BOUNDARY BEHAVIOUR OF ANALYTIC FUNCTIONS IN SPACES OF DIRICHLET TYPE

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ABSTRACT. For $0 < p < \infty$ and $\alpha > -1$ we let $\mathcal{D}^p_\alpha$ be the space of all analytic functions f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that f' belongs to the weighted Bergman space A^p_α. We obtain a number of sharp results concerning the existence of tangential limits for functions in the spaces $\mathcal{D}^p_\alpha$. We also study the size of the exceptional set $E(f) = \{e^{i\theta} \in \partial\mathbb{D} : V(f,\theta) = \infty\}$, where $V(f,\theta)$ denotes the radial variation of f along the radius $[0, e^{i\theta})$, for functions $f \in \mathcal{D}^p_\alpha$.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. If $0 < r < 1$ and f is an analytic function in $\mathbb{D}$ (abbreviated $f \in \mathcal{H}ol(\mathbb{D})$), we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad I_p(r,f) = M_p^p(r,f),\ (0 < p < \infty),$$

$$M_\infty(r,f) = \sup_{0\le t\le 2\pi} |f(re^{it})|.$$

For $0 < p \le \infty$ the Hardy space H^p consists of those functions $f \in \mathcal{H}ol(\mathbb{D})$ for which $||f||_{H^p} \overset{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty$. We refer to [10] for the theory of Hardy spaces.

The weighted Bergman space A^p_α $(0 < p < \infty,\ \alpha > -1)$ is the space of all functions $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \overset{\text{def}}{=} \left(\int_{\mathbb{D}} (1-|z|)^\alpha |f(z)|^p\,dA(z)\right)^{1/p} < \infty,$$

where $dA(z) = \frac{1}{\pi}dx\,dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We mention [11] and [16] as general references for the theory of Bergman spaces.

We shall write $\mathcal{D}^p_\alpha$ $(0 < p < \infty,\ \alpha > -1)$ for the space of all functions $f \in \mathcal{H}ol(\mathbb{D})$ such that $\int_{\mathbb{D}}(1-|z|)^\alpha |f'(z)|^p\,dA(z) < \infty$. In other words,

$$f \in \mathcal{D}^p_\alpha \Leftrightarrow f' \in A^p_\alpha.$$

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If $p < \alpha + 1$, it is well known that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$ with equivalence of norms (see Theorem 6 of [12]). If $p > 1$ and $\alpha = p-2$ we are considering the Besov spaces $\mathcal{B}^p$ which have been extensively studied in [3], [9] and [29]. Specially relevant is the space $\mathcal{B}^2 = \mathcal{D}^2_0$, which coincides the classical Dirichlet space $\mathcal{D}$.

The space $\mathcal{D}^p_\alpha$ is said to be a Dirichlet space if $p \geq \alpha + 1$. Specially interesting are the spaces in the "limit case" $p = \alpha + 1$, that is, the spaces $\mathcal{D}^p_{p-1}$, $0 < p < \infty$. These spaces are closely related to Hardy spaces. Indeed, a direct calculation with Taylor coefficients gives that $H^2 = \mathcal{D}^2_1$. Furthermore, we have

$$H^p \subset \mathcal{D}^p_{p-1}, \quad 2 \leq p < \infty, \tag{1}$$

and

$$\mathcal{D}^p_{p-1} \subset H^p \quad 0 < p \leq 2. \tag{2}$$

The relation (1) is a classical result of Littlewood and Paley [21], and (2) can be found in [28]. A good number of results on the spaces $\mathcal{D}^p_{p-1}$ have been recently obtained in [4], [13], [14], [15] and [28]. We remark that the spaces $\mathcal{D}^p_{p-1}$ are not nested. Actually, it is easy to see that if $p \neq q$ then there is no relation of inclusion between $\mathcal{D}^p_{p-1}$ and $\mathcal{D}^q_{q-1}$.

Fatou's theorem asserts that if $0 < p \leq \infty$ and $f \in H^p$ then f has a finite non-tangential limit $f(e^{i\theta})$ for a.e. $e^{i\theta} \in \partial\mathbb{D}$. Bearing in mind (2), we see that this is true if $f \in \mathcal{D}^p_{p-1}$ and $0 < p \leq 2$. In view of (1), it is natural to ask whether or not Fatou's theorem remains true for the spaces $\mathcal{D}^p_{p-1}$, $2 < p < \infty$. The answer to this question is negative. Indeed, Theorem 3.5 of [13] asserts that if $2 < p < \infty$, then there exists a function $f \in \mathcal{D}^p_{p-1}$ such that

$$\lim_{r\to 1^-} \frac{|f(re^{it})|}{\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}}\left(\log\log\frac{1}{1-r}\right)^{-1}} = \infty, \quad \text{for a.e. } e^{it} \in \partial\mathbb{D}. \tag{3}$$

This function has a non-tangential limit almost nowhere in ∂D.

Fatou's theorem is best possible for Hardy spaces, in the sense that it cannot be extended further to give the existence of "tangential limits". Indeed, Lohwater and Piranian [22] (see also p. 43 of [8] and [20], [31] and p. 280 in Vol. I of [32] for some related results) proved that if γ_0 is a Jordan curve, internally tangent to $\partial\mathbb{D}$ at $z = 1$, and having no other point in common with $\partial\mathbb{D}$, and γ_θ ($\theta \in \mathbb{R}$) denotes the rotation of γ_0 through an angle θ around the origin, then there exists a function $f \in H^\infty$ such that, for every $\theta \in \mathbb{R}$, f does not approach a limit as $z \to e^{i\theta}$ along γ_θ.

In spite of this, a number of "tangential-Fatou's theorems" have been proved for certain spaces of Dirichlet type.

For $A > 0$, $\gamma \geq 1$ and $\xi \in \partial\mathbb{D}$, we define

$$R(A, \gamma, \xi) = \{z \in \mathbb{D} : |1 - \overline{\xi}z|^\gamma \leq A(1 - |z|)\}.$$

When $\gamma = 1$ and $A > 1$, the region $R(A, \gamma, \xi)$ is basically a Stolz angle. When $\gamma > 1$, $R(A, \gamma, \xi)$ is a region contained in $\mathbb{D}$ which touches $\partial\mathbb{D}$ at ξ tangentially. As γ increases, the degree of tangency increases.

We define also, for $A > 1$ and $\beta > 0$,

$$R_{\exp}(A, \beta, \xi) = \left\{ z \in \mathbb{D} : \exp\left(-|1 - \overline{\xi} z|^{-\beta}\right) \leq \frac{(1 - |z|)}{A} \right\},$$

and

$$R_{\log}(A, \beta, \xi) = \left\{ z \in \mathbb{D} : |1 - \overline{\xi} z| \leq A(1 - |z|) \left(\log \frac{2}{1 - |z|} \right)^{\beta} \right\}.$$

As β increases, the degree of tangency increases in both types of tangential regions.

If $f \in \mathcal{H}ol(\mathbb{D})$ we say that f has the γ-limit L at $e^{i\theta}$, if $f(z) \to L$ as $z \to e^{i\theta}$ within $R(A, \gamma, \xi)$ for every A. Notice that saying that f has the 1-limit L at $e^{i\theta}$ is the same as saying that f has the non-tangential limit L at $e^{i\theta}$. Substituting the regions $R(A, \gamma, \xi)$ with the regions $R_{\exp}(A, \beta, \xi)$ and $R_{\log}(A, \beta, \xi)$, we have the notions of $\beta_{\exp}$-limits and $\beta_{\log}$-limits. We observe that these definitions of tangential limits are equivalent to those considered in [2], [7], [23] and [26].

Among other results Kinney [19] and Nagel, Rudin and Shapiro [23] (see also [26]) proved the following:

(i) If $0 < \alpha < 1$ and $f \in D^2_\alpha$, then f has a finite α^{-1}-limit at a.e $e^{i\theta} \in \partial\mathbb{D}$.

(ii) If $f \in D^2_0 = \mathcal{D}$, then f has a finite $1_{\exp}$-limit almost everywhere.

In view of these results, it is natural to ask whether results of this kind can be proved for the spaces $\mathcal{D}^p_\alpha$ for other choices of p and α. We start with a negative result.

Theorem 1. *(a) Suppose that $A > 1$ and $\beta > 1$. Then there exists a function $f \in \cap_{1 \leq p < \infty} \mathcal{D}^p_{p-1}$ such that for almost every $e^{i\theta} \in \partial\mathbb{D}$, f does not approach a limit as $z \to e^{i\theta}$ inside $R_{\log}(A, \beta, e^{i\theta})$.*

(b) Suppose that $A > 0$ and $\gamma > 1$. Then there exists a function $f \in \cap_{0 < p < \infty} \mathcal{D}^p_{p-1}$ such that for almost every $e^{i\theta} \in \partial\mathbb{D}$, f does not approach a limit as $z \to e^{i\theta}$ inside $R(A, \gamma, e^{i\theta})$

Next we turn our attention to the spaces $\mathcal{D}^p_\alpha$ with $1 \leq p \leq 2$ and $-1 < \alpha \leq p - 1$. We will prove the following theorem.

Theorem 2. *(a) Suppose that $1 \leq p \leq 2$, $p - 2 < \alpha \leq p - 1$ and $f \in \mathcal{D}^p_\alpha$. Then f has an $(\alpha - p + 2)^{-1}$-limit at a.e. $e^{i\theta} \in \partial\mathbb{D}$.*

(b) Suppose that $1 < p \leq 2$ and $f \in \mathcal{D}^p_{p-2} = \mathcal{B}^p$. Then f has a $(p' - 1)_{\exp}$-limit at a.e. $e^{i\theta} \in \partial\mathbb{D}$.

Here and throughout the paper, if $p > 1$ we write p' for the exponent conjugate of p, $\frac{1}{p} + \frac{1}{p'} = 1$.

We will prove that part (a) of Theorem 2 is sharp in the sense that the degree of potential tangency $(\alpha - p + 2)^{-1}$ cannot be substituted by any larger one.

THEOREM 3. *Suppose that* $1 \le p \le 2$, $p-2 < \alpha \le p-1$, $A > 0$ *and* $\gamma > (\alpha - p + 2)^{-1}$ *Then there exists a function* $f \in \mathcal{D}^p_\alpha$ *such that for almost every* $e^{i\theta} \in \partial\mathbb{D}$, f *does not approach a limit as* $z \to e^{i\theta}$ *inside* $R(A, \gamma, e^{i\theta})$

Now we turn to questions related to radial variation of analytic functions. If $f \in \mathcal{H}ol(\mathbb{D})$ and $\theta \in [-\pi, \pi)$, we define

$$V(f,\theta) \overset{\text{def}}{=} \int_0^1 |f'(re^{i\theta})|\, dr. \tag{4}$$

Then $V(f,\theta)$ denotes the radial variation of f along the radius $[0, e^{i\theta})$, that is, the length of the image of this radius under the mapping f. We define the exceptional set $E(f)$ associated to f as

$$E(f) = \{e^{i\theta} \in \partial\mathbb{D} : V(f,\theta) = \infty\}. \tag{5}$$

It is clear that if f has finite radial variation at $e^{i\theta}$ then f has a finite radial limit at $e^{i\theta}$. Even though every H^p-function, $0 < p \le \infty$, has finite radial limits a.e., if we take $f \in \mathcal{H}ol(\mathbb{D})$ given by a power series with Hadamard gaps

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \quad \text{with } n_{k+1} \ge \lambda n_k, \quad \text{for all } k \quad (\lambda > 1), \tag{6}$$

such that

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty \quad \text{but} \quad \sum_{k=1}^{\infty} |a_k| = \infty,$$

then $f \in \cap_{0<p<\infty} H^p$, but a result of Zygmund (see Theorem 1 on p. 194 of [30]) shows that $V(f,\theta) = \infty$ for every $\theta \in [-\pi, \pi)$.

We will prove a positive result for $\mathcal{D}^p_{p-1}$-functions, $0 < p \le 1$

THEOREM 4. *If* $0 < p \le 1$ *and* $f \in \mathcal{D}^p_{p-1}$ *then* $E(f)$ *has measure* 0.

We note that this result cannot be extended to $p > 1$. Indeed, if we take f given by a power series with Hadamard gaps as in (6) with $\sum_{k=1}^{\infty} |a_k|^p < \infty$ and $\sum_{k=1}^{\infty} |a_k| = \infty$, we have that $f \in \mathcal{D}^p_{p-1}$ (see Proposition A of [13]) and so $V(f,\theta) = \infty$ for every $\theta \in [-\pi, \pi)$.

On the other hand, we have the following well known result of Beurling [5] for functions in $\mathcal{D}^2_\alpha$.

THEOREM A. *Let* f *be an analytic function in* $\mathbb{D}$.

(a) If $f \in \mathcal{D}$, *then* $E(f)$ *has logarithmic capacity* 0.

(b) If $0 < \alpha < 1$ *and* $f \in \mathcal{D}^2_\alpha$, *then* $E(f)$ *has* α*-capacity* 0.

See [17] for the definitions of logarithmic capacity and α-capacity and [27] for an extension of Theorem A.

We will prove the following result for other values of p.

THEOREM 5. *Suppose that* $f \in \mathcal{D}^p_\alpha$.

(a) If $0 < p \leq 1$ *and* $-1 < \alpha < p-1$, *then* $E(f)$ *has Lebesgue measure* 0.

(b) If $1 < p < 2$ *and* $p-2 < \alpha < p-1$, *then* $E(f)$ *has Lebesgue measure* 0.

(c) If $1 < p \leq 2$ *and* $\alpha = p-2$, *then* $E(f)$ *has logarithmic capacity* 0.

(d) If $2 < p < \infty$ *and* $p-1 > \alpha \geq \frac{p}{2} - 1$, *then* $E(f)$ *has* β*-capacity* 0 *for all* $\beta > \frac{2}{p}(1+\alpha) - 1$.

(e) If $2 < p < \infty$ *and* $\alpha < \frac{p}{2} - 1$, *then* $E(f)$ *has logarithmic capacity* 0.

2. On the membership of Blaschke products in spaces of Dirichlet type

We remark that $H^\infty \not\subset \mathcal{D}^p_\alpha$, if $0 < p < \infty$ and $-1 < \alpha < p-1$ (see, e.g., section 3 of [14] for explicit examples). Clearly, (1) gives that $H^\infty \subset \mathcal{D}^p_{p-1}$, if $2 \leq p < \infty$. However, this does not remain true for $0 < p < 2$. Indeed, Vinogradov [28, p. 3822-3823] has shown that there exist Blaschke products B which do not belong to $\cup_{0<p<2}\mathcal{D}^p_{p-1}$. In this section we shall find a number of sufficient conditions for the membership of a Blaschke product in some of the spaces $\mathcal{D}^p_\alpha$. These results will be basic in the proofs of Theorem 1 and Theorem 3.

We recall that if a sequence of points $\{a_n\}$ in $\mathbb{D}$ satisfies the *Blaschke condition*: $\sum_{n=1}^\infty (1-|a_n|) < \infty$, the corresponding Blaschke product B is defined as

$$B(z) = \prod_{n=1}^\infty \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}.$$

Such a product is analytic in $\mathbb{D}$, bounded by one, and with non-tangential limits of modulus one almost everywhere on the unit circle. We start obtaining sufficient conditions for the membership of a Blaschke product in the spaces $\mathcal{D}^p_{p-1}$, improving the first part of Lemma 2.11 of [28].

LEMMA 1. *Let* B *be a Blaschke product with sequence of zeros* $\{a_n\}$.

(a) If $\{a_n\}$ *satisfies*

$$\sum_{n=1}^\infty (1-|a_n|) \log\left(\frac{1}{1-|a_n|}\right) < \infty, \tag{7}$$

then $B \in \cap_{1\leq p<\infty}\mathcal{D}^p_{p-1}$.

(b) If there exists $q \in (0,1)$ *such that*

$$\sum_{n=1}^\infty (1-|a_n|)^q < \infty, \tag{8}$$

then $B \in \cap_{0<p<\infty} \mathcal{D}^p_{p-1}$.

Proof. A result of Rudin's ([25, Theorem I]) shows that (7) implies that $B \in \mathcal{D}^1_0$. Then (a) follows from the Cauchy estimate $|B'(z)| \le 1/(1-|z|)$.

We turn now to part (b). Suppose that $\{a_n\}$ satisfies (8) for a certain $q \in (0,1)$. Assume for now that $p \in (0,1]$. Using Theorem 3.1 of [18] we see that $B' \in A^{2-q}$. Using this, Hölder's inequality with exponents $\frac{2-q}{p}$ and $\frac{2-q}{2-q-p}$ and the fact that $\frac{(2-q)(1-p)}{2-q-p} < 1$, we obtain we obtain

$$\int_{\mathbb{D}} |B'(z)|^p (1-|z|^2)^{p-1}\, dA(z)$$
$$\le \left(\int_{\mathbb{D}} |B'(z)|^{2-q}\, dA(z) \right)^{\frac{p}{2-q}} \left(\int_{\mathbb{D}} (1-|z|^2)^{\frac{(2-q)(p-1)}{2-q-p}}\, dA(z) \right)^{\frac{2-q-p}{2-q}} < \infty.$$

Hence, we have shown that $B \in \mathcal{D}^p_{p-1}$, for all $p \in (0,1]$. Using the Cauchy estimate again, we obtain that $B \in \mathcal{D}^p_{p-1}$ for all $p \in (0,\infty)$, as desired. □

We next give a simplified proof of a result that essentially is Theorem 3.1(i) for $\beta = 1$ and $p \ge 1$ in [18].

Lemma 2. *Let p and α be such that $p \ge 1$ and $p-2 < \alpha < p-1$. If B is a Blaschke product whose sequence of zeros $\{a_n\}$ satisfies*

$$\sum_{n=1}^{\infty} (1-|a_n|)^{\alpha+2-p} < \infty, \tag{9}$$

then $B \in \mathcal{D}^p_\alpha$.

Proof. We shall use the notation and terminology of [1, pp. 332-333].

Let p, α and B be as in the statement. Notice that $0 < \alpha + 2 - p < 1$, and then, using Theorem 1 of [24], we deduce that $B' \in \mathrm{B}^{1/(\alpha-p+3)}$ or, equivalently, $B \in \mathcal{D}^1_{\alpha-p+1}$. Then as in the proof of Lemma 1, the Cauchy estimate implies $B \in \mathcal{D}^p_\alpha$ since $p-1 \ge 0$. □

3. Tangential limits for $\mathcal{D}^p_\alpha$-functions

Proof of Theorem 1(a). We are going to use an argument which is similar to the one used in the proof of Theorem 7.44 of [32], Vol I, Chapter VII.

Take M with $1 < M < A$ and let Let C_θ be the boundary of $R_{\log}(M,\beta,e^{i\theta})$, $(\theta \in [0,2\pi))$. For all sufficiently large n, let l_n denote the length of the arc of the circle $|z| = 1 - \frac{1}{n}$ which lies in $R_{\log}(M,\beta,1)$ and let $m_n = E[\frac{2\pi}{l_n}] + 1$, where, for $x \in \mathbb{R}$, $E[x]$ denotes the greatest integer that is smaller than or equal to x. Let $S_n = \{z_{n,1}, z_{n,2}, \dots z_{n,m_n}\}$ be any collection of m_n points equally spaced on $|z| = 1 - \frac{1}{n}$. Since the circular distance between any two consecutive points of S_n is smaller than l_n, for every θ the set $R_{\log}(M,\beta,e^{i\theta})$ contains a point of S_n.

We define

$$\sigma_n = \sum_{k=1}^{m_n}(1-|z_{n,k}|)\log\left(\frac{1}{1-|z_{n,k}|}\right) = \frac{m_n\log(n)}{n}.$$

Notice that $l_n \asymp \frac{1}{n}\log^\beta n$. Then it is easy to see that there exists a positive constant C (which does not depend on n) such that

$$\begin{aligned}\sigma_n =& \frac{m_n\log(n)}{n} \le \frac{(1+\frac{2\pi}{l_n})\log(n)}{n} \le C\frac{\log(n)}{nl_n}\\ \le& C\frac{1}{\log^{\beta-1} n} \to 0, \quad \text{as } n\to\infty.\end{aligned}$$

Let us take then an increasing sequence n_k satisfying that $\sum_{k=1}^\infty \sigma_{n_k} < \infty$ and let B be the Blaschke product with zeros at the points of $\cup_{k=1}^\infty S_{n_k}$. By part (a) of Lemma 1, $B \in \cap_{1\le p<\infty}\mathcal{D}^p_{p-1}$. Notice that for each $\theta\in\mathbb{R}$, B has infinitely many zeros in the set $R_{\log}(M,\beta,e^{i\theta})$. Thus for every θ, the limit of $B(z)$ as $z\to e^{i\theta}$ inside of $R_{\log}(M,\beta,e^{i\theta})$ must be zero if it exists at all. Since the radial limit of B has absolute value 1 a.e., it follows that for almost every $e^{i\theta}\in\partial\mathbb{D}$, the the limit of $B(z)$ as $z\to e^{i\theta}$ inside of $R_{\log}(M,\beta,e^{i\theta})$ does not exist. □

Part (b) of Theorem 1 can be proved in a similar way using part (b) of Lemma 1. We omit the details.

Next we shall obtain a representation formula for functions f in the space $\mathcal{D}^p_\alpha$, $-1<\alpha,\ 1\le p\le 2$ which will play a basic role in the proof of Theorem 2.

Theorem 6. *Suppose that either* $1\le p\le 2$ *and* $-1<\alpha<p-1$ *or* $1<p\le 2$ *and* $\alpha=p-2$, *and that* $f\in\mathcal{D}^p_\alpha$. *Then there exists a function* $h(e^{i\theta})\in L^p(\partial\mathbb{D})$ *such that*

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{\frac{\alpha+1}{p}}}\,d\theta, \quad z\in\mathbb{D}. \tag{10}$$

Proof. Let p and α be as in the statement and $f(z)=\sum_{n=0}^\infty a_nz^n\in\mathcal{D}^p_\alpha$. Then $zf'(z)=\sum_{n=0}^\infty na_nz^n\in A^p_\alpha$. Since $\mathcal{D}^p_\alpha\subset A^p_\alpha$ we also have that $f\in A^p_\alpha$. Then it follows that

$$zf'(z)+\frac{\alpha+1}{p}f(z) = \sum_{n=0}^\infty\left(n+\frac{\alpha+1}{p}\right)a_nz^n\in A^p_\alpha.$$

So using Lemma 1.1 of [6] (see also part (iii) of Theorem 5 of [12]) and Corollary 3.5 of [6], we deduce that the fractional integral

$$\begin{aligned}h(z) &\overset{\text{def}}{=} \tilde{I}^{\frac{\alpha+1}{p}}\left(zf'(z)+\frac{\alpha+1}{p}f(z)\right)\\ &= \sum_{n=0}^\infty\left(n+\frac{\alpha+1}{p}\right)B\left(n+1,\frac{\alpha+1}{p}\right)a_nz^n\end{aligned}$$

belongs to H^p since $p \leq 2$. Here$B(.,.)$ is the classical beta function. Note that

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

and recall that $\Gamma(s+1) = s\Gamma(s)$, for all $s \neq 0, -1, \dots$. Then it is easy to see that

$$h(z) = \sum_{n=0}^{\infty} \frac{n!\Gamma(\frac{\alpha+1}{p})}{\Gamma(n+\frac{\alpha+1}{p})} a_n z^n.$$

Then,

$$h(e^{i\theta}) = \sum_{n=0}^{\infty} \frac{n!\Gamma(\frac{\alpha+1}{p})}{\Gamma(n+\frac{\alpha+1}{p})} a_n e^{in\theta} \in L^p(\partial\mathbb{D}).$$

By the Binomial Theorem,

$$\frac{1}{(1-e^{-i\theta}z)^{\frac{\alpha+1}{p}}} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{\alpha+1}{p})}{k!\Gamma(\frac{\alpha+1}{p})} e^{-ik\theta} z^k.$$

Thus,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{\frac{\alpha+1}{p}}}\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} \frac{n!\Gamma(\frac{\alpha+1}{p})}{\Gamma(n+\frac{\alpha+1}{p})} a_n e^{in\theta}\right)\left(\sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{\alpha+1}{p})}{k!\Gamma(\frac{\alpha+1}{p})} e^{-ik\theta} z^k\right) d\theta \\
&= \sum_{n=0}^{\infty} a_n z^n = f(z).
\end{aligned}$$

This finishes the proof. □

Proof of Theorem 2. We need to consider three cases.

Case a1: $1 \leq p \leq 2$ and $\alpha = p-1$. Then $\mathcal{D}^p_\alpha = \mathcal{D}^p_{p-1} \subset H^p$ and the result in this case follows from Fatou's theorem for H^p.

Case a2: $1 \leq p \leq 2$ and $p-2 < \alpha < p-1$. If $f \in \mathcal{D}^p_\alpha$ then, using Theorem 6 we have that there exists $h(e^{i\theta}) \in L^p(\partial\mathbb{D})$ such that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{\frac{\alpha+1}{p}}}\, dt, = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{1-\frac{p-\alpha-1}{p}}}\, dt.$$

Notice that $p\left(\frac{p-\alpha-1}{p}\right) < 1$, so by part (a) of Theorem A of [23] we have that f has $(\alpha-p+2)^{-1}$-limit at a.e. $e^{i\theta} \in \partial\mathbb{D}$.

Case b: $1 < p \leq 2$ and $\alpha = p-2$. Using again Theorem 6 we have that if $f \in \mathcal{D}^p_\alpha$ then there exists $h(e^{i\theta}) \in \mathrm{L}^p(\partial\mathbb{D})$ such that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{1-\frac{1}{p}}}\, dt.$$

Using part (b) of Theorem A of [23] we deduce that f has $(p'-1)_{\exp}$-limit at a.e. $e^{i\theta} \in \partial\mathbb{D}$. □

Theorem 3 can be proved arguing as in the proof of part (a) of Theorem 1, using Lemma 2 instead of Lemma 1. Again, we shall omit the details.

4. Radial Variation of functions in the spaces $\mathcal{D}^p_\alpha$

Proof of Theorem 4. Let $0 < p < 1$ and $f \in \mathcal{D}^p_{p-1}$. Set

$$F_f = \{\theta \in [-\pi, \pi] : f \text{ has a finite non-tangential limit at } e^{i\theta}\}.$$

By (2) and Fatou's theorem, $[-\pi, \pi] \setminus F_f$ has Lebesgue measure 0. On the other hand, Zygmund proved in p. 81 of [30] that

$$(1-r)|f'(re^{i\theta})| \to 0, \quad \text{as } r \to 1^-,$$

for all $\theta \in F_f$. Consequently the set

$$F_f^\star = \{\theta \in [-\pi, \pi] : (1-r)|f'(re^{i\theta})| \to 0\}$$

is such that $[-\pi, \pi] \setminus F_f^\star$ has Lebesgue measure 0. Since $f \in \mathcal{D}^p_{p-1}$, we deduce that the set

$$T_f = \{\theta \in [-\pi, \pi] : \int_0^1 (1-r)^{p-1} |f'(re^{i\theta})|^p \, dr < \infty\}$$

is such that $[-\pi, \pi] \setminus T_f$ has Lebesgue measure 0. Thus, $[-\pi, \pi] \setminus \left(F_f^\star \cap T_f\right)$ has Lebesgue measure 0. Furthermore, if $\theta \in F_f^\star \cap T_f$ there exists a positive constant C_θ such that

$$V(f, \theta) = \int_0^1 |f'(re^{i\theta})|^p |f'(re^{i\theta})|^{1-p} \, dr \le C_\theta \int_0^1 (1-r)^{p-1} |f'(re^{i\theta})|^p \, dr < \infty.$$

□

Proof of Theorem 5. Since

$$\mathcal{D}^p_\alpha \subset \mathcal{D}^p_\beta, \quad -1 < \alpha \le \beta, \quad 0 < p < \infty,$$

(a) follows from Theorem 4.

Suppose now that $1 < p < 2$, $p - 2 < \alpha < p - 1$ and $f \in \mathcal{D}^p_\alpha$. Then the set

$$T_f^\alpha = \{\theta \in [-\pi, \pi] : \int_0^1 (1-r)^\alpha |f'(re^{i\theta})|^p \, dr < \infty\}$$

is such that $[-\pi, \pi] \setminus T_f^\alpha$ has Lebesgue measure 0. Now, using Hölder's inequality we see that there exists a positive constant $C_{\alpha,p}$ such that

$$\begin{aligned} V(f, \theta) &= \int_0^1 (1-r)^{\alpha/p} |f'(re^{i\theta})| (1-r)^{-\alpha/p} \, dr \\ &\le \left(\int_0^1 (1-r)^\alpha |f'(re^{i\theta})|^p \, dr\right)^{1/p} \left(\int_0^1 (1-r)^{-p'\alpha/p} \, dr\right)^{1/p'} \\ &\le C_{\alpha,p} \left(\int_0^1 (1-r)^\alpha |f'(re^{i\theta})|^p \, dr\right)^{1/p} < \infty, \end{aligned}$$

for all $\theta \in T_f^\alpha$. (We have used that $-p'\alpha/p > -1$ since $\alpha < p-1$.) Thus, (b) is proved.

(c) follows from the well known inclusion

$$\mathcal{D}^p_{p-2} = \mathcal{B}^p \subset \mathcal{B}^q = \mathcal{D}^q_{q-2}, \quad 1 < p < q < \infty,$$

(see, e. g., [3, p 112]), Theorem A and the fact that $\mathcal{B}^2 = \mathcal{D}$.

Finally, suppose that $2 < p < \infty$ and $f \in \mathcal{D}^p_\alpha$. Using Hölder's inequality with exponents $\frac{p}{p-2}$ and $p/2$, we have that

$$\begin{aligned} &\int_{\mathbb{D}} (1-|z|)^\beta |f'(z)|^2 \, dA(z) = \int_{\mathbb{D}} (1-|z|)^{\beta - \frac{2\alpha}{p}} |f'(z)|^2 \, (1-|z|)^{\frac{2\alpha}{p}} \, dA(z) \\ &\le \left(\int_{\mathbb{D}} (1-|z|)^{\frac{p\beta - 2\alpha}{p-2}} \, dA(z) \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{D}} (1-|z|)^\alpha |f'(z)|^p \, dA(z) \right)^{2/p}. \end{aligned} \tag{11}$$

Letting $\beta = 0$, we see that the condition $\alpha < \frac{p}{2} - 1$ implies that $f \in \mathcal{D}$. Hence, (e) follows from part (a) of Theorem A. On the other hand, if $p - 1 > \alpha \ge \frac{p}{2} - 1$ then β can be chosen so that $\beta > \frac{2}{p}(1+\alpha) - 1$ and $0 < \beta < 1$. Then (11) implies that $f \in \mathcal{D}^2_\beta$, and (d) follows from part (b) of Theorem A. □

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