, then $T_g$ is bounded on $A^p_\omega$ if and only if $g \in C^1(\omega^*)$. Since $H^p \subset A^p_\omega \subset A^p_\alpha$ for each $\omega \in \mathcal{I}$ and $\alpha > -1$, it is natural to expect that $C^1(\omega^*)$ lies somewhere between $BMOA$ and $B^{\alpha}$. In this chapter we will give some insight to the structural properties of the spaces $C^1(\omega^*)$ and $C^1_0(\omega^*)$. We will also study their relations to several classical spaces of analytic functions on $D$. In particular, we will confirm the expected inclusions $BMOA \subset C^1(\omega^*) \subset B^{\alpha}$. Moreover, we will show that whenever a rapidly increasing weight $\omega$ admits certain regularity, then $C^1(\omega^*)$ is not conformally invariant, but contains non-trivial inner functions.

5.1. Inclusion relations

The first result shows the basic relations between $BMOA$, $C^1(\omega^*)$, $B$, and $A^p_\omega$, when $\omega \in \mathcal{I} \cup \mathcal{R}$. In some parts we are forced to require additional regularity on $\omega \in \mathcal{I}$ due to technical reasons induced by the fact that rapidly increasing weights may admit a strong oscillatory behavior, as was seen in Chapter 1. Recall that $h : [0, 1) \to (0, \infty)$ is essentially increasing on $[0, 1)$ if there exists a constant $C > 0$ such that $h(r) \leq C h(t)$ for all $0 \leq r < t < 1$.

Proposition 5.1. (A) If $\omega \in \mathcal{I} \cup \mathcal{R}$, then $C^1(\omega^*) \subset \bigcap_{0 < p < \infty} A^p_\omega$.

(B) If $\omega \in \mathcal{I} \cup \mathcal{R}$, then $BMOA \subset C^1(\omega^*) \subset B$.

(C) If $\omega \in \mathcal{R}$, then $C^1(\omega^*) = B$.

(D) If $\omega \in \mathcal{I}$, then $C^1(\omega^*) \subset B$.

(E) If $\omega \in \mathcal{I}$ and both $\omega(r)$ and $\frac{\psi_\omega(r)}{1-r}$ are essentially increasing on $[0, 1)$, then $BMOA \subset C^1(\omega^*)$.

Proof. (A). Let $g \in C^1(\omega^*)$. Theorem 2.1 shows that $|g'(z)|^2 \omega^*(z) \, dA(z)$ is a $p$-Carleson measure for $A^p_\omega$ for all $0 < p < \infty$. In particular, $|g'(z)|^2 \omega^*(z) \, dA(z)$ is a finite measure and hence $g \in A^p_\omega$ by (4.5). Therefore (4.3) yields

$$\|g\|_{A^p_\omega}^4 = 4^2 \int_D |g(z)|^2 |g'(z)|^2 \omega^*(z) \, dA(z) + |g(0)|^4 \lesssim \|g\|_{A^2_\omega}^2 + |g(0)|^4,$$

and thus $g \in A^4_\omega$. Continuing in this fashion, we deduce $g \in A^{2n}_\omega$ for all $n \in \mathbb{N}$, and the assertion follows.
(B). If \( g \in \text{BMOA} \), then \( |g'(z)|^2 \log \frac{1}{|z|} \, dA(z) \) is a classical Carleson measure \([31]\) (or \([33, \text{Section 8}]\)), that is,
\[
\sup_{I \subset \mathbb{T}} \int_{S(I)} |g'(z)|^2 \log \frac{1}{|z|} \, dA(z) < \infty.
\]
Therefore
\[
\int_{S(I)} |g'(z)|^2 \omega^*(z) \, dA(z) \leq \int_{S(I)} |g'(z)|^2 \log \frac{1}{|z|} \left( \int_{|z|}^1 \omega(s) s \, ds \right) \, dA(z)
\leq \left( \int_{1-|I|}^1 \omega(s) s \, ds \right) \int_{S(I)} |g'(z)|^2 \log \frac{1}{|z|} \, dA(z)
\leq \left( \int_{1-|I|}^1 \omega(s) s \, ds \right) |I| \approx \omega(S(I)),
\]
which together with Theorem 2.1 gives \( g \in C^1(\omega^*) \) for all \( \omega \in \mathcal{I} \cup \mathcal{R} \).

Let now \( g \in C^1(\omega^*) \) with \( \omega \in \mathcal{I} \cup \mathcal{R} \). It is well known that \( g \in \mathcal{H}(\mathbb{D}) \) is a Bloch function if and only if
\[
\int_{S(I)} |g'(z)|^2 (1 - |z|^2)^\gamma \, dA(z) \lesssim |I|^\gamma, \quad I \subset \mathbb{T},
\]
for some (equivalently for all) \( \gamma > 1 \), see \([82]\). Fix \( \beta = \beta(\omega) > 0 \) and \( C = C(\beta, \omega) > 0 \) as in Lemma 1.1. Then Lemma 1.6 and Lemma 1.1 yield
\[
\int_{S(I)} |g'(z)|^2 (1 - |z|)^{\beta+1} \, dA(z) = \int_{S(I)} |g'(z)|^2 \omega^*(z) \frac{(1 - |z|)^{\beta+1}}{\omega^*(z)} \, dA(z)
\approx \int_{S(I)} |g'(z)|^2 \omega^*(z) \frac{(1 - |z|)^{\beta}}{\int_{|z|}^1 \omega(s) s \, ds} \, dA(z)
\leq \frac{C |I|^\beta}{\int_{1-|I|}^1 \omega(s) s \, ds} \int_{S(I)} |g'(z)|^2 \omega^*(z) \, dA(z)
\lesssim |I|^{\beta+1}, \quad |I| \leq \frac{1}{2},
\]
and so \( g \in \mathcal{B} \).

(C). By Part (B) it suffices to show that \( \mathcal{B} \subset C^1(\omega^*) \) for \( \omega \in \mathcal{R} \). To see this, let \( g \in \mathcal{B} \) and \( \omega \in \mathcal{R} \). By (1.4) there exists \( \alpha = \alpha(\omega) > 0 \) such that \( h(r) = \frac{\int_0^r \omega(s) \, ds}{(1-r)^\alpha} \) is decreasing on \([0, 1]\). This together with Lemma 1.6 gives
\[
\int_{S(I)} |g'(z)|^2 \omega^*(z) \, dA(z) = \int_{S(I)} |g'(z)|^2 \frac{\omega^*(z)}{(1 - |z|)^{\alpha+1}} (1 - |z|)^{\alpha+1} \, dA(z)
\approx \int_{S(I)} |g'(z)|^2 \frac{\int_{|z|}^1 \omega(s) s \, ds \, ds}{(1 - |z|)^\alpha} (1 - |z|)^{\alpha+1} \, dA(z)
\leq \frac{\int_{1-|I|}^1 \omega(s) \, ds}{|I|^{\alpha}} \int_{S(I)} |g'(z)|^2 (1 - |z|)^{\alpha+1} \, dA(z)
\lesssim \omega(S(I)), \quad |I| \leq \frac{1}{2},
\]
and therefore $g \in C^1(\omega^*)$.

(D). Let $\omega \in \mathcal{I}$, and assume on the contrary to the assertion that $B \subset C^1(\omega^*)$. Ramey and Ullrich [70, Proposition 5.4] constructed $g_1, g_2 \in B$ such that $|g_1'(z)| + |g_2'(z)| \geq (1-|z|)^{-1}$ for all $z \in \mathbb{D}$. Since $g_1, g_2 \in C^1(\omega^*)$ by the antithesis, Lemma 1.6 yields

$$\|f\|_{A^2_\omega}^2 \geq \int_{\mathbb{D}} |f(z)|^2 \left(|g_1'(z)|^2 + |g_2'(z)|^2\right) \omega^*(z) \, dA(z)$$

$$\geq \frac{1}{2} \int_{\mathbb{D}} |f(z)|^2 \left(|g_1'(z)| + |g_2'(z)|\right)^2 \omega^*(z) \, dA(z)$$

$$(5.1)$$

$$\geq \frac{1}{2} \int_{\mathbb{D}} |f(z)|^2 \frac{\omega^*(z)}{(1-|z|)^2} \, dA(z) = \int_{\mathbb{D}} |f(z)|^2 \frac{\int_0^1 |\omega(s)| \, ds}{(1-|z|)} \, dA(z)$$

$$= \int_{\mathbb{D}} |f(z)|^2 \frac{\psi(\omega(z))}{1-|z|} \, dA(z)$$

for all $f \in \mathcal{H}(\mathbb{D})$. If $\int_{\mathbb{D}} \frac{\psi(\omega(z))}{1-|z|} \, dA(z) = \infty$, we choose $f \equiv 1$ to obtain a contradiction. Assume now that $\int_{\mathbb{D}} \frac{\psi(\omega(z))}{1-|z|} \, dA(z) < \infty$, and replace $f$ in (5.1) by the test function $F_{a,2}$ from Lemma 2.4. Then (2.6) and Lemma 1.6 yield

$$\omega^*(a) \geq \int_0^1 \frac{(1-|a|)^{\gamma+1}}{(1-|a|r)^\gamma} \frac{\psi(r)}{1-r} \omega(r) \, dr \geq (1-|a|) \int_{|a|}^1 \frac{\psi(r)}{1-r} \omega(r) \, dr,$$

and hence

$$\int_{|a|}^1 \frac{\psi(r)}{1-r} \omega(r) \, dr \leq \int_{|a|}^1 \omega(r) \, dr, \quad a \in \mathbb{D}.$$

By letting $|a| \to 1^-$, Bernouilli-l'Hôpital theorem and the assumption $\omega \in \mathcal{I}$ yield a contradiction. For completeness with regards to (5.1) we next construct $f \in A^2_\omega$ such that

$$(5.2)$$

$$\int_{\mathbb{D}} |f(z)|^2 \frac{\psi(\omega(z))}{1-|z|} \, dA(z) = \infty,$$

provided $\omega \in \mathcal{I}$ such that $\frac{\psi(r)}{1-r}$ is essentially increasing on $[0,1)$. To do this, denote

$$\omega_k = \int_0^1 s^{2k} \omega(s) s \, ds, \quad M_k = \int_0^1 s^{2k} \frac{\psi(s)}{1-s} \omega(s) s \, ds, \quad k \in \mathbb{N}.$$

Since $\omega \in \mathcal{I}$, there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that

$$(5.3)$$

$$n_k \psi(1-n_k^{-1}) \geq k^2, \quad k \in \mathbb{N}.$$

We claim that the analytic function

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{n_k}}{M_{n_k}^{1/2}}$$

has the desired properties. Parseval's identity shows that $f$ satisfies (5.2). To prove $f \in A^2_\omega$, note first that by the proof of Lemma 1.1, there exists $r_0 \in (0,1)$ such that $h(r) = \int_{r_0}^r \omega(s) s \, ds$ is increasing on $[r_0,1)$, and hence $h$ is essentially increasing on $[0,1)$. This together with the assumption on $\frac{\psi(r)}{1-r}$ and Lemmas 1.1 and 1.3 yields

$$(5.4)$$

$$\omega_k \simeq \int_{1-\frac{1}{k}}^1 \omega(s) s \, ds, \quad M_k \simeq \int_{1-\frac{1}{k}}^1 \frac{\psi(s)}{1-s} \omega(s) s \, ds, \quad k \in \mathbb{N}.$$
Parseval’s identity, (5.4), the assumption on $\frac{\psi_\omega(r)}{r^{1/p}}$ and (5.3) finally give
\[
\|f\|_{A^2}^2 = \sum_k \frac{\omega_k}{M_{n_k}} \geq \sum_{k=1}^\infty \frac{\int_1^1 \frac{\psi_\omega(s)}{1-s} s \omega(s) ds}{\int_1^1 \frac{1}{1-s} \omega(s) ds} \sum_{k=1}^\infty \frac{1}{k^2} \leq \|f\|_{C^1}^2.
\]

(E) Recall that $\text{BMOA} \subset C^1(\omega^*)$ by Part (B). In Proposition 5.2(E) below we will prove that $C^1_0(\omega^*) \not\subseteq \text{BMOA}$, which yields the desired strict inclusion $\text{BMOA} \subset C^1(\omega^*)$.

Results analogous to those given in Proposition 5.1 are valid for $C^1_0(\omega^*)$. Recall that $\text{VMOA}$ consists of those functions in the Hardy space $H^1$ that have vanishing mean oscillation on the boundary $\Gamma$, and $f \in \mathcal{H}(\Delta)$ belongs to the little Bloch space $B_0$ if $f'(z)(1-|z|^2) \to 0$, as $|z| \to 1^-$. It is well known that $\text{VMOA} \subset B_0$.

**Proposition 5.2.**

(A) $C^1_0(\omega^*)$ is a closed subspace of $(C^1(\omega^*), \| \cdot \|_{C^1(\omega^*)})$.

(B) If $\omega \in \mathcal{I} \cup \mathcal{R}$, then $\text{VMOA} \subset C^1_0(\omega^*) \subset B_0$.

(C) If $\omega \in \mathcal{R}$, then $C^1_0(\omega^*) = B_0$.

(D) If $\omega \in \mathcal{I}$, then $C^1_0(\omega^*) \not\subseteq B_0$.

(E) If $\omega \in \mathcal{I}$ and both $\omega(r)$ and $\frac{\psi_\omega(r)}{r}$ are essentially increasing, then $\text{VMOA} \subset C^1_0(\omega^*)$. Moreover, there exists a function $g \in C^1_0(\omega^*)$ such that $g \notin \text{BMOA}$.

**Proof.** Parts (A), (B) and (C) follow readily from the proof of Proposition 5.1. Part (D) can be proved by following the argument in Proposition 5.1(D), with $g_1, g_2 \in \mathcal{B}$ being replaced by $(g_1)_r, (g_2)_r \in B_0$, and using Fatou’s lemma at the end of the proof. It remains to prove the second assertion in (E), which combined with (B) implies $\text{VMOA} \subset C^1_0(\omega^*)$. To do this, we assume without loss of generality that $\int_1^1 \omega(s) ds < 1$. Consider the lacunary series
\[
g(z) = \sum_{k=0}^\infty \frac{z^{2^k}}{2^{k/2}} \left( \psi_\omega(1 - 2^{-k}) \log \left( \frac{e}{\int_1^1 \omega(s) ds} \right) \right)^{1/2}.
\]

The assumption on $\frac{\psi_\omega(r)}{r^{1/p}}$ implies
\[
\infty = \lim_{p \to 1^-} \log \log \left( \frac{e}{\int_0^1 \omega(s) ds} \right) \leq \int_0^1 \frac{dr}{\psi_\omega(r) \log \left( \frac{e}{\int_0^1 \omega(s) ds} \right)} \leq \sum_{k=0}^\infty \int_1^1 2^{-k} \frac{dr}{\psi_\omega(r) \log \left( \frac{e}{\int_0^1 \omega(s) ds} \right)} \leq \sum_{k=0}^\infty \frac{1}{2^k \psi_\omega(1 - 2^{-k}) \log \left( \frac{e}{\int_1^1 \omega(s) ds} \right)} = \|g\|_{C^1}^2,
\]

and hence $g \notin \text{BMOA}$.
It remains to show that \( g \in \mathcal{C}_0^1(\omega^*) \). Consider \( z = re^{it} \in \mathbb{D} \) with \( r \geq \frac{1}{2} \), and take \( N = N(r) \in \mathbb{N} \) such that \( 1 - \frac{1}{2N} \leq r < 1 - \frac{1}{2N+1} \). Since \( \omega(r) \) is essentially increasing on \([0, 1)\) and \( h(x) = x \log \frac{x}{2} \) is increasing on \((0, 1] \), we deduce

\[
\sum_{k=0}^{N} 2^{k/2} r^{2^k} \left( \psi_{\omega}(1 - 2^{-k}) \log \left( \frac{e}{\int_{1-2^{-k}} \omega(s) \, ds} \right) \right)^{1/2}
\]

\[
\leq \frac{1}{2^{N/2}} \sum_{k=0}^{N} 2^{k/2} \left( \psi_{\omega}(1 - 2^{-N}) \log \left( \frac{e}{\int_{1-2^{-N}} \omega(s) \, ds} \right) \right)^{1/2}
\]

\[
\leq \frac{1}{2^{N/2}} \sum_{k=0}^{N} 2^{k/2} \left( \psi_{\omega}(r) \log \left( \frac{e}{\int_{1} \omega(s) \, ds} \right) \right)^{1/2}
\]

Moreover, the assumptions on \( \omega(r) \) and \( \frac{\psi_{\omega}(r)}{r} \) and Lemma 1.1(ii) yield

\[
\sum_{k=N+1}^{\infty} 2^{k/2} r^{2^k} \left( \psi_{\omega}(1 - 2^{-k}) \log \left( \frac{e}{\int_{1-2^{-k}} \omega(s) \, ds} \right) \right)^{1/2}
\]

\[
\leq \frac{1}{2^{N/2}} \sum_{k=N+1}^{\infty} 2^{k/2} \left( \psi_{\omega}(1 - 2^{-N}) \log \left( \frac{e}{\int_{1-2^{-N}} \omega(s) \, ds} \right) \right)^{1/2}
\]

\[
\leq \frac{1}{2^{N/2}} \sum_{k=N+1}^{\infty} 2^{k/2} \left( \psi_{\omega}(r) \log \left( \frac{e}{\int_{1} \omega(s) \, ds} \right) \right)^{1/2}
\]

Consequently, joining (5.6) and (5.7), we deduce

\[
M_\infty(r, g') \leq \frac{1}{(1 - r) \psi_{\omega}(r) \log \left( \frac{e}{\int_{1} \omega(s) \, ds} \right)} \left( \psi_{\omega}(r) \right)^{1/2}, \quad r \geq \frac{1}{2},
\]

in particular,

\[
M_\infty^2(r, g') = o \left( \frac{1}{\psi_{\omega}(r)(1 - r)} \right) \asymp \left( \frac{\omega(r)}{\omega^*(r)} \right), \quad r \to 1^-,
\]

by Lemma 1.6. Hence, for each \( \varepsilon > 0 \), there exists \( r_0 \in (0, 1) \) such that \( M_\infty^2(r, g') \leq \varepsilon \frac{\omega(r)}{\omega^*(r)} \) for all \( r \in [r_0, 1) \). It follows that

\[
\int_{S(I)} |g'(z)|^2 \omega^*(z) \, dA(z) \leq \int_{S(I)} M_\infty^2(|z|, g') \omega^*(z) \, dA(z)
\]

\[
\leq \varepsilon \int_{S(I)} \omega(z) \, dA(z) = \varepsilon \omega(S(I)), \quad |I| \leq 1 - r_0,
\]

and thus \( g \in \mathcal{C}_0^1(\omega^*) \) by Theorem 2.1(ii). \( \square \)
The proof of Part (E) shows that for any $\omega \in \mathcal{I}$, with both $\omega(r)$ and $\frac{\psi(r)}{r}$ essentially increasing, there are functions in $C_0^1(\omega^*)$ which have finite radial limits almost nowhere on $\mathbb{T}$.

5.2. Structural properties of $C^1(\omega^*)$

We begin with the following auxiliary result which, in view of Theorem 2.1, yields a global integral characterization of $q$-Carleson measures for $A^p_\omega$ when $0 < p \leq q < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$.

**Lemma 5.3.** Let $0 < s < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$, and let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then there exists $\eta = \eta(\omega) > 1$ such that

$$I_1(\mu) = \sup_{a \in \mathbb{D}} \frac{\mu(S(a))}{(\omega(S(a)))^s} \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left( \frac{1}{\omega(S(a))} \left( \frac{1 - |a|}{|1 - \overline{az}|} \right)^\eta \right)^s d\mu(z) = I_2(\mu)$$

and, for $s \geq 1$,

$$\lim_{|a| \to 1} \frac{\mu(S(a))}{(\omega(S(a)))^s} = 0 \Leftrightarrow \lim_{|a| \to 1} \int_{\mathbb{D}} \left( \frac{1}{\omega(S(a))} \left( \frac{1 - |a|}{|1 - \overline{az}|} \right)^\eta \right)^s d\mu(z) = 0.$$

In particular, if $\omega \in \mathcal{I}$, then the above assertions are valid for all $1 < \eta < \infty$.

**Proof.** Clearly,

$$\int_{\mathbb{D}} \left( \frac{1}{\omega(S(a))} \left( \frac{1 - |a|}{|1 - \overline{az}|} \right)^\eta \right)^s d\mu(z) \geq \int_{S(a)} \left( \frac{1}{\omega(S(a))} \left( \frac{1 - |a|}{|1 - \overline{az}|} \right)^\eta \right)^s d\mu(z) \approx \frac{\mu(S(a))}{(\omega(S(a)))^s},$$

and consequently $I_1(\mu) \lesssim I_2(\mu)$.

For $a \in \mathbb{D} \setminus \{0\}$ and $k \in \mathbb{N} \cup \{0\}$, denote

$$S_k(a) = \{ z \in \mathbb{D} : |z - a/|a|| < 2^k(1 - |a|) \}.$$
Then \(|1-\pi z| \approx 1-|a|\), if \(z \in S_0(a)\), and \(|1-\pi z| \approx 2^k(1-|a|)\), if \(z \in S_k(a)\setminus S_{k-1}(a)\) and \(k \geq 1\). Let \(\eta > \beta + 1\), where \(\beta = \beta(\omega) > 0\) is from Lemma 1.1. Then
\[
I_2(\mu) \leq \int_D \left(\frac{1}{\omega(S(a))} \left(\frac{1-|a|}{|1-\pi z|}\right)^\eta\right)^s d\mu(z)
\]
\[
= \frac{1}{\omega(S(a))^s} \sum_{k} \int_{S_k(a)\setminus S_{k-1}(a)} \left(\frac{1-|a|}{|1-\pi z|}\right)^\eta d\mu(z)
\]
\[
\leq \frac{1}{\omega(S(a))^s} \sum_{k} \frac{1}{2^{k\eta}} \int_{S_k(a)} d\mu(z)
\]
(5.9)
\[
\leq \frac{I_1(\mu)}{\omega(S(a))^s} \sum_{k} \frac{1}{2^{k\eta}} (2^k(1-|a|))^s \left(\int_{1-2^k(1-|a|)} \omega(s)s ds\right)^s
\]
\[
= \frac{I_1(\mu)}{\omega(S(a))^s} \sum_{k} \frac{1}{2^{k\eta}} (1-|a|)^s \left(\int_{1-2^k(1-|a|)} \omega(s)s ds\right)^s
\]
\[
\leq \frac{I_1(\mu)}{\omega(S(a))^s} \sum_{k} \frac{1}{2^{k\eta}} (1-|a|)^s \left(\int |\omega(s)s ds\right)^s
\]
\[
\leq I_1(\mu) \sum_{k=0}^{\infty} \frac{1}{2^{k\eta}} \leq I_1(\mu),
\]
because \(\eta > \beta + 1\). It follows that \(I_2(\mu) \lesssim I_1(\mu)\), and thus the first assertion is proved.

Assume now that
\[
(5.10) \quad \lim_{|a| \to 1} \frac{\mu(S(a))}{\omega(S(a)))^s} = 0,
\]
and hence \(I_1(\mu) < \infty\). Then (5.9) implies
\[
\int_D \left(\frac{1}{\omega(S(a))} \left(\frac{1-|a|}{|1-\pi z|}\right)^\eta\right)^s d\mu(z)
\]
\[
\leq \frac{1}{\omega(S(a))^s} \left(1-|a|\right)\left(1-1^s\right) \mu(D) + \sup_{a \in D} \frac{\mu(S(a) \setminus D(0, r))}{\omega(S(a))^s}
\]
\[
\leq \left(\int_{1} \omega(s)s ds\right)^s \frac{\mu(D)}{(1-r)\eta} + \sup_{|a| \geq r} \frac{\mu(S(a))}{\omega(S(a))^s},
\]
where the obtained upper bound can be made arbitrarily small by fixing sufficiently large \(r\) by (5.10) first and then choosing \(|a|\) to be close enough to 1, see the proof of Lemma 1.1 for an argument similar to the last step. The opposite implication follows by (5.8).

Lemma 5.3 yields an alternative description of \(C^\alpha(\omega^*)\) when \(\alpha \geq 1\). Indeed, for each \(\omega \in I \cup R\) and \(\eta = \eta(\omega) > 1\) large enough, the quantity
\[
\|g\|_{C^\alpha(\omega^*), \eta}^2 = |g(0)|^2 + \sup_{a \in D} \int_D \left(\frac{1}{\omega(S(a))} \left(\frac{1-|a|}{|1-\pi z|}\right)^\eta\right)^\alpha |g'(z)|^2 \omega^*(z) dA(z)
\]
is equivalent to \( \|g\|_{C^\alpha(\omega^*)}^2 \). We also deduce that \( g \in C^\alpha_0(\omega^*) \) if and only if
\[
\lim_{|a| \to 1} \int_{|z| < 1} \left( \frac{1}{\omega(S(a))} \left( \frac{1 - |a|}{1 - |z|} \right)^\alpha \right) |g'(z)|^2 \omega^*(z) dA(z) = 0.
\]

Recall that an inner function is called \textit{trivial} if it is a finite Blaschke product. It is known that only inner functions in VMOA are the trivial ones \cite{33}, but this is not true in the case of \( B_0 \) \cite{19}. Therefore it is natural to ask whether or not \( C^\alpha_0(\omega^*) \) contains non-trivial inner functions when \( \omega \in I \)? The next result gives an affirmative answer to this question and offers an alternative proof for the first assertion in Proposition 5.2(E) under the additional assumption on the monotonicity of \( \psi_\omega(r)/(1-r) \).

**Proposition 5.4.** If \( \omega \in I \) such that \( \omega(r) \) is essentially increasing and \( \psi_\omega(r)/(1-r) \) is increasing, then there are non-trivial inner functions in \( C^\alpha_0(\omega^*) \).

**Proof.** We may assume without loss of generality that \( \int_0^1 \omega(r) \, dr < 1 \). We will show that there exists a non-trivial inner function \( I \) such that
\[
M_\infty(r, I') \lesssim \frac{1}{\left( (1-r)\psi_\omega(r) \log \left( \frac{e}{\int_0^r \omega(\varsigma) \, d\varsigma} \right) \right)^{1/2}}, \quad r \in (0, 1),
\]
and therefore \( I \in C^\alpha_0(\omega^*) \) by Theorem 2.1.

Consider the auxiliary function \( \Phi \) defined by
\[
\Phi^2(r) = \frac{r}{\psi_\omega(1-r) \log \left( \frac{e}{\int_{1-r}^1 \omega(\varsigma) \, d\varsigma} \right)}, \quad r \in (0, 1], \quad \Phi(0) = 0.
\]
It is clear that \( \Phi \) is continuous and increasing since \( \psi_\omega(r)/(1-r) \) is increasing on \( (0, 1) \). Further,
\[
\int_0^1 \frac{\Phi^2(t)}{t} \, dt = \int_0^1 \frac{1}{\psi_\omega(r) \log \left( \frac{e}{\int_0^r \omega(\varsigma) \, d\varsigma} \right)} \, dr = \infty.
\]
By using the assumption on \( \omega \) and the monotonicity of \( h(r) = r \log \frac{e}{r} \), we obtain
\[
\begin{align*}
t \int_t^1 \frac{\Phi(s)}{s^2} \, ds + t \Phi(1) & \leq t \int_t^1 \frac{\Phi(s)}{s^2} \, ds + \Phi(t) \\
& \lesssim t \int_0^{1-t} \left( (1-r)^3 \psi_\omega(r) \log \left( \frac{e}{\int_r^1 \omega(\varsigma) \, d\varsigma} \right) \right)^{1/2} \, dr + \Phi(t) \\
& \lesssim t \omega^{1/2}(1-t) \int_0^{1-t} \left( (1-r)^3 \left( \int_r^1 \omega(\varsigma) \, d\varsigma \right) \log \left( \frac{e}{\int_r^1 \omega(\varsigma) \, d\varsigma} \right) \right)^{1/2} \, dr + \Phi(t) \\
& \leq \frac{t}{\left( \psi_\omega(1-t) \log \left( \frac{e}{\int_0^1 \omega(\varsigma) \, d\varsigma} \right) \right)^{1/2}} \int_0^{1-t} (1-r)^{-3/2} \, dr + \Phi(t) \lesssim \Phi(t).
\end{align*}
\]
5.2. STRUCTURAL PROPERTIES OF $C^1(\omega^*)$

Consequently, [5, Theorem 6(b)] implies that there exists a non-trivial inner function $I$ such that

$$(1 - |z|^2)\frac{|I'(z)|}{1 - |I(z)|^2} \lesssim \Phi(1 - |z|), \quad z \in \mathbb{D}.$$

This implies (5.11) and finishes the proof.

It is natural to expect that polynomials are not dense in $C^1(\omega^*)$. Indeed we will prove that the closure of polynomials in $(C^1(\omega^*), \|c^1(\omega^*)\|$ is nothing else but $C^1_0(\omega^*)$.

**Proposition 5.5.** Let $\omega \in \mathcal{I} \cup \mathcal{R}$ and $g \in C^1(\omega^*)$. Then the following assertions are equivalent:

(i) $g \in C^1_0(\omega^*)$;

(ii) $\lim_{r \to 1^-} \|g - g_r\|_{C^1(\omega^*)} = 0$;

(iii) There is a sequence of polynomials $\{P_n\}$ such that $\lim_{n \to \infty} \|g - P_n\|_{C^1(\omega^*)} = 0$.

**Proof.** Since $C^1_0(\omega^*)$ is a closed subspace of $C^1(\omega^*)$ by Proposition 5.2(A), and $H^\infty$ is continuously embedded in $C^1(\omega^*)$, standard arguments give the implications (ii)⇒(iii)⇒(i). It remains to prove (i)⇒(ii). To do this, take $\gamma = \gamma(\omega) > 0$ large enough and consider the test functions

$$f_{a,2}(z) = \left( \frac{1}{\omega(S(a))} \left( \frac{1 - |a|}{|1 - az|} \right) \right)^{\gamma+1},$$

defined in (2.13). Since $g \in C^1_0(\omega^*)$ by the assumption, Theorem 2.1, Lemma 5.3 and Theorem 4.2 yield

$$\lim_{|a| \to 1^-} \|T_g(f_{a,2})\|_{A^2_{\omega}}^2 \simeq \lim_{|a| \to 1^-} \int_{\mathbb{D}} |f_{a,2}(z)|^2 |g'(z)|^2 \omega^*(z) dA(z) = 0. \quad (5.12)$$

Since

$$\lim_{r \to 1^-} \|g - g_r\|_{C^1(\omega^*)} \simeq \lim_{r \to 1^-} \sup_{a \in \mathbb{D}} \|T_{g-g_r}(f_{a,2})\|_{A^2_{\omega}}^2,$$

it is enough to prove

$$\lim_{r \to 1^-} \sup_{a \in \mathbb{D}} \|T_{g-g_r}(f_{a,2})\|_{A^2_{\omega}}^2 = 0. \quad (5.13)$$

Let $\varepsilon > 0$ be given. By (5.12) and Proposition 5.2(B), there exists $r_0 \in (1/2, 1)$ such that

$$\|T_g(f_{a,2})\|_{A^2_{\omega}}^2 < \varepsilon, \quad |a| \geq r_0, \quad (5.14)$$

and

$$|g'(a)|^2(1 - |a|)^2 < \varepsilon, \quad |a| \geq r_0. \quad (5.15)$$

If $r_0 \leq |a| \leq 1/(2 - r)$, then (4.20) and (5.14) yield

$$\|T_{g-g_r}(f_{a,2})\|_{A^2_{\omega}}^2 \lesssim \|T_g(f_{a,2})\|_{A^2_{\omega}}^2 + \|T_{g_r}(f_{a,2})\|_{A^2_{\omega}}^2 \lesssim \|T_g(f_{a,2})\|_{A^2_{\omega}}^2 < \varepsilon.$$
If \(|a| \geq \max \{r_0, 1/(2 - r)\}\), then (5.15) and Lemma 2.3(i), applied to \(\omega^* \in \mathcal{R}\), give
\[
\|T_{g - g_r}(f_{a, 2})\|_{A^2_\omega}^2 \lesssim \|T_g(f_{a, 2})\|_{A^2_\omega}^2 + \|T_{g_r}(f_{a, 2})\|_{A^2_\omega}^2 \\
\lesssim \varepsilon + M^2_\omega(r, g') \int_{D} |f_{a, 2}(z)|^2 \omega^*(z) \, dA(z) \\
\lesssim \varepsilon + M^2_\omega \left(2 - \frac{1}{|a|}g'\right)(1 - |a|)^2 \lesssim \varepsilon.
\]

Consequently,
\[
(5.16) \quad \sup_{r \in (0, 1), |a| \geq r_0} \|T_{g - g_r}(f_{a, 2})\|_{A^2_\omega}^2 \lesssim \varepsilon,
\]
and since clearly,
\[
\lim_{r \to 1^-} \sup_{|a| < r_0} \|T_{g - g_r}(f_{a, 2})\|_{A^2_\omega}^2 \lesssim \frac{1}{\omega(S(r_0))} \lim_{r \to 1^-} \|g - g_r\|_{A^2_\omega}^2 = 0,
\]
we obtain (5.13). \(\square\)

A space \(X \subset \mathcal{H}(D)\) equipped with a seminorm \(\rho\) is called \textit{conformally invariant} or \textit{Möbius invariant} if there exists a constant \(C > 0\) such that
\[
\sup_{\varphi} \rho(g \circ \varphi) \leq C \rho(g), \quad g \in X,
\]
where the supremum is taken on all Möbius transformations \(\varphi\) of \(D\) onto itself.

\(\text{BMOA and} \ \mathcal{B}\) are conformally invariant spaces. This is not necessarily true for \(C^1(\omega^*)\) if \(\omega \in \mathcal{I}\) as the following result shows. Typical examples satisfying the hypothesis of Proposition 5.6 are the weights (1.8) and \(v_\alpha\), \(1 < \alpha < \infty\).

**Proposition 5.6.** Let \(\omega \in \mathcal{I}\) such that both \(\omega(r)\) and \(\frac{\psi_{\omega}(r)}{1 - r}\) are essentially increasing on \([0, 1]\), and
\[
(5.17) \quad \int_r^1 \omega(s)s \, ds \lesssim \int_r^{\frac{2}{1 - r}} \omega(s)s \, ds, \quad 0 \leq r < 1.
\]

Then \(C^1(\omega^*)\) is not conformally invariant.

**Proof.** Let \(\omega \in \mathcal{I}\) be as in the assumptions. Recall first that \(g \in C^1(\omega^*)\) if and only if
\[
\sup_{b \in \mathbb{D}} \frac{(1 - |b|)^2}{\omega(S(b))} \int_{D} \frac{|g'(z)|^2}{1 - \overline{b}z} \omega(S(z)) \, dA(z) < \infty
\]
by Lemma 1.6 and Lemma 5.3 with \(s = 1\) and \(\eta = 2\). Let \(g \in C^1(\omega^*) \setminus H^2\) be the function constructed in the proof of Proposition 5.1(E). Then
\[
\sup_{b \in \mathbb{D}} \frac{(1 - |b|)^2}{\omega(S(b))} \int_{D} \frac{|(g \circ \varphi_a)'(z)|^2}{1 - \overline{b}z} \omega(S(z)) \, dA(z) \\
\geq \frac{(1 - |a|)^2}{\omega(S(a))} \int_{D} \frac{|g'(\zeta)|^2}{1 - \overline{a}\zeta} \omega(S(\varphi_a(\zeta))) \, dA(\zeta) \\
\geq \int_{D_{[0,|a|]}} |g'(\zeta)|^2 (1 - |\zeta|) \left(\frac{\omega(S(\varphi_a(\zeta)))}{\omega(S(a))} \frac{|1 - \overline{a}\zeta|^2}{1 - |\zeta|^2}\right) \, dA(\zeta),
\]
where \(\varphi_a(\zeta) = \frac{\zeta - a}{1 - \overline{a}\zeta}\).
where

\[
\frac{\omega(S(\varphi_a(\zeta)))}{\omega(S(a))} \frac{|1 - a\zeta|^2}{1 - |\zeta|} = \frac{(1 - |\varphi_a(\zeta)|)}{(1 - |a|)} \int_{|\varphi_a(\zeta)|}^{1} \omega(s)s \, ds \frac{|1 - a\zeta|^2}{1 - |\zeta|}
\]

\[
\geq \frac{\int_{|a|}^{1 + |a|^2} \omega(s)s \, ds}{\int_{|a|}^{1} \omega(s)s \, ds} \geq 1, \quad |\zeta| \leq |a|,
\]

by Lemma 1.6 and (5.17). Since \( g \notin H^2 \), the assertion follows by letting \( |a| \to 1^- \) in (5.18). \( \square \)
CHAPTER 6

Schatten Classes of the Integral Operator $T_g$ on $A^2_\omega$

Let $H$ be a separable Hilbert space. For any non-negative integer $n$, the $n$:th
singular value of a bounded operator $T : H \to H$ is defined by

$$\lambda_n(T) = \inf \{ \| T - R \| : \text{rank}(R) \leq n \},$$

where $\| \cdot \|$ denotes the operator norm. It is clear that

$$\|T\| = \lambda_0(T) \geq \lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq 0.$$  

For $0 < p < \infty$, the Schatten $p$-class $S_p(H)$ consists of those compact operators
$T : H \to H$ whose sequence of singular values $\{\lambda_n\}_{n=0}^\infty$ belongs to the space $\ell^p$ of $p$-summable sequences. For $1 \leq p < \infty$, the Schatten $p$-class $S_p(H)$ is a Banach space
with respect to the norm $\|T\|_p = \|\{\lambda_n\}_{n=0}^\infty\|_{\ell^p}$. Therefore all finite rank operators belong to every $S_p(H)$, and the membership of an operator in $S_p(H)$ measures in
some sense the size of the operator. We refer to [27] and [84, Chapter 1] for more
information about $S_p(H)$.

The membership of the integral operator $T_g$ in the Schatten $p$-class $S_p(H)$ has
been characterized when $H$ is either the Hardy space $H^2$ [10], the classical weighted
Bergman space $A^2_\alpha$ [11], or the weighted Bergman space $A^2_\omega$, where $\omega$ is a rapidly
decreasing weight [64, 65] or a Bekollé-Bonami weight [24] (the case $p \geq 2$). We
note that if $\omega$ is regular and $p \geq 2$, then [24, Theorem 5.1] yields a characterization
of those $g \in H(\mathbb{D})$ for which $T_g \in S_p(A^2_\omega)$. This because by Lemma 1.4(i), for each
$p_0 = p_0(\omega) > 1$ there exists $\eta = \eta(p_0, \omega) > -1$ such that $\omega(z) \frac{\zeta(z)}{1-|z|^2}$ belongs to $B_{p_0}(\eta)$.

The main purpose of this chapter is to provide a complete description of those
symbols $g \in H(\mathbb{D})$ for which the integral operator $T_g$ belongs to the Schatten $p$-class
$S_p(A^2_\omega)$, where $\omega \in \mathcal{I} \cup \mathcal{R}$. For this aim, we recall that, for $1 < p < \infty$, the Besov
space $B_p$ consists of those $g \in H(\mathbb{D})$ such that

$$\|g\|_{B_p} = \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-2} dA(z) + |g(0)|^p < \infty.$$  

**Theorem 6.1.** Let $\omega \in \mathcal{I} \cup \mathcal{R}$ and $g \in H(\mathbb{D})$. If $p > 1$, then $T_g \in S_p(A^2_\omega)$ if and
only if $g \in B_p$. If $0 < p \leq 1$, then $T_g \in S_p(A^2_\omega)$ if and only if $g$ is constant.

The assertion in Theorem 6.1 is by no means a surprise. This because of the
inclusions $H^2 \subset A^2_\omega \subset A^2_\alpha$, where $\alpha = \alpha(\omega) \in (-1, \infty)$, and the fact that the
membership of $T_g$ in both $S_p(H^2)$ and $S_p(A^2_\omega)$ is characterized by the condition
$g \in B_p$.

This chapter is mainly devoted to proving Theorem 6.1. It appears that the
hardest task here is to make the proof work for the rapidly increasing weights, the
regular weights could be treated in an alternative way. This again illustrates the
phenomenon that in the natural boundary between $H^2$ and each $A^2_\omega$, given by the
spaces $A^2_\omega$ with $\omega \in \mathcal{I}$, things work in a different way than in the classical weighted
Bergman space $A_2^α$. It appears that on the way to the proof of Theorem 6.1 for $ω ∈ I$, we are forced to study the Toeplitz operator $T_μ$, induced by a complex Borel measure $μ$ and a reproducing kernel, in certain Dirichlet type spaces that are defined by using the associated weight $ω^α$. Our principal findings on $T_μ$ are gathered in Theorem 6.11, whose proof occupies an important part of the chapter.

6.1. Preliminary results

We will need several definitions and auxiliary lemmas. We first observe that by Lemma 2.5 each point evaluation $L_a(f) = f(a)$ is a bounded linear functional on $A_2^ω$ for all $0 < p < ∞$ and $ω ∈ I ∪ R$. Therefore there exist reproducing kernels $B_ω^a ∈ A_2^ω$ with $\|L_a\| = \|B_ω^a\|_{A_2^ω}$ such that

$$f(a) = \langle f, B_ω^a \rangle_{A_2^ω} = \int_D f(z) B_ω^a(z) \omega(z) dA(z), \quad f ∈ A_2^ω.$$  \hspace{1cm} (6.1)

We will write $b_ω^a = \frac{B_ω^a}{\|B_ω^a\|_{A_2^ω}}$ for the normalized reproducing kernels. In fact, Lemma 2.5 shows that the norm convergence implies the uniform convergence on compact subsets of $D$. It follows that the space $A_p^ω$ is a Banach space for all $1 ≤ p < ∞$ and $ω ∈ I ∪ R$.

Lemma 6.2. If $ω ∈ I ∪ R$, then the reproducing kernel $B_ω^a ∈ A_2^ω$ satisfies

$$\|B_ω^a\|_{A_2^ω}^2 \asymp \frac{1}{ω(S(a))}$$  \hspace{1cm} (6.2)

for all $a ∈ D$.

Proof. Consider the functions $F_{a,2}$ of Lemma 2.4. The relations (2.5) and (2.6) yield

$$\|B_ω^a\|_{A_2^ω} = \|L_a\| ≥ \frac{|F_{a,2}(a)|}{\|F_{a,2}\|_{A_2^ω}} \asymp \frac{1}{ω(S(a))^{1/2}}$$  \hspace{1cm} (6.3)

for all $a ∈ D$. Moreover, Lemma 2.5 gives

$$|f(a)|^2 \lesssim M_ω(|f|^2)(a) ≤ \left( \sup_{I : a ∈ S(I)} \frac{1}{ω(S(I))} \right) \|f\|_{A_2^ω}^2 = \frac{\|f\|_{A_2^ω}^2}{ω(S(a))},$$

and so

$$\|B_ω^a\|_{A_2^ω} = \|L_a\| \lesssim \frac{1}{ω(S(a))^{1/2}}.$$

This together with (6.3) yields (6.2). \hspace{1cm} \Box

It is known that if $\{e_n\}$ is an orthonormal basis of a Hilbert space $H \subset H(D)$ with reproducing kernel $K_z$, then

$$K_z(ζ) = \sum_n e_n(ζ) \overline{e_n(z)}$$  \hspace{1cm} (6.4)

for all $z, ζ ∈ D$, see e.g. [84, Theorem 4.19]. It follows that

$$\sum_n |e_n(z)|^2 ≤ \|K_z\|_H^2$$  \hspace{1cm} (6.5)

for any orthonormal set $\{e_n\}$ of $H$, and equality in (6.5) holds if $\{e_n\}$ is a basis of $H$. 

For \( n \in \mathbb{N} \cup \{0\} \) and a radial weight \( \omega \), set

\[ \omega_n = \int_0^1 r^{2n+1} \omega(r) \, dr. \]

**Lemma 6.3.** If \( 0 < \alpha < \infty \), \( n \in \mathbb{N} \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), then

\[
\int_0^1 r^n \omega_{\alpha-2}(r) \, dr \asymp \frac{\omega^* \left( 1 - \frac{1}{n+1} \right)}{(n+1)^{\alpha-1}},
\]

and

\[
(n+1)^{2-\alpha} \omega^* \asymp \int_0^1 r^{2n+1} \omega_{\alpha-2}(r) \, dr.
\]

**Proof.** Now \( \omega_{\alpha-2}^* \in \mathcal{R} \) by Lemma 1.7, and so (1.4) yields

\[
\int_{1-\frac{1}{n+1}}^1 r^n (1-r)^{\alpha-2} \omega^*(r) \, dr \lesssim \frac{\omega^* \left( 1 - \frac{1}{n+1} \right)}{(n+1)^{\alpha-1}}.
\]

Moreover, by Lemma 1.6 and the proof of Lemma 1.1, there exists \( \beta = \beta(\omega) > 1 \) such that \((1-r)^{-\beta} \omega^*(r)\) is essentially increasing on \([1/2, 1]\). It follows that

\[
\int_0^1 r^n (1-r)^{\alpha-2} \omega^*(r) \, dr
\]

\[
\lesssim (n+1)^\beta \omega^* \left( 1 - \frac{1}{n+1} \right) \int_0^{1-\frac{1}{n+1}} r^n (1-r)^{\beta+\alpha-2} \, dr
\]

\[
\lesssim (n+1)^\beta \omega^* \left( 1 - \frac{1}{n+1} \right) \int_0^1 r^n (1-r)^{\beta+\alpha-2} \, dr \asymp \frac{\omega^* \left( 1 - \frac{1}{n+1} \right)}{(n+1)^{\alpha-1}}
\]

and

\[
\int_0^1 r^n (1-r)^{\alpha-2} \omega^*(r) \, dr \geq \int_{1-\frac{1}{n+1}}^1 r^n (1-r)^{\alpha-2} \omega^*(r) \, dr
\]

\[
\gtrsim (n+1)^\beta \omega^* \left( 1 - \frac{1}{n+1} \right) \int_{1-\frac{1}{n+1}}^1 (1-r)^{\beta+\alpha-2} \, dr \asymp \frac{\omega^* \left( 1 - \frac{1}{n+1} \right)}{(n+1)^{\alpha-1}}.
\]

Combining (6.8)–(6.10) we obtain (6.6).

The second assertion is a consequence of (6.6). Namely, (6.6) with \( \alpha = 2 \) gives

\[
\omega^* \left( 1 - \frac{1}{n+1} \right) \asymp (n+1) \int_0^1 r^n \omega^*(r) \, dr,
\]

which combined with (6.6) yields

\[
\int_0^1 r^n \omega_{\alpha-2}(r) \, dr \asymp (n+1)^{2-\alpha} \int_0^1 r^n \omega^*(r) \, dr
\]

for any \( n \in \mathbb{N} \). By replacing \( n \) by \( 2n+1 \) we obtain (6.7). \( \square \)

**Lemma 6.4.** For \( a \in \mathbb{D} \), \( -\infty < \alpha < 1 \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), define

\[
\Phi_\alpha^\omega(0) = 0, \quad \Phi_\alpha^\omega(z) = \pi \int_0^z B_{\alpha-\alpha}^\omega(\xi) \, d\xi,
\]

\[
(6.11) \quad \Phi_\alpha^\omega(0) = 0, \quad \Phi_\alpha^\omega(z) = \pi \int_0^z B_{\alpha-\alpha}^\omega(\xi) \, d\xi.
\]

\[
(6.11) \quad \Phi_\alpha^\omega(z) = \pi \int_0^z B_{\alpha-\alpha}^\omega(\xi) \, d\xi,
\]

\[
(6.11) \quad \Phi_\alpha^\omega(0) = 0, \quad \Phi_\alpha^\omega(z) = \pi \int_0^z B_{\alpha-\alpha}^\omega(\xi) \, d\xi.
\]
Then there exists $\delta = \delta(\alpha, \omega) > 0$ such that

$$|\Phi_a^{\omega, \alpha}(z)| \geq \frac{1}{(1 - |a|)\omega_{\alpha}(a)}, \quad z \in D(a, \delta(1 - |a|)),$$

for all $|a| \geq \frac{1}{2}$.

**Proof.** Taking the orthonormal basis $\{2(\omega_{\alpha}^*)^{-1/2}z^n\}_{n=0}^{\infty}$ in $A^2_{\omega_{\alpha}^*}$ and using (6.4) we deduce

$$\Phi_a^{\omega}(z) = \sum_{n=0}^{\infty} \frac{(\overline{a}z)^{n+1}}{2(n+1)(\omega_{\alpha}^*)_n}, \quad z \in \mathbb{D}.$$ Let $|a| \geq \frac{1}{2}$ be given and fix the integer $N \geq 2$ such that $1 - \frac{1}{N} < |a| \leq 1 - \frac{1}{N+1}$. Now $\omega_{\alpha}^* \in \mathcal{R}$ by Lemma 1.7, and hence Lemma 6.3 and (1.2) yield $(n+1)(\omega_{\alpha}^*)_n \approx (\omega_{\alpha}^*_{-1})_n \approx \omega_{\alpha}^*(1 - \frac{1}{N+1})$. Since $\omega^*$ is decreasing, we deduce

$$\Phi_a^{\omega, \alpha}(z) \gtrsim \sum_{n=N}^{\infty} \frac{|a|^{2n+2}}{(\omega_{\alpha}^*_{-1})_n} \geq \frac{1}{(\omega_{\alpha}^*_{-1})_N} \sum_{n=N}^{\infty} |a|^{2n+2} \approx \frac{1}{(1 - |a|)\omega_{\alpha}(a)}$$

for all $|a| \geq \frac{1}{2}$.

Next, bearing in mind (6.4), Lemma 6.2, (1.29) and Lemma 1.7, we deduce

$$B_{\omega, \alpha}(\zeta) = \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{2(\omega_{\alpha}^*)_n} = \|B_{\omega, \alpha} \|_{A^2_{\omega_{\alpha}^*}} \approx \frac{1}{(1 - |\zeta|)2\omega_{\alpha}(\zeta)}$$

for each $\zeta \in \mathbb{D}$. Let now $z \in D(a, \delta(1 - |a|))$, where $|a| \geq \frac{1}{2}$ and $0 < \delta < 1$. Then the Cauchy-Schwarz inequality, (6.12), the relation $1 - |a| \approx 1 - |\zeta|$ for all $\zeta \in [a, z]$, and the observation (i) to Lemma 1.1 yield

$$|B_{\omega, \alpha}^{\omega, \alpha}(\zeta)| \leq \left( \sum_{n=0}^{\infty} \frac{|a|^{2n}}{2(\omega_{\alpha}^*)_n} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{2(\omega_{\alpha}^*)_n} \right)^{\frac{1}{2}} \approx \frac{1}{(1 - |\zeta|)2\omega_{\alpha}(\zeta)} \approx |B_{\omega, \alpha}^{\omega, \alpha}(a)|.$$ Consequently,

$$|\Phi_a^{\omega, \alpha}(z) - \Phi_a^{\omega, \alpha}(a)| \leq \max_{\zeta \in [a, z]} |(\Phi_a^{\omega, \alpha})'(\zeta)| |z - a| \leq \max_{\zeta \in [a, z]} |B_{\omega, \alpha}^{\omega, \alpha}(\zeta)| \delta(1 - |a|) \approx \frac{\delta}{(1 - |a|)\omega_{\alpha}(a)}.$$ By choosing $\delta > 0$ sufficiently small, we deduce

$$|\Phi_a^{\omega, \alpha}(z)| \geq \Phi_a^{\omega, \alpha}(a) - |\Phi_a^{\omega, \alpha}(z) - \Phi_a^{\omega, \alpha}(a)| \gtrsim \frac{1}{(1 - |a|)\omega_{\alpha}(a)}$$

for all $z \in D(a, \delta(1 - |a|))$. \qed
6.1. PRELIMINARY RESULTS

A sequence \( \{a_k\}_{k=0}^{\infty} \) of points in \( \mathbb{D} \) is called **uniformly discrete** if it is separated in the pseudohyperbolic metric, that is, if there exists a constant \( \gamma > 0 \) such that
\[
\varrho(a_j, a_k) = \frac{|a_j - a_k|}{1 - |a_j a_k|} \geq \gamma \quad \text{for all } k.
\]
For \( 0 < \varepsilon < 1 \), a sequence \( \{a_k\}_{k=0}^{\infty} \) is called an \( \varepsilon \)-net if \( \mathbb{D} = \bigcup_{k=0}^{\infty} \Delta(a_k, \varepsilon) \). A sequence \( \{a_k\}_{k=0}^{\infty} \subset \mathbb{D} \) is a \( \delta \)-lattice if it is uniformly discrete with constant \( \gamma = \delta/5 \) and if it is a \( 5\delta \)-net. With these preparations we are ready for the next lemma.

**Lemma 6.5.** For \( a \in \mathbb{D} \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), define

\[
\phi^\omega_a(z) = \frac{\Phi^\omega_a(z)}{\|B^\omega_a\|_{A^2_\omega}}, \quad z \in \mathbb{D}.
\]

Let \( \{a_j\} \) be a uniformly discrete sequence and \( \{e_j\} \) be an orthonormal set in \( A^2_\omega \). Let \( E = \{e_j\} \) denote the subspace generated by \( \{e_j\} \) and equipped with the norm of \( A^2_\omega \), and consider the linear operator \( J : E \to A^2_\omega \), defined by \( J(e_j) = \phi^\omega_{a_j} \). If \( P \) is the orthogonal projection from \( A^2_\omega \) to \( E \), then \( J = J \circ P \) is bounded on \( A^2_\omega \).

**Proof.** Let \( \omega \in \mathcal{I} \cup \mathcal{R} \). Since \( P : A^2_\omega \to E \) is bounded, it suffices to show that

\[
\|J \left( \sum_j c_j e_j \right) \|_{A^2_\omega} \lesssim \left( \sum_j |c_j|^2 \right)^{1/2}
\]

for all sequences \( \{c_j\} \in l^2 \). To prove (6.13), note first that \( \omega^* \in \mathcal{R} \) by Lemma 1.7, and hence (6.1) can be applied for \( f' \in A^2_\omega^* \). Using this, the definitions of \( J \) and \( \phi^\omega_{a_j} \), the polarization of the identity (4.5) and the Cauchy-Schwarz inequality, we obtain

\[
\left| \left< J \left( \sum_j c_j e_j \right), f \right>_{A^2_\omega} \right| = \left| \left< \sum_j c_j \phi^\omega_{a_j}, f \right>_{A^2_\omega} \right| = 4 \left\| \sum_j \omega_j c_j (b^\omega_{a_j}, f')_{A^2_\omega} \right\| \leq 4 \sum_j |c_j| \frac{|f'(a_j)|}{\|B^\omega_{a_j}\|_{A^2_\omega^*}} \left( \sum_j |c_j|^2 \right)^{1/2} \left( \sum_j |f'(a_j)|^2 \|B^\omega_{a_j}\|_{A^2_\omega^*}^{-2} \right)^{1/2}
\]

(6.14)

for all \( f \in A^2_\omega \). Applying now Lemma 6.2 and Lemma 1.6 to \( \omega^* \in \mathcal{R} \), and Lemma 1.7, with \( \alpha = 2 \), we deduce

\[
\|B^\omega_{a_j}\|_{A^2_\omega^*}^{-2} \approx \omega^*(S(a_j)) \approx \omega^{**}(a_j) \times (1 - |a_j|^2) \omega^*(a_j), \quad |a_j| \geq \frac{1}{2}.
\]

(6.15)
Let \( \gamma > 0 \) be the pseudohyperbolic separation constant of \( \{a_j\} \). Then (6.15) together with the subharmonicity of \( |f'|^2 \), (1.2) for \( \omega^* \) and Theorem 4.2 yield

\[
\sum_{|a_j| \geq \frac{1}{2}} |f'(a_j)|^2 \|B_{a_j}^{-\star}\|_{A^2_{\omega^*}}^{-2} \geq \sum_{|a_j| \geq \frac{1}{2}} |f'(a_j)|^2 (1 - |a_j|)2 \omega^*(a_j) \\
\lesssim \sum_{|a_j| \geq \frac{1}{2}} \omega^*(a_j) \int_{\Delta(a_j, \frac{1}{2})} |f'(z)|^2 dA(z) \\
(6.16) \\
\lesssim \sum_{|a_j| \geq \frac{1}{2}} \int_{\Delta(a_j, \frac{1}{2})} |f'(z)|^2 \omega^*(z) dA(z) \\
\lesssim \|f'\|^2_{A^2_{\omega^*}} \lesssim \|f\|^2_{A^2_{\omega}}.
\]

By combining (6.14) and (6.16), we finally obtain

\[
\left\langle \overline{J} \left( \sum_j c_j e_j \right), f \right\rangle_{A^2_{\omega}} \lesssim \left( \sum_j |c_j|^2 \right)^{1/2} \|f\|_{A^2_{\omega}}, \quad f \in A^2_{\omega},
\]

which in turn implies (6.13).

\[ \Box \]

### 6.2. Proofs of the main results

With the lemmas of the previous section in hand we are ready to start proving Theorem 6.1. We first deal with the case \( 0 < p \leq 1 \), and show that \( g \in B_p \) is a necessary condition for \( T_g \) to belong to \( S_p(A^2_{\omega}) \) for all \( 1 < p < \infty \).

**Proposition 6.6.** Let \( \omega \in \mathcal{I} \cup \mathcal{R} \) and \( g \in \mathcal{H}(\mathbb{D}) \). If \( p > 1 \) and \( T_g \in S_p(A^2_{\omega}) \), then \( g \in B_p \). If \( 0 < p \leq 1 \), then \( T_g \in S_p(A^2_{\omega}) \) if and only if \( g \) is constant.

**Proof.** Let first \( p \geq 1 \) and \( T_g \in S_p(A^2_{\omega}) \). Let \( \{e_j\} \) be an orthonormal set in \( A^2_{\omega} \). Let \( \{a_j\} \) be a uniformly discrete sequence and consider the linear operator \( J = \overline{J} \circ P \), where \( P \) is the orthogonal projection from \( A^2_{\omega} \) to \( \{\{e_j\}\} \) and the linear operator \( \overline{J} : \{\{e_j\}\} \to A^2_{\omega} \) is defined by \( \overline{J}(e_j) = \phi_{a_j}^\omega \). The operator \( J \) is bounded on \( A^2_{\omega} \) by Lemma 6.5. Since \( S_p(A^2_{\omega}) \) is a two-sided ideal in the space of bounded linear operators on \( A^2_{\omega} \), we have \( J^* T_g J \in S_p(A^2_{\omega}) \) by [84, p. 27]. Hence [84, Theorem 1.27] yields

\[
\sum_j \left| \left\langle T_g(\phi_{a_j}^\omega), \phi_{a_j}^\omega \right\rangle_{A^2_{\omega}} \right|^p = \sum_j \left| \left\langle (J^* T_g J)(e_j), e_j \right\rangle_{A^2_{\omega}} \right|^p < \infty.
\]

Then, by the polarization of the identity (4.5), (6.1) for \( \omega^* \), Lemma 6.4 and (6.15), we obtain

\[
\infty > \sum_j \left| \left\langle T_g(\phi_{a_j}^\omega), \phi_{a_j}^\omega \right\rangle_{A^2_{\omega}} \right|^p = 4 \sum_j |a_j|^p \left| \left\langle g' \phi_{a_j}^\omega, b_{a_j}^\omega \right\rangle_{A^2_{\omega}} \right|^p \\
(6.17) \\
\lesssim \sum_{|a_j| \geq \frac{1}{k}} \left| \left| B_{a_j}^{-\star} \right|_{A^2_{\omega^*}} \right|^p \lesssim \sum_{|a_j| \geq \frac{1}{k}} |g'(a_j)|^p (1 - |a_j|)^p.
\]

Therefore for any uniformly discrete sequence \( \{a_j\} \), and hence in particular for any \( \delta \)-lattice, we have

\[
\sum_{|a_j| \geq \frac{1}{k}} |g'(a_j)|^p (1 - |a_j|)^p < \infty.
\]
Arguing as in [72, p. 917], this in turn implies

\[(6.18) \quad \int_{\mathbb{D}} |g'(z)|^p(1 - |z|^2)^{p-2} dA(z) < \infty, \]

which is the assertion for \( p > 1 \).

If \( 0 < p \leq 1 \) and \( T_g \in \mathcal{S}_p(\mathbb{A}_2^2) \), then \( T_g \in \mathcal{S}_1(\mathbb{A}_2^2) \), and hence the first part of the proof gives (6.18) with \( p = 1 \), which implies that \( g \) is constant. Since \( T_g = 0 \) if \( g \) is constant, we deduce the assertion also for \( 0 < p \leq 1 \). \( \square \)

We note that if \( \omega \) is regular and \( 1 < p < 2 \), then the assertion in Proposition 6.6 can be proved in an alternative way by following the argument in [84, Proposition 7.15] and using [84, Theorem 1.26], Lemmas 1.6 and 6.2, (1.29) and Theorem 4.2.

The proof of the fact that \( g \in B_p \) is a sufficient condition for \( T_g \) to belong to \( \mathcal{S}_p(\mathbb{A}_2^2) \) is more involved, in particular when \( p > 2 \). We begin with the case \( 1 < p < 2 \) which will be proved by using ideas from [64]. We need an auxiliary result.

**Lemma 6.7.** If \( \omega \in \mathcal{I} \cup \mathcal{R} \), then

\[
\left\| \frac{\partial}{\partial \bar{z}} B^\omega \right\|_{\mathbb{A}_2^2} \lesssim \frac{\| B^\omega \|_{\mathbb{A}_2^2}}{1 - |z|^2}, \quad z \in \mathbb{D}.
\]

**Proof.** Let \( \{ e_n \}_{n=0}^{\infty} \) be the orthonormal basis of \( \mathbb{A}_2^2 \) given by \( e_n(z) = (2 \omega_n)^{-1/2} z^n \), where \( \omega_n = \int_0^1 r^{2n+1} \omega(r) \, dr \). If \( f(z) = \sum_{n=0}^{\infty} e_n(z) \), then Parseval’s identity yields

\[
M_2^2(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} \frac{r^{2n}}{2\omega_n} = \| B^\omega \|_{\mathbb{A}_2^2}^2, \quad |z| = r.
\]

But now

\[
\left\| \frac{\partial}{\partial \bar{z}} B^\omega \right\|_{\mathbb{A}_2^2}^2 = \sum_{n=1}^{\infty} |e'_n(z)|^2 = \sum_{n=1}^{\infty} \frac{n^2 r^{2n-2}}{2 \omega_n} = M_2^2(r, f'), \quad |z| = r,
\]

and hence an application of the Cauchy integral formula, Lemma 6.2 and Lemma 1.1, with \( t = \frac{1+r}{2} \), give

\[
\left\| \frac{\partial}{\partial \bar{z}} B^\omega \right\|_{\mathbb{A}_2^2} = M_2(r, f') \lesssim \frac{M_2 \left( \frac{1+r}{2} \right)}{1 - r} \approx \frac{\| B^\omega \|_{\mathbb{A}_2^2}}{1 - |z|^2}, \quad |z| = r,
\]

which is the desired estimate. \( \square \)

**Proposition 6.8.** Let \( 1 < p < 2 \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \). If \( g \in B_p \), then \( T_g \in \mathcal{S}_p(\mathbb{A}_2^2) \).

**Proof.** If \( 1 < p < \infty \), then \( T_g \in \mathcal{S}_p(\mathbb{A}_2^2) \) if and only if

\[
\sum_n |\langle T_g(e_n), e_n \rangle_{\mathbb{A}_2^2} |^p < \infty
\]

for any orthonormal set \( \{ e_n \} \), see [84, Theorem 1.27]. Let \( 1 < p < 2 \) and let \( \{ e_n \} \) be an orthonormal set in \( \mathbb{A}_2^2 \). Then the polarization of the identity (4.5), two
applications of Hölder’s inequality, Lemma 6.7, Lemma 6.2 and Lemma 1.6 yield

\[ \sum_n \left| (T_g(e_n), e_n)_{A^2_\omega} \right|^p \]

\[ \lesssim \sum_n \left( \int_D |g'(z)||e_n(z)||e_n'(z)|\omega^*(z)\,dA(z) \right)^p \]

\[ \leq \sum_n \left( \int_D |g'(z)|^p|e_n(z)|^p|e_n'(z)|^{2-p}\omega^*(z)\,dA(z) \right) \]

\[ \cdot \left( \int_D |e_n'(z)|^2\omega^*(z)\,dA(z) \right)^{p-1} \]

(6.19)

\[ \lesssim \int_D |g'(z)|^p \left( \sum_n |e_n(z)|^p|e_n'(z)|^{2-p} \right) \omega^*(z)\,dA(z) \]

\[ \leq \int_D |g'(z)|^p \left( \sum_n |e_n(z)|^2 \right)^{\frac{p}{2}} \left( \sum_n |e_n'(z)|^2 \right)^{1-\frac{p}{2}} \omega^*(z)\,dA(z) \]

\[ \lesssim \int_D |g'(z)|^p \frac{\|B^*\|_{A^2}^2}{(1-|z|)^{\frac{p}{2}-2}} \omega^*(z)\,dA(z) \approx \|g\|^p_{B^p}. \]

Thus, \( T_g \in \mathcal{S}_p(A^2_\omega) \) if \( g \in B_p \). \qed

We have now proved Theorem 6.1 when \( \omega \in \mathcal{I} \cup \mathcal{R} \) and \( 0 < p < 2 \). As mentioned earlier, the case \( \omega \in \mathcal{R} \) and \( 2 \leq p < \infty \) follows by [24, Theorem 5.1] and Lemma 1.4(i). We now proceed to the remaining case \( \omega \in \mathcal{I} \) and \( 2 \leq p < \infty \). The proof we are going to present works also for \( \omega \in \mathcal{R} \).

6.2.1. Dirichlet type spaces induced by \( \omega^* \). For \( \alpha \in \mathbb{R} \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), let us consider the Hilbert space \( H_\alpha(\omega^*) \), which consists of those \( f \in \mathcal{H}(\mathbb{D}) \) whose Maclaurin series \( \sum_{n=0}^{\infty} a_n z^n \) satisfies

\[ \sum_{n=0}^{\infty} (n+1)^{\alpha+2}\omega_n^*|a_{n+1}|^2 < \infty. \]

Bearing in mind Theorem 4.2, we deduce the identity \( H_0(\omega^*) = A^2_\omega \), which is a special case of the following result. Recall that \( \omega_\alpha(z) = (1-|z|)^{\alpha}\omega(z) \) for all \( \alpha \in \mathbb{R} \).

Lemma 6.9. If \( -\infty < \alpha < 2 \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), then

(6.20) \[ H_\alpha(\omega^*) = \left\{ f \in \mathcal{H}(\mathbb{D}) : \int_D |f'(z)|^2 \omega_\alpha^*(z)\,dA(z) < \infty \right\}. \]

In particular, if \( \alpha < 0 \), then \( H_\alpha(\omega^*) = A^2_{\omega^*_{-\alpha-2}} \).
6.2. PROOFS OF THE MAIN RESULTS

PROOF. If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) \), then Lemma 6.3 and Parseval’s identity give
\[
\sum_{n=0}^{\infty} (n+1)^{\alpha+2} \omega^a_n |a_{n+1}|^2 = \sum_{n=0}^{\infty} (n+1)^2 (n+1)^{2-\alpha} \omega^a_n |a_{n+1}|^2 \\
\approx 2 \sum_{n=0}^{\infty} (n+1)^2 |a_{n+1}|^2 \int_0^1 r^{2(n+1)} \omega^a_\alpha(r) \ dr \\
= \int_\mathbb{D} |f'(z)|^2 \omega^a_\alpha(z) \ dA(z),
\]
which proves (6.20).

The identity \( H_\alpha(\omega^*) = A^2_\omega, -\infty < \alpha < 0 \), is an immediate consequence of (6.20), Theorem 4.2 and Lemma 1.7.

Lemma 6.9 allows us to define an inner product on \( H_\alpha(\omega^*) \), \(-\infty < \alpha < 2\), by
\[
\langle f, g \rangle_{H_\alpha(\omega^*)} = f(0)g(0)\omega^a_\alpha(\mathbb{D}) + 2 \sum_{n=0}^{\infty} (n+1)^2 a_{n+1}b_{n+1} \int_0^1 r^{2(n+1)} \omega^a_\alpha(r) \ dr
\]
\[
= f(0)g(0)\omega^a_\alpha(\mathbb{D}) + \int_\mathbb{D} f'(z)g'(z) \omega^a_\alpha(z) \ dA(z),
\]
where \( \sum_{n=0}^{\infty} a_n z^n \) and \( \sum_{n=0}^{\infty} b_n z^n \) are the Maclaurin series of \( f \) and \( g \) in \( \mathbb{D} \), respectively. It also follows from Lemma 6.9 that each point evaluation \( L_\alpha(f) = f(a) \) is a bounded linear functional on \( H_\alpha(\omega^*) \) for all \(-\infty < \alpha < 2\). Therefore there exist reproducing kernels \( K_\alpha^a \in H_\alpha(\omega^*) \) with \( \|L_\alpha\| = \|K_\alpha^a\|_{H_\alpha(\omega^*)} \) such that \( L_\alpha(f) = \langle f, K_\alpha^a \rangle_{H_\alpha(\omega^*)} \), and thus
\[
(f(a) = f(0)K_\alpha^a(0)\omega^a_\alpha(\mathbb{D}) + \int_\mathbb{D} f'(z)\frac{\partial K_\alpha^a(z)}{\partial z} \omega^a_\alpha(z) \ dA(z)
\]
for all \( f \in H_\alpha(\omega^*) \).

Lemma 6.10. If \(-\infty < \alpha < 1 \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), then
\[
\|K_\alpha^a\|^2_{H_\alpha(\omega^*)} \approx \frac{1}{\omega^a_\alpha(a)}, \quad |a| \geq \frac{1}{2}
\]

PROOF. If \(-\infty < \alpha < 0 \), then Lemma 6.9, Lemma 6.2, Lemma 1.6, and Lemma 1.7 yield
\[
\|K_\alpha^a\|^2_{H_\alpha(\omega^*)} \approx \|B_\alpha^a\|^2_{A^2_\omega} \|A^2_\omega\|_{L^1(\mathbb{D})} \approx \frac{1}{\omega^a_\alpha(S(a))}
\]
\[
\approx \frac{1}{(\omega^a_\alpha)^*(a)}, \quad |a| \geq \frac{1}{2}
\]
A similar reasoning involving Theorem 4.2 and Lemma 1.6 gives the assertion for \( \alpha = 0 \).

To prove the case \( 0 < \alpha < 1 \), consider the orthonormal basis \( \{e_n\}_{n=0}^{\infty} \) of \( H_\alpha(\omega^*) \) given by
\[
e_0(z) = \left( \frac{1}{\omega^a_\alpha(\mathbb{D})} \right)^{1/2}, \quad e_n(z) = \frac{z^n}{n \left( \int_0^1 \int_0^1 r^{2n-1} \omega^a_\alpha(r) \ dr \right)^{1/2}}, \quad n \in \mathbb{N}.
\]
Lemma 6.3 yields

\[ \| K_a^{\alpha} \|_{H_{n}(\omega^{*})}^2 = K_a^{\alpha}(a) = \frac{1}{\omega_{-\alpha}(D)} + \sum_{n=0}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{2+\alpha}\omega_n^*} \]

(6.24)

\[ \asymp 1 + \sum_{n=0}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{2+\alpha}\omega_n^*}, \quad a \in \mathbb{D}. \]

Take \( N \in \mathbb{N} \) such that \( 1 - \frac{1}{N} \leq |a| < 1 - \frac{1}{N+1} \). Using (4.5), with \( f(z) = z^{n+1} \), we deduce

\[ 4(n+1)^{2}\omega_n^* = \int_0^1 r^{2n+3}\omega(r) \, dr, \quad n \in \mathbb{N} \cup \{0\}, \]

and hence \( \{\omega_n^*(n+1)^2\}_{n=0}^{\infty} \) is a decreasing sequence. This together with the fact \( \omega^* \in \mathcal{R} \), which follows by Lemma 1.7, and Lemma 6.3 gives

\[ \sum_{n=0}^{N} \frac{|a|^{2n+2}}{(n+1)^{2+\alpha}\omega_n^*} \leq \frac{1}{(N+1)^{2}\omega_n^*} \sum_{n=0}^{N} \frac{|a|^{2n+2}}{(n+1)^{\alpha}} \]

(6.25)

\[ \leq \frac{1}{(N+1)^{2}\omega_n^*} \left( 1 - \frac{1}{N+1} \right) \sum_{n=0}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{\alpha}} \]

\[ \leq \frac{1 - |a|}{\omega^*(a)} (1 - |a|)^{\alpha-1} = \frac{1}{\omega_{-\alpha}^*(a)}, \quad a \in \mathbb{D}, \]

and

\[ \sum_{n=N}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{2+\alpha}\omega_n^*} \geq \frac{1}{(N+1)^{2}\omega_n^*} \sum_{n=N}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{\alpha}} \]

(6.26)

\[ \asymp \frac{1}{(N+1)^{2}\omega_n^*} \left( 1 - \frac{1}{N+1} \right) \sum_{n=N}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{\alpha}} \]

\[ \geq \frac{1 - |a|}{\omega^*(a)} (1 - |a|)^{\alpha-1} = \frac{1}{\omega_{-\alpha}^*(a)}, \quad |a| \geq \frac{1}{2}. \]

Moreover, (6.23) with \( \alpha = 0 \) yields

\[ \sum_{n=N+1}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{2+\alpha}\omega_n^*} \lesssim \frac{1}{(N+1)^{2}\omega_n^*} \sum_{n=0}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{2}\omega_n^*} \]

(6.27)

\[ \lesssim \frac{1}{(N+1)^{2}\omega_n^*} \sum_{n=0}^{\infty} \frac{|a|^{2n+2}}{(n+1)^{2}\omega_n^*} \]

\[ \lesssim (1 - |a|)^{\alpha} \|K_a^{\alpha}\|_{H_{o}(\omega^{*})}^2 \asymp \frac{1}{\omega_{-\alpha}^*(a)}. \]

By combining (6.24)–(6.27) we obtain the assertion for \( 0 < \alpha < 1 \). \( \square \)

### 6.2.2. Toeplitz operator \( T_{\mu} \) and complex interpolation technique

We next consider a decomposition of \( \mathbb{D} \) into disjoint sets of roughly equal size in the hyperbolic sense. Let \( \Upsilon \) denote the family of all dyadic arcs of \( \mathbb{T} \). Every dyadic arc \( I \subset \mathbb{T} \) is of the form

\[ I_{n,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2n} \leq \theta < \frac{2\pi(k+1)}{2n} \right\}, \]
where \( k = 0, 1, 2, \ldots, 2^n - 1 \) and \( n \in \mathbb{N} \cup \{0\} \). For each \( I \subset \mathbb{T} \), set

\[
R(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I, \ 1 - \frac{|I|}{2\pi} \leq r < 1 - \frac{|I|}{4\pi} \right\}.
\]

Then the family \( \{R(I) : I \in \mathbb{Y}\} \) consists of pairwise disjoint sets whose union covers \( \mathbb{D} \). For \( I_j \in \mathbb{Y} \setminus \{I_{0,0}\} \), we will write \( z_j \) for the unique point in \( \mathbb{D} \) such that \( z_j = (1 - |I_j|/2\pi)a_j \), where \( a_j \in \mathbb{T} \) is the midpoint of \( I_j \). For convenience, we associate the arc \( I_{0,0} \) with the point \( 1/2 \).

Now, if \( \mu \) is a complex Borel measure on \( \mathbb{D} \), let us consider the operator

\[
T_\mu(f)(w) = \int_\mathbb{D} f(z)K^\alpha(w, z) \, d\mu(z), \quad f \in H_\alpha(\omega^*),
\]

where \( K^\alpha(w, z) = K^\alpha_w(z) \). This operator has been studied, for instance, in [57].

**Theorem 6.11.** Let \( 1 \leq p < \infty \) and \( -\infty < \alpha < 1 \) such that \( p\alpha < 1 \). Let \( \omega \in \mathcal{I} \cup \mathcal{R} \), and let \( \mu \) be a complex Borel measure on \( \mathbb{D} \). If

\[
\sum_{R_j \in \mathbb{Y}} \left( \frac{\mu(|R_j|)}{\omega^* \alpha(z_j)} \right)^p < \infty,
\]

then \( T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*)) \), and there exists a constant \( C > 0 \) such that

\[
|T_\mu|^p \leq C \sum_{R_j \in \mathbb{Y}} \left( \frac{\mu(|R_j|)}{\omega^* \alpha(z_j)} \right)^p.
\]

Conversely, if \( \mu \) is a positive Borel measure on \( \mathbb{D} \) and \( T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*)) \), then (6.28) is satisfied.

We consider this result of its own interest because it allows us to complete the proof of Theorem 6.1 for rapidly increasing weights.

**Corollary 6.12.** Let \( 2 \leq p < \infty \), \( \omega \in \mathcal{I} \cup \mathcal{R} \) and \( g \in \mathcal{H}(\mathbb{D}) \). Then, \( T_g \in \mathcal{S}_p(A^\alpha_\omega) \) if and only if \( g \in B_p \).

**Proof.** By the proof of [10, Theorem 2], \( g \in B_p \) if and only if

\[
\sum_{R_j \in \mathbb{Y}} \left( \int_{R_j} |g'(z)|^2 \, dA(z) \right)^{p/2} < \infty,
\]

which is equivalent to

\[
\sum_{R_j \in \mathbb{Y}} \left( \int_{R_j} \frac{|g'(z)|^2 \omega^*(z) \, dA(z)}{\omega^*(z_j)} \right)^{p/2} < \infty
\]

since \( \omega^* \in \mathcal{R} \) by Lemma 1.7. Therefore we may apply Theorem 6.11 with \( \alpha = 0 \) to the measure \( d\mu_g(z) = |g'(z)|^2 \omega^*(z) \, dA(z) \) to deduce \( T_{\mu_g} \in \mathcal{S}_{p/2}(H_0(\omega^*)) \) if and only if \( g \in B_p \). But \( H_0(\omega^*) = A^\alpha_\omega \) by Theorem 4.2, and hence \( T_{\mu_g} \in \mathcal{S}_{p/2}(A^\alpha_\omega) \). On the other hand, (6.22) gives

\[
\langle (T_\mu^* T_\mu)(f), h \rangle_{H_0(\omega^*)} = \langle T_g(f), T_g(h) \rangle_{H_0(\omega^*)} = \int_{\mathbb{D}} f(z) \overline{h(z)} |g'(z)|^2 \omega^*(z) \, dA(z),
\]
so, by taking \( h(z) = K_0^0(z) \), we deduce

\[
(T_g^* T_g)(f)(w) = \int_{\mathbb{D}} f(z) K_0^0(w, z) |g'(z)|^2 \omega^*(z) \, dA(z) = T_{\mu_g}(f)(w).
\]

Therefore \( g \in B_p \iff T_g^* T_g \in S_{p/2}(A_2^2) \iff T_g \in S_p(A_2^2) \), where in the last equivalence we use [84, Theorem 1.26].

Propositions 6.6 and 6.8, and Corollary 6.12 yield Theorem 6.1. Finally, we prove Theorem 6.11.

**Proof of Theorem 6.11.** We borrow the argument from the proof of the main theorem in [57, p. 352–355]. First, we will show that (6.28) implies \( T_{\mu} \in S_p(H_{\alpha}(\omega^*)) \) and (6.29). This part of the proof will be divided into two steps.

**First Step.** We begin with showing by a limiting argument that it is enough to prove the assertion for measures with compact support. For simplicity we only consider positive Borel measures, a similar reasoning works also for complex Borel measures.

Assume that there exists a constant \( C > 0 \) such that (6.29) is satisfied for all positive compactly supported Borel measures on \( \mathbb{D} \) that fulfill (6.28). Let \( \mu \) be a positive Borel measure on \( \mathbb{D} \) such that

\[
M_p = \sum_{R_j \in \mathcal{Y}} \left( \frac{\mu(R_j)}{\omega^*_{-\alpha}(z_j)} \right)^p < \infty.
\]

For each \( k \in \mathbb{N} \), let \( \mu_k \) denote the measure defined by

\[
\mu_k(E) = \mu \left( E \cap D \left( 0, 1 - \frac{1}{k} \right) \right)
\]

for any Borel set \( E \subset \mathbb{D} \), and set

\[
M_p^k = \sum_{R_j \in \mathcal{Y}} \left( \frac{\mu_k(R_j)}{\omega^*_{-\alpha}(z_j)} \right)^p.
\]

Since \( \mu_k \) has compact support and \( M_p^k \leq M_p < \infty \) for all \( k \in \mathbb{N} \), the assumption yields \( T_{\mu_k} \in S_p(H_{\alpha}(\omega^*)) \) for all \( k \in \mathbb{N} \), and

\[
|T_{\mu_k}|_p^p \leq C M_p^k \leq C M_p < \infty.
\]

In particular, \( T_{\mu_k} \) is a compact operator on \( H_{\alpha}(\omega^*) \) with \( \|T_{\mu_k}\|_p^p \leq C M_p \) for all \( k \in \mathbb{N} \).
Consider the identity operator $I_d : H_\alpha(\omega^*) \to L^2(\mu)$. The definition (6.21), Fubini’s theorem and (6.22) yield
\[
(T_\mu(g), f)_{H_\alpha(\omega^*)} = T_\mu(g)(0)\overline{f(0)} \omega^*_{\alpha}(D)
\]
\[
+ \int_D \left( \int_D g(z) \frac{\partial K^\alpha(\zeta, z)}{\partial \zeta} d\mu(z) \right) \frac{f(\zeta)}{\overline{\omega^*_{\alpha}(\zeta)}} dA(\zeta)
\]
\[
= \int_D g(z) \left( \overline{f(0)K^\alpha(0, z)} \omega^*_{\alpha}(D) \right) d\mu(z)
\]
\[
+ \int_D g(z) \left( \int_D \frac{\partial K^\alpha(\zeta, z)}{\partial \zeta} \omega^*_{\alpha}(\zeta) dA(\zeta) \right) d\mu(z)
\]
\[
= \int_D g(z)\overline{f(z)} d\mu(z) = \langle I_d(g), I_d(f) \rangle_{L^2(\mu)}
\]
(6.32)
for all $g$ and $f$ in $H_\alpha(\omega^*)$, and thus $T_\mu = I_d^* I_d$. Therefore $T_\mu$ is bounded (resp. compact) on $H_\alpha(\omega^*)$ if and only if $I_d : H_\alpha(\omega^*) \to L^2(\mu)$ is bounded (resp. compact). Since the same is true if $\mu$ is replaced by $\mu_k$, we deduce that $I_d : H_\alpha(\omega^*) \to L^2(\mu_k)$ is compact with $\|I_d\|^2 \leq \|T_{\mu_k}\| \leq CM_p$ for all $k \in \mathbb{N}$ by the previous paragraph. Now, by using the monotone convergence theorem, we obtain
\[
\int_D |f(z)|^2 d\mu(z) = \int_D \left( \lim_{k \to \infty} |f(z)|^2 \chi_{D(0,1-\frac{1}{k})}(z) \right) d\mu(z)
\]
\[
\leq \left( \lim_{k \to \infty} \|T_{\mu_k}\| \right) \|f\|_{H_\alpha(\omega^*)}^2 \leq CM_p \|f\|_{H_\alpha(\omega^*)}^2
\]
for all $f \in H_\alpha(\omega^*)$. Therefore $I_d : H_\alpha(\omega^*) \to L^2(\mu)$ is bounded and so is $T_\mu$ on $H_\alpha(\omega^*)$. Furthermore, by using standard tools and (6.32), we deduce
\[
\|T_\mu - R\|
\]
\[
= \sup_{\|f\|_{H_\alpha(\omega^*)} \leq 1, \|g\|_{H_\alpha(\omega^*)} \leq 1} |\langle (T_\mu - R)(f), g \rangle_{H_\alpha(\omega^*)}|
\]
\[
= \sup_{\|f\|_{H_\alpha(\omega^*)} \leq 1, \|g\|_{H_\alpha(\omega^*)} \leq 1} \left[ \int_D f(z) \overline{g(z)} d\mu(z) - \langle R(f), g \rangle_{H_\alpha(\omega^*)} \right]
\]
(6.33)
\[
= \sup_{\|f\|_{H_\alpha(\omega^*)} \leq 1, \|g\|_{H_\alpha(\omega^*)} \leq 1} \lim_{k \to \infty} \left[ \int_D f(z) \overline{g(z)} d\mu_k(z) - \langle R(f), g \rangle_{H_\alpha(\omega^*)} \right]
\]
\[
= \lim_{k \to \infty} \sup_{\|f\|_{H_\alpha(\omega^*)} \leq 1, \|g\|_{H_\alpha(\omega^*)} \leq 1} \left[ \int_D f(z) \overline{g(z)} d\mu_k(z) - \langle R(f), g \rangle_{H_\alpha(\omega^*)} \right]
\]
\[
= \lim_{k \to \infty} \|T_{\mu_k} - R\|
\]
for any bounded operator $R$ on $H_\alpha(\omega^*)$. Therefore $\lim_{k \to \infty} \|T_{\mu_k} - T_\mu\| = 0$, which implies that $T_\mu$ is compact. We also deduce from (6.33) that $\lambda_n(T_\mu) = \lim_{k \to \infty} \lambda_n(T_{\mu_k})$, which together with (6.31) and Fatou’s lemma gives
\[
|T_{\mu_k}|_p^p = \sum_{n=0}^{\infty} \lambda_n(T_{\mu_k})^p \leq \sum_{n=0}^{\infty} \lim_{k \to \infty} \lambda_n(T_{\mu_k})^p
\]
\[
\leq \lim \inf_{k \to \infty} \left( \sum_{n=0}^{\infty} \lambda_n(T_{\mu_k})^p \right) \leq \lim \sup_{k \to \infty} |T_{\mu_k}|_p^p \leq CM_p.
\]
This completes the proof of the first step.

**Second Step.** Assume that \( \mu \) is a compactly supported complex Borel measure on \( \mathbb{D} \). The proof is by complex interpolation. First, we will obtain the assertion for \( p = 1 \). Since \( T_\mu \) is compact on \( H_\alpha(\omega^*) \), there are orthonormal sets \( \{e_n\} \) and \( \{f_n\} \) on \( H_\alpha(\omega^*) \) such that

\[
T_\mu(f) = \sum_n \lambda_n \langle f, e_n \rangle_{H_\alpha(\omega^*)} f_n
\]

for all \( f \in H_\alpha(\omega^*) \). Therefore the formula (6.21), Fubini’s theorem, the Cauchy-Schwarz inequality, (6.5) and Lemma 6.10 yield

\[
|T_\mu|_{S_1(H_\alpha(\omega^*))} = \sum_n \left| \langle T_\mu(e_n), f_n \rangle_{H_\alpha(\omega^*)} \right| = \sum_n \left| T_\mu(e_n)(0) f_n(0) \omega^\alpha_\omega(\mathbb{D}) \right|
\]

\[
+ \int_{\mathbb{D}} \left( \int_{\mathbb{D}} e_n(z) \frac{\partial K_\alpha(w, z)}{\partial w} f_n(w) \omega^\alpha_\omega(w) dA(w) \right) d\mu(z) = \sum_n \left| \int_{\mathbb{D}} e_n(z) f_n(0) K_\alpha(0, z) \omega^\alpha_\omega(\mathbb{D}) d\mu(z) \right|
\]

\[
+ \int_{\mathbb{D}} \frac{\partial K_\alpha(w, z)}{\partial w} f_n(w) \omega^\alpha_\omega(w) dA(w) \left. \right|_{w=z} d\mu(z) = \sum_n \left| \int_{\mathbb{D}} e_n(z) f_n(z) d\mu(z) \right|
\]

\[
\leq \int_{\mathbb{D}} \left( \sum_n |e_n(z)|^2 \right)^{1/2} \left( \sum_n |f_n(z)|^2 \right)^{1/2} d|\mu|(z)
\]

\[
\leq \int_{\mathbb{D}} \|K_\alpha\|^2_{H_\alpha(\omega^*)} d|\mu|(z) \geq \int_{\mathbb{D}} \frac{d|\mu|(z)}{\omega^\alpha_\omega(z)} \sum_{R_j \in \mathcal{Y}} |\mu(R_j)| \omega^\alpha_\omega(z_j),
\]

because \( \omega^\alpha_\omega \in \mathcal{R} \) by Lemma 1.7. Thus the assertion is proved for \( p = 1 \) and \( -\infty < \alpha < 1 \).

Let now \( 1 < p < \infty \) and \( -\infty < \alpha < 1 \) with \( \rho \alpha < 1 \). Take \( \varepsilon > 0 \) such that \( \alpha < 2\varepsilon < \frac{1-\rho}{\rho} \). For \( \zeta \) in the strip \( \Lambda = \{w : 0 \leq \Re w \leq 1\} \), define the differentiation operator \( Q_\zeta \) on \( H(\mathbb{D}) \) by

\[
Q_\zeta \left( \sum_{n=0}^{\infty} a_n z^n \right) = \sum_{n=0}^{\infty} (n+1)^{\varepsilon(1-\rho\zeta)} a_n z^n,
\]

and let \( \gamma = \alpha - 2\varepsilon(1 - p\Re \zeta) \). It is easy to see that \( Q_\zeta \) is a bounded invertible operator from \( H_\alpha(\omega^*) \) to \( H_\gamma(\omega^*) \) and \( \|Q_\zeta\| = \|Q_{\Re \zeta}\| \). If \( \Re \zeta = 0 \), then \( H_\gamma(\omega^*) = H_{\gamma-2\varepsilon}(\omega^*) = A_{\omega^\alpha(\alpha-2\varepsilon)}^\omega \) by Lemma 6.9, because \( \alpha - 2\varepsilon < 0 \). If \( \Re \zeta = 1 \), then \( \gamma < 1 \) because \( 2\varepsilon < \frac{1-\alpha}{\rho} \). For each \( \zeta \in \Lambda \), define the measure-valued function \( \mu_\zeta \) by

\[
\mu_\zeta = \sum_{R_j \in \mathcal{Y}} \left( \frac{|\mu(R_j)|}{\omega^\alpha_\omega(z_j)} \right)^{1-\rho\zeta} \chi_{R_j} \mu,
\]

where the coefficient of \( \chi_{R_j} \) is taken to be zero if \( \mu(R_j) = 0 \). Further, define the analytic operator-valued function \( S_\zeta \) on \( H_\alpha(\omega^*) \) by

\[
S_\zeta(f)(w) = \int_{\mathbb{D}} Q_\zeta(f)(z) \overline{Q_\zeta(K_\omega^\alpha(z)(1-|z|)^{2\varepsilon(1-\rho\zeta)}} d\mu_\zeta(z).
\]
Since \( \mu \) has compact support, \( \|S_\zeta\| \) is uniformly bounded on \( \Lambda \). Moreover, \( S_\Lambda = T_\mu \).

By [34, Theorem 13.1], it suffices to find constants \( M_0 > 0 \) and \( M_1 > 0 \) such that

\[
\|S_\zeta\| \leq M_0 \quad \text{when} \quad \Re \zeta = 0, \quad \text{and} \quad |S_\zeta| \leq M_1 \quad \text{when} \quad \Re \zeta = 1,
\]

because then it follows that \( T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*)) \) and \( |T_\mu|_p \leq M_0^{-1/p}M_1^{1/p} \).

Let us find \( M_0 \) first. By using Fubini’s theorem, (6.22), the identity

\[
\mathcal{Q}_\zeta(K_\alpha^p)(w) = \mathcal{Q}_\zeta(K_\mu^p)(z)
\]

arguing as in (6.32) we obtain

\[
(6.35) \quad \langle S_\zeta(f), g \rangle_{H_\alpha(\omega^*)} = \int_\mathbb{D} Q_\zeta(f)(z)\overline{Q_\zeta(g)(z)}(1 - |z|)^{2\epsilon(1-p\zeta)} \, d\mu_\zeta(z)
\]

for all \( f \) and \( g \) in \( H_\alpha(\omega^*) \), and consequently,

\[
(6.36) \quad |\langle S_\zeta(f), g \rangle_{H_\alpha(\omega^*)}| \leq \left( \int_\mathbb{D} |Q_\zeta(f)(z)|^2(1 - |z|)^{2\epsilon} \, d\mu_\zeta(z) \right)^{1/2}
\]

\[
\cdot \left( \int_\mathbb{D} |Q_\zeta(g)(z)|^2(1 - |z|)^{2\epsilon} \, d\mu_\zeta(z) \right)^{1/2}
\]

if \( \Re \zeta = 0 \). Now \( \alpha - 2\epsilon < 0 \) by our choice, so Lemma 6.9 yields \( H_{\alpha - 2\epsilon}(\omega^*) = A_{\omega^*_{-(\alpha - 2\epsilon) - 2}}^2 \), where \( \omega^*_{-(\alpha - 2\epsilon) - 2} \in \mathcal{R} \) by Lemma 1.7. Moreover, both \( Q_\zeta(f) \) and \( Q_\zeta(g) \) belong to \( H_{\alpha - 2\epsilon}(\omega^*) = A_{\omega^*_{-(\alpha - 2\epsilon) - 2}}^2 \) because \( \gamma = \alpha - 2\epsilon \) if \( \Re \zeta = 0 \). Since \( |Q_\zeta(f)|^2 \) is subharmonic and \( \omega^*_{-(\alpha - 2\epsilon) - 2} \in \mathcal{R} \), we obtain

\[
(6.37) \quad |Q_\zeta(f)(z)|^2 \leq \frac{\int_{\Delta(z,r)} |Q_\zeta(f)(\xi)|^2 \omega^*_{-(\alpha - 2\epsilon) - 2}(\xi) \, dA(\xi)}{\omega^*_{-(\alpha - 2\epsilon)}(z)}
\]

for any fixed \( r \in (0, 1) \). This together with Fubini’s theorem yields

\[
\int_\mathbb{D} |Q_\zeta(f)(z)|^2(1 - |z|)^{2\epsilon} \, d\mu_\zeta(z)
\]

\[
\leq \int_\mathbb{D} |Q_\zeta(f)(\xi)|^2 \omega^*_{-(\alpha - 2\epsilon) - 2}(\xi) \left( \int_{\Delta(\xi,r)} \frac{d\mu_\zeta(z)}{\omega^*_{\alpha}(z)} \right) \, dA(\xi)
\]

\[
\leq \sup_{\xi \in \mathbb{D}} \left( \int_{\Delta(\xi,r)} \frac{d\mu_\zeta(z)}{\omega^*_{\alpha}(z)} \right) \int_\mathbb{D} |Q_\zeta(f)(\xi)|^2 \omega^*_{-(\alpha - 2\epsilon) - 2}(\xi) \, dA(\xi)
\]

\[
\leq \sup_{R_j \in \mathcal{T}} \frac{\|\mu_\zeta(R_j)\|_{L^1(\omega^*_{\alpha}(z_j))}}{\omega^*_{\alpha}(z_j)} \|Q_\zeta(f)\|_{A_{\omega^*_{-(\alpha - 2\epsilon) - 2}}^2}^2
\]

and the same is true for \( Q_\zeta(g) \) in place of \( Q_\zeta(f) \). Therefore (6.36) gives

\[
(6.39) \quad |\langle S_\zeta(f), g \rangle_{H_\alpha(\omega^*)}| \leq C^2(\mu_\zeta) \|Q_\zeta(f)\|_{A_{\omega^*_{-(\alpha - 2\epsilon) - 2}}^2} \|Q_\zeta(g)\|_{A_{\omega^*_{-(\alpha - 2\epsilon) - 2}}^2}
\]

\[
\leq C^2(\mu_\zeta) \|Q_0\|_p^2 \|f\|_{H_\alpha(\omega^*)} \|g\|_{H_\alpha(\omega^*)},
\]

where

\[
C(\mu_\zeta) = \sup_{R_j \in \mathcal{T}} \frac{\|\mu_\zeta(R_j)\|_{L^1(\omega^*_{\alpha}(z_j))}}{\omega^*_{\alpha}(z_j)}.
\]

Since

\[
(6.40) \quad \frac{\|\mu_\zeta(R_j)\|_{L^1(\omega^*_{\alpha}(z_j))}}{\omega^*_{\alpha}(z_j)} = \left( \frac{\|\mu(R_j)\|_{L^1(\omega^*_{\alpha}(z_j))}}{\omega^*_{\alpha}(z_j)} \right)^{p\Re \zeta - 1} = \left( \frac{\|\mu(R_j)\|_{L^1(\omega^*_{\alpha}(z_j))}}{\omega^*_{\alpha}(z_j)} \right)^{p\Re \zeta},
\]
we obtain $|\mu_\zeta(R_j)| \leq \omega_{*\alpha}(z_j)$ if $\Re \zeta = 0$, and hence $C(\mu_\zeta) \lesssim 1$. Therefore the existence of $M_0 > 0$ with the desired properties follows by (6.39).

Finally, we will show that $S_\zeta \in S_1$ if $\Re \zeta = 1$. In this case the fact $\omega_{*\alpha} \in \mathcal{R}$ and (6.40) yield

$$\int d|\mu_\zeta|(z) = \sum_{R_j \in \mathcal{Y}} \frac{|\mu_\zeta|(R_j)}{\omega_{*\alpha}(z_j)} = \sum_{R_j \in \mathcal{Y}} \left( \frac{|\mu|(R_j)}{\omega_{*\alpha}(z_j)} \right)^p. \quad (6.41)$$

Let $\zeta \in \Lambda$ with $\Re \zeta = 1$ be fixed, and choose bounded invertible operators $A$ and $B$ on $H_n(\omega^*)$ so that $V = Q_\zeta A$ and $W = Q_\zeta B$ are unitary operators from $H_\alpha(\omega^*)$ to $H_\gamma(\omega^*)$, where $\gamma = \alpha + 2\varepsilon(p - 1)$. Let $T = B^* S_\zeta A$. By applying (6.35), with $f$ and $g$ being replaced by $A(f)$ and $B(g)$, respectively, we obtain

$$\langle (T(f), g) H_n(\omega^*) \rangle = \langle (S_\zeta A(f), B(g)) H_n(\omega^*) \rangle = \int \int V(f)(z) W(g)(z)(1 - |z|)^{2\varepsilon(1-p)} d|\mu_\zeta|(z). \quad (6.42)$$

Let now $\{f_n\}$ and $\{g_n\}$ be orthonormal sets on $H_\alpha(\omega^*)$. Since $V$ and $W$ are unitary operators, the sets $\{e_n = V(f_n)\}$ and $\{h_n = W(g_n)\}$ are orthonormal on $H_\gamma(\omega^*)$. Since $-\infty < \gamma < 1$, the inequality (6.5) and Lemma 6.10 yield

$$\sum_n |e_n(z)|^2 \leq \|K_\alpha^\gamma \|^2 H_\gamma(\omega^*) \approx \frac{1}{\omega_{*\gamma}(z)},$$

and similarly for $\{h_n\}$. These inequalities together with (6.42), the Cauchy-Schwarz inequality and (6.41) give

$$\sum_n |\langle T(f_n), g_n H_n(\omega^*) \rangle| \leq \sum_n \int |e_n(z)||h_n(z)|(1 - |z|)^{2\varepsilon(1-p)} d|\mu_\zeta|(z)$$

$$\leq \int \left( \sum_n |e_n(z)|^2 \right)^{1/2} \left( \sum_n |h_n(z)|^2 \right)^{1/2} (1 - |z|)^{2\varepsilon(1-p)} d|\mu_\zeta|(z)$$

$$\lesssim \int \left( \frac{1 - |z|}{\omega_{*\gamma}(z)} \right)^{2\varepsilon(1-p)} d|\mu_\zeta|(z) = \int \frac{d|\mu_\zeta|(z)}{\omega_{*\alpha}(z)} \approx \sum_{R_j \in \mathcal{Y}} \left( \frac{|\mu|(R_j)}{\omega_{*\alpha}(z_j)} \right)^p. \quad (6.43)$$

By [84, Theorem 1.27] this implies $T \in S_1(H_\alpha(\omega^*))$ with

$$|T|_1 \lesssim \sum_{R_j \in \mathcal{Y}} \left( \frac{|\mu(R_j)|}{\omega_{*\alpha}(z_j)} \right)^p,$$

from which the existence of $M_1 > 0$ with the desired properties follows by the inequality

$$|S_\zeta|_1 \leq \|(B^*)^{-1}\| |T|_1 \|A^{-1}\|,$$

see [84, p. 27]. This finishes the proof of the first assertion in Theorem 6.11.

It remains to show that if $\mu$ is a positive Borel measure such that $T_\mu \in S_{p}(H_\alpha(\omega^*))$, then (6.28) is satisfied. We only give an outline of the proof. A
similar reasoning as in the beginning of Proposition 6.6 together with (6.32) shows that the assumption $T_\mu \in \mathcal{S}_p(H_\alpha(\omega^*))$ yields

$$\sum_n \left( \int_{\mathbb{B}} |v_n(z)|^2 \, d\mu(z) \right)^p < \infty,$$

where $\{v_n\}$ are the images of orthonormal vectors $\{e_n\}$ in $H_\alpha(\omega^*)$ under any bounded linear operator $\tilde{J} : [\{e_n\}] \to H_\alpha(\omega^*)$. By choosing

$$v_n(z) = \frac{\Phi_{\alpha_n}(z)}{\|B_{\alpha_n}\|_{A_{\omega^*}^{\alpha_n}}},$$

where $\{a_n\}$ is uniformly discrete and $a_n \neq 0$ for all $n$, a similar reasoning as in the proof of Lemma 6.5, with $\omega^* - \alpha$ in place of $\omega^*$, shows that $\tilde{J} : [\{e_n\}] \to H_\alpha(\omega^*)$ is bounded.

Now, take $\delta = \delta(\alpha, \omega) > 0$ obtained in Lemma 6.4 and consider the covering $\Omega = \{D(a_n, \delta(1 - |a_n|))\}_{n \geq 0}$ of $\mathbb{D}$, induced by a uniformly discrete sequence $\{a_n\}$ with $a_n \neq 0$, such that

(i) Each $R_j \in \Upsilon$ intersects at most $N = N(\delta)$ discs of $\Omega$.

(ii) Each disc of $\Omega$ intersects at most $M = M(\delta)$ squares of $\Upsilon$.

Then, by combining (6.43) and (6.44), and using Lemmas 6.4 and 6.2 as well as (1.29) and (1.2) for $\omega^* - \alpha \in \mathbb{R}$ we obtain

$$\sum_{R_j \in \Upsilon} \left( \frac{|\mu|(R_j)}{\omega^* - \alpha(z_j)} \right)^p \lesssim C(N, M) \sum_n \left( \int_{\mathbb{B}} \frac{|\Phi_{\alpha_n}^{\omega^* - \alpha}(z)|^2}{\|B_{\alpha_n}\|_{A_{\omega^*}^{\alpha_n}}} \, d\mu(z) \right)^p < \infty,$$

which finishes the proof. □

We finish the chapter by the following result concerning the case $p = \infty$ in Theorem 6.11. The proof of this proposition is straightforward and its principal ingredients can be found in the proof of Theorem 6.11.

**Proposition 6.13.** Let $-\infty < \alpha < 0$ and $\omega \in \mathcal{I} \cup R$, and let $\mu$ be a complex Borel measure on $\mathbb{D}$. If

$$\sup_{R_j \in \Upsilon} \frac{|\mu|(R_j)}{\omega^* - \alpha(z_j)} < \infty,$$

then $T_\mu$ is bounded on $H_\alpha(\omega^*)$, and there exists a constant $C > 0$ such that

$$\|T_\mu\| \leq C \sup_{R_j \in \Upsilon} \frac{|\mu|(R_j)}{\omega^* - \alpha(z_j)}.$$

Conversely, if $\mu$ is a positive Borel measure on $\mathbb{D}$ and $T_\mu$ is bounded on $H_\alpha(\omega^*)$, then (6.45) is satisfied.
CHAPTER 7

Applications to Differential Equations

In this chapter we study linear differential equations with solutions in either the weighted Bergman space $A^p_\omega$ or the Bergman-Nevalinna class $BN_\omega$. Our primary interest is to relate the growth of coefficients to the growth and the zero distribution of solutions. Apart from tools commonly used in the theory of complex differential equations in the unit disc, this chapter relies also strongly on results and techniques from Chapters 1–5.

7.1. Solutions in the weighted Bergman space $A^p_\omega$

The aim of this section is to show how the results and techniques developed in the preceding chapters can be used to find a set of sufficient conditions for the analytic coefficients $B_j$ of the linear differential equation

$$
(7.1) \quad f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_1 f' + B_0 f = B_k
$$

forcing all solutions to be in the weighted Bergman space $A^p_\omega$. In fact, the linearity of the equation is not in essence, yet it immediately guarantees the analyticity of all solutions if the coefficients are analytic. At the end of the section we will shortly discuss certain non-linear equations, and obtain results on their analytic solutions.

The approach taken here originates from Pommerenke’s work [69] in which he uses Parseval’s identity and Carleson’s theorem to study the question of when all solutions of the equation $f'' + Bf = 0$ belong to the Hardy space $H^2$. For the case of (7.1) with solutions in $H^p$, we refer to [71]. For the theory of complex differential equations the reader is invited to see [53, 54].

The main result of this section is Theorem 7.1. Its proof is based on Theorem 2.1, the equivalent norms given in Theorem 4.2, the proof of Theorem 4.1, and the estimate on the zero distribution of functions in $A^p_\omega$ given in Theorem 3.14.

**Theorem 7.1.** Let $0 < p < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$. Then there exists a constant $\alpha = \alpha(p,k,\omega) > 0$ such that if the coefficients $B_j \in \mathcal{H}(\mathbb{D})$ of (7.1) satisfy

$$
(7.2) \quad \sup_{I \subset \mathbb{T}} \frac{\int_{S(I)} |B_0(z)|^2 (1 - |z|)^{2k-2} \omega^*(z) \, dA(z)}{\omega(S(I))} \leq \alpha,
$$

and

$$
(7.3) \quad \sup_{z \in \mathbb{D}} |B_j(z)|(1 - |z|)^{k-j} \leq \alpha, \quad j = 1, \ldots, k - 1,
$$

and a $k$:th primitive $B_k^{(-k)}$ of $B_k$ belongs to $A^p_\omega$, then all solutions $f$ of (7.1) belong to $A^p_\omega$. Moreover, the ordered zero sequence $\{z_k\}$ of each solution satisfies (3.44).

**Proof.** Let $f$ be a solution of (7.1), and let first $0 < p < 2$. We may assume that $f$ is continuous up to the boundary; if this is not the case we replace $f(z)$ by...
Let $f(rz)$ and let $r \to 1^-$ at the end of the proof. The equivalent norm (4.4) in $A^{p\varepsilon}_\omega$, the equation (7.1) and the assumption (7.3) yield

$$
\|f\|_{A^{p\varepsilon}_\omega} \geq \int_{D} \left( \int_{T(u)} |f^{(k)}(z)|^2 \left( 1 - \left| \frac{z}{u} \right| \right)^{2k-2} dA(z) \right)^{\frac{q}{2}} \omega(u) dA(u)
+ \sum_{j=0}^{k-1} |f^{(j)}(0)|^p

\lesssim \sum_{j=0}^{k-1} \int_{D} \left( \int_{T(u)} |f^{(j)}(z)|^2 |B_j(z)|^2 \left( 1 - \left| \frac{z}{u} \right| \right)^{2k-2} dA(z) \right)^{\frac{q}{2}} \omega(u) dA(u)
+ ||B_k^{(-k)}||^p_{A^{p\varepsilon}_\omega} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p

\lesssim \int_{D} \left( \int_{T(u)} |f(z)|^2 |B_0(z)|^2 \left( 1 - \left| \frac{z}{u} \right| \right)^{2k-2} dA(z) \right)^{\frac{q}{2}} \omega(u) dA(u)
+ \alpha^p \|f\|_{A^{p\varepsilon}_\omega}^p + ||B_k^{(-k)}||^p_{A^{p\varepsilon}_\omega} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p.
\]

Arguing as in the last part of the proof of Theorem 4.1(ii), with $q = p < 2$, we obtain

$$
\int_{D} \left( \int_{T(u)} |f(z)|^2 |B_0(z)|^2 \left( 1 - \left| \frac{z}{u} \right| \right)^{2k-2} dA(z) \right)^{\frac{q}{2}} \omega(u) dA(u)
\lesssim ||f||_{A^{p(2-p)}_{r\omega}} \left( \int_{D} |f(z)|^p |B_0(z)|^{2p} \int_{T(z)} \omega(u) \left( 1 - \left| \frac{z}{u} \right| \right)^{2k-2} dA(u) dA(z) \right)^{\frac{p}{2}}
= ||f||_{A^{p(2-p)}_{r\omega}}^p \left( \int_{D} |f(z)|^p |B_0(z)|^2 \left( \int_{|z|}^1 \omega(s) \left( 1 - \left| \frac{z}{s} \right| \right)^{2k-1} ds \right) dA(z) \right)^{\frac{q}{2}}
\lesssim ||f||_{A^{p(2-p)}_{r\omega}}^p \left( \int_{D} |f(z)|^p |B_0(z)|^2 (1 - |z|)^{2k-2} \omega^*(z) dA(z) \right)^{\frac{q}{2}},
\]

where the last step follows by Lemma 1.6. Putting everything together, and applying Theorem 2.1 and the assumption (7.2), we get

$$
\|f\|_{A^{p\varepsilon}_\omega}^p \leq C \left( \alpha^p \|f\|_{A^{p\varepsilon}_\omega}^p + ||B_k^{(-k)}||^p_{A^{p\varepsilon}_\omega} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)
$$

for some constant $C = C(p,k,\omega) > 0$. By choosing $\alpha > 0$ sufficiently small, and rearranging terms, we deduce $f \in A^{p\varepsilon}_\omega$.

A proof of the case $p = 2$ can be easily constructed by imitating the reasoning above and taking into account Theorem 2.1 and the Littlewood-Paley formula (4.5).

Let now $2 < p < \infty$. Since the reasoning in (7.4) remains valid, we only need to deal with the last integral in there. However, we have already considered a similar situation in the proof of Theorem 4.1(ii). A careful inspection of that proof, with
Let $q = p > 2$ and $|B_0(z)|^2 (1 - |z|)^{2k-2}$ in place of $|g'(z)|^2$, shows that (7.5) remains valid for some $C = C(p, k, \omega) > 0$, provided $2 < p < \infty$ and $\omega \in \mathcal{I} \cup \mathcal{R}$. It follows that $f \in A^p_\omega$.

Finally, we complete the proof by observing that the assertion on the zeros of solutions is an immediate consequence of Theorem 3.14. \hfill \square

Theorem 7.1 is of very general nature because it is valid for all weights $\omega \in \mathcal{I} \cup \mathcal{R}$. If $\omega$ is one of the exponential type radial weights $w_{\gamma,\alpha}(z)$, defined in (1.6), or tends to zero even faster as $|z|$ approaches 1, then the maximum modulus of a solution $f$ might be of exponential growth, and hence a natural way to measure the growth of $f$ is by means of the terminology of Nevalinna theory. This situation will be discussed in Section 7.2.

If $\omega$ is rapidly increasing, then (7.2) is more restrictive than the other assumptions (7.3) in the sense that the supremum in (7.3) with $j = 0$ is finite if (7.2) is satisfied, but the converse is not true in general. Moreover, if $\omega$ is regular, then (7.2) reduces to a radial condition as the following consequence of Theorem 7.1 shows. These facts can be deduced by carefully observing the proofs of Parts (B)-(D) in Proposition 5.1.

**Corollary 7.2.** Let $0 < p < \infty$ and $\omega \in \mathcal{R}$. Then there exists a constant $\alpha = \alpha(p, k, \omega) > 0$ such that if the coefficients $B_j \in \mathcal{H}(\mathbb{D})$ of (7.1) satisfy

\[
\sup_{z \in \mathbb{D}} |B_j(z)|(1 - |z|^2)^{k-j} \leq \alpha, \quad j = 0, \ldots, k-1,
\]

and a $k$:th primitive $B_k^{(-k)}$ of $B_k$ belongs to $A^p_\omega$, then all solutions of (7.1) belong to $A^p_\omega$.

It is clear that the assertions in Theorem 7.1 and Corollary 7.2 remain valid if the assumptions are satisfied only near the boundary. By this we mean that in (7.2) it is enough to take the supremum over the intervals whose length does not exceed a pregiven $\delta \in (0, 1)$, and that in (7.3) we may replace $\sup_{z \in \mathbb{D}}$ by $\sup_{|z| \geq 1-\delta}$. In the proofs one just needs to split the integral over $\mathbb{D}$ into two pieces, $D(0, 1-\delta)$ and the remainder, and proceed in a natural manner. This observation yields the following result.

**Corollary 7.3.** Let $\omega \in \mathcal{I} \cup \mathcal{R}$. If the coefficients $B_j \in \mathcal{H}(\mathbb{D})$ of (7.1) satisfy

\[
\lim_{|I| \to 0} \frac{\int_{S(I)} |B_0(z)|^2 (1 - |z|^2)^{2k-2} \omega^*(z) dA(z)}{\omega(S(I))} = 0,
\]

and

\[
\lim_{|z| \to 1^-} |B_j(z)|(1 - |z|^2)^{k-j} = 0, \quad j = 1, \ldots, k-1,
\]

and a $k$:th primitive $B_k^{(-k)}$ of $B_k$ belongs to $\cap_{0<p<\infty} A^p_\omega$, then all solutions of (7.1) belong to $\cap_{0<p<\infty} A^p_\omega$.

For simplicity we now settle to analyze the assertion in Theorem 7.1 in the case of the most simple, yet generally explicitly unsolvable, homogeneous second order linear differential equation

\[
f'' + B f = 0, \quad B \in \mathcal{H}(\mathbb{D}).
\]
In this case the condition (7.2) reduces to

\[ \sup_{I \subset \mathbb{T}} \frac{\int_{S(I)} |B(z)|^2(1 - |z|)^2 \omega^*(z) \, dA(z)}{\omega(S(I))} \leq \alpha. \tag{7.10} \]

We concentrate on the most interesting case \( \omega \in \mathcal{I} \) so that the assumption on the only coefficient \( B \) does not reduce to a radial condition as in Corollary 7.2. Since the weight \( v_\alpha \) are one of the typical members in \( \mathcal{I} \), we choose it as an example. To this end, let \( 0 < p < \infty \) and \( 1 < \beta \leq 2 \). Then the function \( f(z) = (1 - z)^{-\frac{\beta}{2}} (\log \frac{1 - e}{1 - z})^{\frac{\beta}{2p}} \) is a solution of (7.9), where

\[ B(z) = - \frac{\frac{1}{p}(\frac{1}{p} + 1)}{(1 - z)^2} - \frac{(\beta - 2)(2 + p)}{p^2(1 - z)^2 \log \frac{1 - e}{1 - z}} - \frac{(\beta - 2)(\beta - 2 + p)}{p^2(1 - z)^2 \left( \log \frac{1 - e}{1 - z} \right)^2} \]

is analytic in \( \mathbb{D} \). Now \( f \notin A_{v_\alpha}^p \), and therefore the constant \( \alpha = \alpha(p, \omega) \) in (7.10) must satisfy \( \alpha(p, \omega) \leq \frac{1}{p} (\frac{1}{p} + 1) \) if \( \omega = v_\beta \) and \( 1 < \beta \leq 2 \). In particular, \( \alpha(p, \omega) \leq \frac{1}{p} \), as \( p \to \infty \).

We finish the section by shortly discussing certain non-linear equations for which the used methods work. For simplicity, let us consider the equation

\[ (f'')^2 + Bf = 0, \quad B \in \mathcal{H}(\mathbb{D}), \tag{7.11} \]

and the question of when its analytic solutions belong to the Hilbert space \( A_2^p \). If \( f \) is such an analytic solution, then the Littlewood-Paley formula (4.5), Lemma 1.7 and Hölder’s inequality yield

\[ \|f\|_{A_2^p}^2 \leq \int_{\mathbb{D}} |f''(z)|^2(1 - |z|)^2 \omega^*(z) \, dA(z) + |f(0)|^2 + |f'(0)|^2 \]

\[ = \int_{\mathbb{D}} |f(z)||B(z)|(1 - |z|)^2 \omega^*(z) \, dA(z) + |f(0)|^2 + |f'(0)|^2 \]

\[ \leq \|f\|_{A_2^p} \left( \int_{\mathbb{D}} \frac{|B(z)|^2(1 - |z|)^4 \omega^*(z) \omega(z)}{\omega(z)} \, dA(z) \right)^{\frac{1}{2}} + |f(0)|^2 + |f'(0)|^2, \]

and we immediately deduce \( f \in A_2^p \) if the last integral is sufficiently small. This method works also for more general equations. Namely, it yields a set of sufficient condition for the coefficients of

\[ (f^{(k)})^{n_k} + B_{k-1}(f^{(k-1)})^{n_{k-1}} + \cdots + B_1(f')^{n_1} + B_0 f = B_k, \]

where \( n_j \geq 1 \) for all \( j = 1, \ldots, k \), forcing all analytic solutions to be in \( A_2^p \). Details are left to the reader.

### 7.2. Solutions in the Bergman-Nevanlinna class \( BN_{\omega} \)

In this section we study the equation

\[ f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_1 f' + B_0 f = 0 \tag{7.12} \]

with solutions in the Bergman-Nevanlinna class \( BN_{\omega} \) induced by a rapidly increasing or regular weight \( \omega \). We will see that in this case it is natural to measure the growth of the coefficients \( B_j \) by the containment in the weighted Bergman spaces depending on the index \( j \) and \( \omega \). The Bergman-Nevanlinna class \( BN_{\omega} \) is much larger than the weighted Bergman space \( A_{v_\alpha}^p \), and therefore we will use methods that are different from those applied in Section 7.1. The tools employed here involve general growth
estimates for solutions [43], a representation of coefficients in terms of ratios of linearly independent solutions [51], integrated generalized logarithmic derivative estimates [22], the order reduction procedure [35] as well as lemmas on weights proved in Section 1.4, and results on zero distribution of functions in $BN_\omega$ from Section 3.2. Also results from the Nevanlinna value distribution theory are explicitly or implicitly present in many instances.

The order reduction produces linear equations with meromorphic coefficients and solutions. Therefore we say that a meromorphic function $f$ in $D$ belongs to the Bergman-Nevanlinna class $BN_\omega$ if
\[
\int_0^1 T(r,f) \omega(r) \, dr < \infty,
\]
where
\[
T(r,f) = m(r,f) + N(r,1/f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta + N(r,1/f)
\]
is the Nevanlinna characteristic\(^1\). If $f \in H(D)$, then the integrated counting function
\[
N(r,1/f) = \int_0^r n(s,1/f) - n(0,1/f) \, ds + n(0,1/f) \log r
\]
of poles vanishes, and thus this definition is in accordance with that given in Section 3.1 for analytic functions. For the Nevanlinna value distribution theory the reader is invited to see [38, 53].

For each radial weight $\omega$, we write $\omega(r) = \omega(r)(1 - r)$ for short, and recall that $\omega(r) = \int_r^1 \omega(s) \, ds$. If $\omega$ is regular, then $\omega(r) \sim \omega(r)$, and $\omega(r)/\omega(r) \to \infty$, as $r \to 1^-$, for each rapidly increasing weight $\omega$.

We begin with applying the growth estimates for solutions to obtain sufficient conditions for coefficients forcing all solutions to belong to $BN_\omega$.

**Proposition 7.4.** Let $\omega$ be a radial weight. If $B_j \in A^{1/(k-j)}_{\omega}$ for all $j = 0, \ldots, k-1$, then all solutions of (7.12) belong to $BN_\omega$, and the zero sequence $\{z_k\}$ of each non-trivial solution satisfies
\[
\sum_k \omega^{*}(z_k) < \infty.
\]

**Proof.** Let $f$ be a non-trivial solution of (7.12). By [43, Corollary 5.3], there exists a constant $C > 0$, depending only on $k$ and initial values of $f$, such that
\[
m(r,f) \leq C \left( \sum_{j=0}^{k-1} \int_0^r \int_0^{2\pi} |B_j(se^{i\theta})|^{1/(k-j)} \, d\theta \, ds + 1 \right).
\]
Therefore Fubini’s theorem yields
\[
\|f\|_{BN_\omega} \lesssim \sum_{j=0}^{k-1} \|B_j\|_{A^{1/(k-j)}_{\omega}} + \int_0^1 \omega(s) \, ds,
\]
and thus $f \in BN_\omega$. By Proposition 3.16 this in turn implies that the zero sequence $\{z_k\}$ of $f$ satisfies (7.13). \qed

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\(^1\)The integrated counting function of poles is usually denoted by $N(r,f)$. Since we have already conserved this symbol for zeros, we will use the unorthodox notation $N(r,1/f)$ for poles.
We next study the growth of coefficients when all solutions belong to \( BN_\omega \) and will find out a clear difference, that is not explicitly present in Proposition 7.4, between regular and rapidly increasing weights. This difference will be underscored in detail at the end of the section by means of examples. At this point it is also worth mentioning that Theorem 7.5 will be needed when the growth of coefficients is studied under the hypothesis \( \sum_k \omega^*(z_k) < \infty \) on all zero sequences \( \{z_k\} \) of solutions.

**Theorem 7.5.** Let \( f_1, \ldots, f_k \) be linearly independent meromorphic solutions of (7.12), where \( B_0, \ldots, B_{k-1} \) are meromorphic in \( \mathbb{D} \), such that \( f_j \in BN_\omega \) for all \( j = 1, \ldots, k-1 \).

(i) If \( \omega \in \mathcal{R} \), then \( \|B_j\|_{L^\infty_{\omega}} < \infty \) for all \( j = 0, \ldots, k-1 \).

(ii) If \( \omega \in \bar{I} \) such that

\[
\int_0^1 \log \frac{1}{1-r} \omega(r) \, dr < \infty,
\]

then \( \|B_j\|_{L^\infty_{\omega}} < \infty \) for all \( j = 0, \ldots, k-1 \).

By combining Proposition 7.4 and Theorem 7.5 we obtain the following immediate consequence.

**Corollary 7.6.** Let \( \omega \in \mathcal{R} \) and \( B_j \in \mathcal{H}(\mathbb{D}) \) for all \( j = 0, \ldots, k-1 \). Then all solutions of (7.12) belong to \( BN_\omega \) if and only if \( B_j \in A^1_{\bar{\omega}} \) for all \( j = 0, \ldots, k-1 \).

Theorem 7.5 is proved by applying the standard order reduction procedure. To do so, we will need the following three lemmas of which the first one is contained in [36, Lemma 6.4].

**Lemma D.** Let \( f_{0,1}, \ldots, f_{0,m} \) be \( m \geq 2 \) linearly independent meromorphic solutions of

\[
f^{(k)} + B_{0,k-1}f^{(k-1)} + \cdots + B_{0,1}f' + B_{0,0}f = 0, \quad k \geq m,
\]

where \( B_{0,0}, \ldots, B_{0,k-1} \) are meromorphic in \( \mathbb{D} \). For \( 1 \leq q \leq m-1 \), set

\[
f_{q,j} = \left( \frac{f_{q-1,j+1}}{f_{q-1,1}} \right)' \quad j = 1, \ldots, m-q.
\]

Then \( f_{q,1}, \ldots, f_{q,m-q} \) are linearly independent meromorphic solutions of

\[
f^{(k-q)} + B_{q,k-1}f^{(k-q-1)} + \cdots + B_{q,1}f' + B_{q,0}f = 0.
\]

where

\[
B_{q,j} = \sum_{l=j+1}^{k-q+1} \binom{l}{j+1} B_{q-l,1} \frac{f_{l-j-1}}{f_{q-1,1}}
\]

for \( j = 0, \ldots, k-1-q \). Here \( B_{j,k-j} = 1 \) for all \( j = 0, \ldots, q \).

The coefficients \( B_{q,j} \) of (7.16) contain terms of the type \( g^{(j)}/g \), where \( g \) is a meromorphic function given by (7.15). These quotients are known as generalized logarithmic derivatives and they will be estimated by using the following result.

**Lemma 7.7.** Let \( g \) be a meromorphic function in \( BN_\omega \), and let \( k > j \geq 0 \) be integers.
If \( \omega \in \mathcal{R} \), then
\[
\left\| \frac{g^{(k)}}{g^{(j)}} \right\|_{L^{1}_{\omega}} \leq \sum_{n=0}^{\infty} \tilde{\omega}(r_n) \int_{A(r_n, r_{n+1})} \left| \frac{g^{(k)}}{g^{(j)}}(z) \right|^{1} dA(z)
\]
\[
\leq \sum_{n=0}^{\infty} \tilde{\omega}(r_n) \left( T(s(r_{n+3}), g) + \log \frac{e}{1 - r_{n+3}} \right)
\]
\[
\leq \sum_{n=0}^{\infty} \tilde{\omega}(r_n) \left( T(s(r_n), g) + \log \frac{e}{1 - r_n} \right).
\]

Now \( \omega \in \mathcal{R} \) by the assumption, and hence the observation (i) to Lemma 1.1(i) gives
\[
\tilde{\omega}(s(r_n)) \asymp \omega(s(r_n))(1 - s(r_n)) = 2\omega(s(r_n)) \int_{s(r_n)}^{s(r_{n+1})} \omega(r) dr \asymp \int_{s(r_n)}^{s(r_{n+1})} \omega(r) dr
\]
for all \( n \in \mathbb{N} \cup \{0\} \). Moreover, \( T(r, g) \) is an increasing function by [38, p. 8], and hence
\[
\left\| \frac{g^{(k)}}{g^{(j)}} \right\|_{L^{1}_{\omega}} \leq \int_{0}^{1} T(r, g) \omega(r) dr + \int_{0}^{1} \log \frac{e}{1 - r} \omega(r) dr.
\]
The first integral in the right hand side is finite by the assumption \( g \in BN_{\omega} \) and the second one converges by the observation (ii) to Lemma 1.1.

(ii) The assertion follows by minor modifications in the proof of (i). \( \square \)

We will also need the following standard result.

**Lemma 7.8.** Let \( \omega \in \mathcal{I} \cup \mathcal{R} \) such that
\[
\int_{0}^{1} \log \frac{1}{1 - r} \omega(r) dr < \infty.
\]
If \( f \) and \( g \) are meromorphic functions in \( BN_{\omega} \), then \( fg, f/g \) and \( f' \) belong to \( BN_{\omega} \).
Let us deduce that if
\[ T(r, fg) \leq T(r, f) + T(r, g) + \log 2, \]
and so \( fg \in BN_\omega \). The inequality (7.22) and the first fundamental theorem of Nevanlinna [53, Theorem 2.1.10] give
\[ T(r, f/g) \lesssim T(r, f) + T(r, g) + 1, \]
and hence \( f/g \in BN_\omega \). Moreover, a standard lemma of the logarithmic derivative [53, Theorem 2.3.1] together with (7.22) yields
\[ T(r, f') \lesssim T(r, f) + \log^+ T(\rho, f) + \log \frac{1}{\rho - r} + 1, \quad \rho = \frac{1 + r}{2}. \]
Since \( \omega \in I \cup R \), we obtain \( \log^+ T(r, f) \lesssim \log \frac{1}{t - r} \), and hence
\[ T(r, f') \lesssim T(r, f) + \log \frac{1}{1 - r} + 1. \]
By using the assumption (7.21) we deduce \( f' \in BN_\omega \). \( \square \)

By Lemma D and Lemma 7.8 we deduce that if \( f_0, \ldots, f_0, m, m \geq 2 \), belong to \( BN_\omega \) for all \( j = 1, \ldots, m \), then \( f_0, j \in BN_\omega \) for all \( j = 1, \ldots, m - q \) and \( q = 1, \ldots, m - 1 \).

**Proof of Theorem 7.5.** (i) Assume that at least one of the coefficients does not satisfy the assertion and denote \( q = \max\{ j : \| B_j \|_{L^\infty} = \infty \} \).

If \( q = 0 \), then (7.12), standard estimates and Hölder’s inequality yield
\[ \| B_0 \|_{L_\frac{1}{2}} \lesssim \left\| \frac{f^{(k)}}{f} \right\|_{L_\frac{1}{2}} + \sum_{j=1}^{k-1} \| B_j \|_{L_\frac{1}{2}} \left\| \frac{f^{(j)}}{f} \right\|_{L_\frac{1}{2}}. \]
Since \( q = 0 \), we have \( \| B_j \|_{L_\frac{1}{2}} < \infty \) for all \( j = 1, \ldots, k - 1 \). Moreover, the norms of the generalized logarithmic derivatives are finite by Lemma 7.7(i). Therefore the right hand side is finite and we have a contradiction.

Assume now that \( q = 1 \). Denote \( B_j = B_{0,j} \) for \( j = 0, \ldots, k - 1 \) so that the notation for coefficients in (7.14) coincide with those in (7.12). The order of (7.14) is now reduced down once to equation (7.16), where the coefficients \( B_{1,0}, \ldots, B_{1,k-2} \) are given by (7.17). Hölder’s inequality and (7.17) now yield
\[ \| B_{0,1} \|_{L_\frac{1}{2}} \left\| \frac{f^{(k)}}{f} \right\|_{L_\frac{1}{2}} + \sum_{j=2}^{k-2} \| B_{0,j} \|_{L_\frac{1}{2}} \left\| \frac{f^{(j-1)}}{f_1} \right\|_{L_\frac{1}{2}} + \| f^{(k-1)} \|_{L_\frac{1}{2}}. \]
All the norms of the generalized logarithmic derivatives are finite by Lemma 7.7(i), and \( \| B_{0,j} \|_{L_\frac{1}{2}} < \infty \) for all \( j = 2, \ldots, k - 1 \). Moreover, (7.16) gives
\[ \| B_{1,0} \|_{L_\frac{1}{2}} \left\| \frac{f^{(k-1)}}{f_1} \right\|_{L_\frac{1}{2}} + \sum_{j=1}^{k-2} \| B_{1,j} \|_{L_\frac{1}{2}} \left\| \frac{f^{(j)}}{f_1} \right\|_{L_\frac{1}{2}} \]

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where again all the norms of the generalized logarithmic derivatives are finite by Lemma 7.7(i) and Lemma 7.8. Applying once again (7.17), we obtain

$$\|B_{1,j}\|_{L_2^{1/(k-j)}} \leq \|B_{0,j+1}\|_{L_2^{1/(k-j)}} \sum_{l=j+2}^{k-1} \|B_{l}\|_{L_2^{1/(k-j)}} \left\|\frac{f^{(l-j-1)}}{f_1}\right\|_{L_2^{1/(k-j)}}$$

(7.23)

for all $j = 1, \ldots, k - 2$. As earlier, we see that this upper bound is finite. Putting everything together we deduce $\|B_{0,1}\|_{L_2^{1/q}} < \infty$ which is in contradiction with the hypothesis $q = 1$.

If $q > 1$ we must do $q$ order reduction steps and proceed as in the case $q = 1$ until the desired contradiction is reached. We omit the details.

(ii) Proof is essentially identical to that of (i) and is therefore omitted. \qed

To study the converse implication with regards to oscillation of solutions in Proposition 7.4, we first observe that a standard substitution shows that the coefficient $B_{k-1}$ can be ignored when studying the zero distribution of solutions of (7.12). Namely, if we replace $f$ by $u$ in (7.12) and let $\Phi$ be a primitive function of $B_{k-1}$, then the substitution $f = ue^{-\frac{1}{k-1}}$, which does not change zeros, shows that $f$ satisfies

$$f^{(k)} + B^*_j f^{(k-j)} + \cdots + B^*_1 f' + B^*_0 f = 0, \quad k \geq 2,$$

(7.24)

with coefficients $B^*_0, \ldots, B^*_j \in \mathcal{H}(D)$. Therefore, from now on we concentrate on (7.24).

The principal findings of this section are gathered in the following result.

**Theorem 7.9.** Let $\omega \in \mathcal{I} \cup \mathcal{R}$ and $B^*_0, \ldots, B^*_{k-2} \in \mathcal{H}(D)$, and consider the assertions:

1. $B^*_j \in A^{1/(k-j)}_\omega$ for all $j = 0, \ldots, k - 2$;
2. All solutions of (7.24) belong to $BN_\omega$;
3. The zero sequence $\{z_k\}$ of each non-trivial solution of (7.24) satisfies (7.13);
4. $B^*_j \in A^{1/(k-j)}_\omega$ for all $j = 0, \ldots, k - 2$.

With this notation the following assertions hold:

(i) If $\omega \in \mathcal{R}$, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

(ii) If $\omega \in \mathcal{I}$, then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Moreover, if $\omega \in \overline{\mathcal{I}}$ satisfies

$$\int_0^1 \log \frac{1}{1-r} \omega(r) < \infty,$$

then (3) $\Rightarrow$ (4).

**Proof.** (i) By Proposition 7.4, Proposition 3.16 and the relation $\varphi(r) \asymp \omega(r)$ for $\omega \in \mathcal{R}$, it suffices to show that (3) implies (1). Let $f_1, \ldots, f_k$ be linearly independent solutions of (7.24), where $B^*_0, \ldots, B^*_{k-2} \in \mathcal{H}(D)$, and denote

$$y_j = \frac{f_{j-1}}{f_k}, \quad j = 1, \ldots, k - 1.$$  

(7.25)
The second main theorem of Nevanlinna [62, p. 84] yields
\[ T(r, y_j) \leq N(r, y_j) + N(r, 1/y_j) + N(r, y_j - 1) + S(r, y_j), \]
where
\[ S(r, y_j) \lesssim \log^+ T(r, y_j) + \log \frac{1}{1-r}, \]
outside of a possible exceptional set \( F \subset [0, 1) \) satisfying \( \int_F \frac{dr}{1-r} < \infty \). By [35, Lemma 5], there exists a constant \( d \in (0, 1) \) such that
\[ T(r, y_j) \lesssim N(s(r), y_j) + N(s(r), 1/y_j) + N(s(r), y_j - 1) \]
\[ + \log^+ T(s(r), y_j)) + \log \frac{1}{1-r} + \log \frac{1}{d}, \quad r \in (0, 1), \]
(7.26)
where \( s(r) = 1 - d(1-r) \).

Now \( \omega^* \in R \) by Lemma 1.7, and hence the observation (ii) to Lemma 1.1 shows that there exists \( \alpha > 1 \) such that \( (1-r)^\alpha \lesssim \omega^*(r) \). Therefore the assumption (3) implies that the zero sequence \( \{z_k\} \) of each solution \( f \) of (7.24) satisfies
\[ \sum_k (1 - |z_k|)^\alpha \lesssim \sum_k \omega^*(z_k) < \infty. \]
Therefore each solution \( f \) satisfies \( T(r, f) \lesssim (1-r)^{-\alpha-\varepsilon} \) for all \( \varepsilon > 0 \) by [44, Corollary 1.4], and thus \( \log^+ T(s(r), y_j)) \lesssim \log \frac{1}{1-r} - \log d \). Therefore the term \( \log^+ T(s(r), y_j)) \) can be erased from (7.26).

The zeros and poles of \( y_j = \frac{1}{f} \) are the zeros of \( f_j \) and \( f_k \), respectively, and the 1-points of \( y_j \) are precisely the zeros of \( f_j - f_k \), which is also a solution of (7.24). Therefore the assumption (3), (7.26) and Lemma 3.17 yield
\[ y_j \in BN_{\omega^*}, \quad j = 1, \ldots, k-1. \]

By [51, Theorem 2.1] each coefficient of (7.24) can be represented in the form
\[ B_j^* = \sum_{i=0}^{k-i} (-1)^{2k-i} \delta_{ki} \begin{pmatrix} k-i & \cdots & k-i-j \end{pmatrix} \frac{W_{k-i}(\sqrt{W_k})^{(k-i-j)}}{\sqrt{W_k}}, \]
where \( \delta_{kk} = 0 \) and \( \delta_{ki} = 1 \) otherwise, and
\[ W_j = \begin{vmatrix} y_1' & y_2' & \cdots & y_{k-1}' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(j-1)} & y_2^{(j-1)} & \cdots & y_{k-1}^{(j-1)} \\ y_1^{(j+1)} & y_2^{(j+1)} & \cdots & y_{k-1}^{(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(k)} & y_2^{(k)} & \cdots & y_{k-1}^{(k)} \end{vmatrix}, \quad j = 1, \ldots, k. \]

By (7.28) and Hölder’s inequality,
\[ \|B_j^*\|_{L^\infty} \lesssim \left\| \frac{\sqrt{W_k}}{\sqrt{W_k}} \right\|_{L^{1/2}}^{1/2} + \left\| \frac{W_j}{W_k} \right\|_{L^{1/2}}^{1/2} \]
\[ + \sum_{i=1}^{k-j-1} \left\| W_{k-i} \right\|_{L^{1/2}}^{1/2} \left\| \frac{\sqrt{W_k}}{\sqrt{W_k}} \right\|_{L^{1/2}}^{1/2}, \]
(7.30)
The function $\sqrt{W_k}$ is a well defined meromorphic function in $\mathbb{D}$ [44, 51]. Therefore \[ \left\| \left( \frac{W_{k-i}}{W_k} \right)^{k-j} \right\|_{L^\infty} \] is finite for each $i = 1, \ldots, k - j$ by Lemma 7.7, (7.29) and Lemma 7.8.

Finally, we note that the norms involving $W_{k-i}$ in (7.30) are finite by Theorem 7.5(i), because the functions $1, y_1, \ldots, y_{k-1}$ satisfy (7.27) and are linearly independent meromorphic solutions of

\[ y^{(k)} - \frac{W_{k-1}(z)}{W_k(z)} y^{(k-1)} + \cdots + (-1)^{k-1} \frac{W_1(z)}{W_k(z)} y' = 0, \]  

see [44, 51].

(ii) The proof is similar and is therefore omitted. \qed

A linear differential equation is called Horowitz-oscillatory, if $\sum |z_k|^2 < \infty$ for the zero sequence $\{z_k\}$ of each non-trivial solution [42, Chapter 13]. Theorem 7.9 with $\omega \equiv 1$ shows that (7.24) is Horowitz-oscillatory if and only if $B_j^*$ belongs to the classical weighted Bergman space $A^{1/2}_{1/2}$ for all $j = 0, \ldots, k - 2$.

The existing literature contains several concrete examples illustrating the correspondence between the growth of coefficients, the growth of solutions and the zero distribution of solutions given in Theorem 7.9(i). See, for example, [22] and the references therein. We next provide two examples regarding to Theorem 7.9(ii) which does not give such a satisfactory correspondence in the case $\omega \in \mathbb{I}$. To this end, consider first the locally univalent analytic function

\[ p(z) = \log \frac{1}{1 - z}, \quad z \in \mathbb{D}. \]

The functions

\[ f_1(z) = \frac{1}{\sqrt{p'(z)}} \sin p(z), \quad f_2(z) = \frac{1}{\sqrt{p'(z)}} \cos p(z) \]

are linearly independent solutions of

\[ f'' + B(z)f = 0, \quad B(z) = (p'(z))^2 + \frac{1}{2} S_p(z), \]

where $S_p = \frac{p'''}{p'} - \frac{3}{2} \left( \frac{p''}{p'} \right)^2$ is the Schwarzian derivative of $p$. Therefore

\[ B(z) = \frac{\left( \log \frac{1}{1 - z} + 1 \right)^2}{(1 - z)^4} - \frac{1}{2} \left( 1 + \log \frac{1}{1 - z} \right) (1 - z)^2 - \frac{3}{4} \left( 1 + \log \frac{1}{1 - z} \right)^2 (1 - z)^2 \]

satisfies

\[ M_r^2(r, B) \asymp \frac{\log \frac{1}{1-r}}{1-r}, \quad r \to 1^-, \]

and hence $B \in A^2_{1/2}$ if and only if $\alpha > 3$, but $B \in A^2_{1/2}$ if and only if $\alpha > 2$. If $\{z_k\}$ is the zero sequence of a solution $f$, then Theorem 7.9(ii) yields

\[ \sum_k v_{\alpha}^*(z_k) \asymp \sum_k \frac{1 - |z_k|}{\log \frac{1}{1 - |z_k|}}^{\alpha-1} < \infty. \]
for all \( \alpha > 3 \). The solution \( f = C_1f_1 + C_2f_2 \), where \( |C_1| + |C_2| \neq 0 \), vanishes at the points \( \{z_k\} \) that satisfy
\[
\log \frac{1}{1 - z_k} = k\pi + \frac{1}{2} \arg \left( \frac{C_1 - C_2i}{C_1 + C_2i} \right), \quad k \in \mathbb{Z}.
\]
Clearly the only possible accumulation point of zeros is 1. If \( |1 - z_k| < \delta \), then \( w_k = \frac{1}{1 - z_k} \) belongs to the right half plane and \( |w_k| > 1/\delta \). Therefore the equation (7.32) does not have solutions when \( k \) is a large negative integer. Moreover, (7.32) yields
\[
\lim_{k \to \infty} |w_k| = +\infty
\]
and
\[
|\arg w_k| = |\arg(\log w_k)| = \left| \arctan \left( \frac{\arg w_k}{\log |w_k|} \right) \right| \lesssim \frac{|\arg w_k|}{\log |w_k|}
\]
for large \( k \). So, if \( \arg w_k \neq 0 \), then \( |\log |w_k|| \lesssim 1 \), which together with (7.33) gives \( \arg(1 - z_k) = -\arg w_k = 0 \), that is, \( z_k \in (0, 1) \) for \( k > 0 \) large enough. It follows that
\[
\sum_k v_\alpha^*(z_k) \asymp \sum_k \frac{1 - |z_k|}{(\log \frac{2}{1 - |z_k|})^{\alpha - 1}} \asymp \sum_{k \geq 2} \frac{1}{k(\log k)^{\alpha - 2}} < \infty
\]
if and only if \( \alpha > 3 \). We deduce that in this case the implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) in Theorem 7.9(ii) are sharp, but (3) \( \Rightarrow \) (4) is not. In view of Proposition 3.16, this makes us questionize the sharpness of Theorem 7.5(ii). The answer, however, is affirmative. Namely, the functions
\[
f_1(z) = \exp \left( \frac{\log \frac{1}{1 - z}}{1 - z} \right), \quad f_2(z) = \exp \left( -\frac{\log \frac{1}{1 - z}}{1 - z} \right)
\]
are linearly independent solutions of
\[
f'' + B_1(z)f' + B_0(z)f = 0,
\]
where
\[
B_1(z) = -\frac{2\log \frac{1}{1 - z} + 3}{(\log \frac{1}{1 - z} + 1)(1 - z)}
\]
and
\[
B_0(z) = -\left( \frac{\log \frac{1}{1 - z} + 1}{1 - z} \right)^2
\]
are analytic in \( \mathbb{D} \). The coefficients satisfy both \( B_1 \in A_{\alpha}^1 \) and \( B_0 \in A_{\alpha}^2 \) if and only if \( \alpha > 2 \). Moreover, it was shown in [22, Example 3] that
\[
T(r, f_1) \asymp \log \frac{1}{1 - r}, \quad r \to 1^-,
\]
and since \( f_2 = 1/f_1 \), all solutions belong to \( BN_{\alpha} \) for all \( \alpha > 2 \) by the first main theorem of Nevanlinna. This shows that the statement in Theorem 7.5(ii) is sharp
in this case, but the first implication in Proposition 7.4 is not. Note also that if \( \{z_k\} \) is the zero sequence of a solution \( f \), then Proposition 3.16 yields

\[
\sum_k v^*_\alpha(z_k) \asymp \sum_k \frac{1 - |z_k|}{(\log \frac{2}{1-|z_k|})^{\alpha-1}} < \infty
\]

for all \( \alpha > 2 \).

There is a clear and significant difference between the geometric zero distribution of solutions in these two examples. Namely, in the latter one the solution \( f = C_1 f_1 + C_2 f_2 \), where \( C_1 \neq 0 \neq C_2 \), vanishes at the points \( \{z_k\} \) that satisfy

\[
\log \frac{1-z_k}{1-z_k^*} = k\pi i + \frac{1}{2} \log \left( \frac{C_2}{C_1} \right), \quad k \in \mathbb{Z}.
\]

It follows that \( f \) has infinitely many zeros whose imaginary part is strictly positive and the same is true for zeros with negative imaginary part. This is in contrast to the first example in which the zeros are real except possibly a finite number of them. Note that the coefficients are essentially of the same growth in both examples, but the leading term in the coefficient \( B \) behaves roughly speaking as \(-B_0\), when \( z \) is close to 1. It would be desirable to establish a result similar to Theorem 7.9(i) for rapidly increasing weight \( \omega \), and, in particular, to see how the geometric zero distribution of solutions fits in the picture.
CHAPTER 8

Further Discussion

This chapter concerns topics that this monograph does not cover. Here we will briefly discuss $q$-Carleson measures for $A^p_\omega$ when $q < p$, generalized area operators acting on $A^p_\omega$ as well as questions related to differential equations and the zero distribution of functions in $A^p_\omega$. We include few open problems that are particularly related to the weighted Bergman spaces induced by rapidly increasing weights.

8.1. Carleson measures

The following result can be proved by using arguments similar to those in the proof of Theorem 4.1(iv).

Theorem 8.1. Let $0 < q < p < \infty$, and let $\mu$ be a positive Borel measure on $\mathbb{D}$. If the function

$$\Psi_\mu(u) = \frac{d\mu(z)}{\omega^*(z)} , \quad u \neq 0,$$

belongs to $L^{p/q}_\omega$, then $\mu$ is a $q$-Carleson measure for $A^p_\omega$.

Even though we do not know if the condition $\Psi_\mu \in L^{p/q}_\omega$ characterizes $q$-Carleson measures for $A^p_\omega$, we can show that this condition is necessary when $A^p_\omega$ is replaced by $L^p_\omega$ and $\omega \in I \cup R$. Namely, if $I_d : L^p_\omega \to L^q(\mu)$ is bounded and $g \in L^p_\omega$ is a positive function, then Fubini’s theorem, Lemma 1.6 and Corollary 2.2 give

$$\int_\mathbb{D} g(u) \Psi_\mu(u) \omega(u) dA(u) = \int_\mathbb{D} \left( \frac{1}{\omega^*(z)} \int_{T(z)} g(u) \omega(u) dA(u) \right) d\mu(z)$$

$$\leq \int_\mathbb{D} M_\omega(g)(z) d\mu(z) = \int_\mathbb{D} \left( (M_\omega(g)(z))^{1/q} \right)^q d\mu(z)$$

$$\leq \int_\mathbb{D} (M_\omega(g)(z))^{q/q} \omega(z) dA(z) \lesssim \|g\|^{p/q}_{L^p_\omega}.$$

Since $L^{q/q}_{\omega}$ is the dual of $L^{p/q}_\omega$, we deduce $\Psi_\mu \in L^{p/q}_{\omega}$.

In view of these results it is natural to ask, if $q$-Carleson measures for $A^p_\omega$, with $\omega \in I \cup R$, are characterized by the condition $\Psi_\mu \in L^{p/q}_{\omega}$?

8.2. Generalized area operators

Let $0 < p < \infty$, $n \in \mathbb{N}$ and $f \in \mathcal{H}(\mathbb{D})$, and let $\omega$ be a radial weight. Then Theorem 4.2 shows that $f \in A^p_\omega$ if and only if

$$\int_\mathbb{D} \left( \int_{T(u)} |f^{(n)}(z)|^2 \left( 1 - \frac{|z|}{u} \right)^{2n-2} dA(z) \right)^{q/q} \omega(u) dA(u) < \infty.$$
Therefore the operator

\[ F_n(f)(u) = \left( \int_{\Gamma(u)} |f^{(n)}(z)|^2 \left( 1 - \left| \frac{z}{u} \right| \right)^{2n-2} dA(z) \right)^{\frac{1}{2}}, \quad u \in \mathbb{D}, \]

is bounded from \( A^p_\omega \) to \( L^q_\omega \) for each \( n \in \mathbb{N} \). To study the case \( n = 0 \), let \( \mu \) be a positive Borel measure on \( \mathbb{D} \), and consider the generalized area operator

\[ G^\omega_{\mu}(f)(\zeta) = \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{\omega^*(z)}, \quad \zeta \in \mathbb{D} \setminus \{0\}, \]

associated with a weight \( \omega \in \mathcal{I} \cup \mathcal{R} \). The method of proof of Theorem 4.1 yields the following result which is related to a study by Cohn on area operators \([25]\). In particular, the reasoning gives an alternative way to establish the implication in \([25, \text{Theorem 1}]\) in which the original proof relies on an argument of John and Nirenberg in conjunction with the Calderon-Zygmund decomposition.

**Theorem 8.2.** Let \( 0 < p \leq q < \infty \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), and let \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then \( G^\omega_{\mu} : A^p_\omega \to L^q_\omega \) is bounded if and only if \( \mu \) is a \((1 + p - \frac{q}{2})\)-Carleson measure for \( A^p_\omega \). Moreover, \( G^\omega_{\mu} : A^p_\omega \to L^q_\omega \) is compact if and only if \( I_d : A^p_\omega \to L^p(\mu) \) is compact.

It would be interesting to find out the condition on \( \mu \) that characterizes the bounded area operators \( G^\omega_{\mu} \) from \( A^p_\omega \) to \( L^q_\omega \), when \( 0 < q < p < \infty \).

### 8.3. Growth and oscillation of solutions

In Chapter 7 we studied the interaction between the growth of coefficients and the growth and the zero distribution of solutions of linear differential equations. In particular, we established a one-to-one correspondence between the growth of coefficients, the growth of solutions and the oscillation of solutions, when all solutions belong to the Bergman-Nevanlinna class \( BN_\omega \) and \( \omega \) is regular. In order to keep our discussion simple here, let us settle to consider the second order equation

\[ f'' + Bf = 0, \quad B \in \mathcal{H}(\mathbb{D}). \tag{8.1} \]

Theorem 7.9(i) states that, if \( \omega \) is regular, then \( B \in A^2_\omega \) if and only if, all solutions \( f \) of (8.1) belong to \( BN_\omega \), if and only if, the zero sequence \( \{z_k\} \) of each non-trivial solution satisfies \( \sum_k \omega^*(z_k) < \infty \). Recall that \( \widetilde{\omega}(r) = \int_0^r \omega(s) \frac{ds}{s} \) and

\[ \omega^*(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} ds \asymp (1 - |z|)\widetilde{\omega}(|z|), \quad |z| \to 1^-, \]

so the convergence of the sum \( \sum_k \omega^*(z_k) \) is clearly a weaker requirement than the Blaschke condition \( \sum_k (1 - |z_k|) < \infty \). If \( \omega \) is rapidly increasing, then Theorem 7.9(ii) and the examples given at the end of Section 7.2 show that the situation is more delicate and it is not enough to measure the distribution of zeros by a condition depending on their moduli only. Needless to say that things get even more complex in general if all solutions belong to the weighted Bergman space \( A^p_\omega \) induces by \( \omega \in \mathcal{I} \cup \mathcal{R} \), see Chapter 3 and Section 7.1. The final end in the oscillation is of course the growth restriction \( |B(z)|(1 - |z|^2)^2 \leq 1 \), which guarantees that each non-trivial solution of (8.1) vanishes at most once in the unit disc. This statement is equivalent to the well known theorem of Nehari that provides a sufficient condition for the injectivity of a locally univalent meromorphic function in \( \mathbb{D} \) in terms of the
size of its Schwarzian derivative. In view of the results in the existing literature and those presented in Chapter 7, it appears that the connection between the growth of the coefficient $B$ and the growth and the oscillation of solutions $f$ of (8.1) is not well understood when Nehari’s condition fails, but Theorem 7.9(i) is too rough. This case seems to call for further research. In particular, it would be desirable to establish a result similar to Theorem 7.9(i) for rapidly increasing weights, and to see how the geometric zero distribution of solutions fits in the picture. Going further, the equations (8.1) with solutions whose zeros constitute Blaschke sequences have attracted attention during the last decade. For the current stage of the theory of these equations as well as open problems we refer to [42].

8.4. Zero distribution

It is known that the density techniques employed in the study of $A^p_\alpha$-zero sets and interpolating and sampling sequences are related [74, 76, 77]. When we were finishing the typesetting of this monograph in July 2011, Jordi Pau kindly brought our attention to the existence of the very recent paper by Seip [73] on interpolation and sampling on the Hilbert space $A^2_\omega$, where $\omega$ is assumed to be continuous such that $\omega(r) \lesssim \omega(\frac{1+r}{1+r})$ for all $0 \leq r < 1$. It is clear that each weight $\omega \in \tilde{T} \cup R$ satisfies this condition, but there are invariant weights $\omega$ that do not have this property. Seip characterizes interpolating and sampling sequences for $A^2_\omega$ by using densities and points out that the techniques used work also for $A^p_\omega$ without any essential changes of the arguments. It is natural to expect that these techniques can be used to obtain information on the zero distribution of functions in $A^p_\omega$.

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