INTEGRABILITY OF THE DERIVATIVE OF A BLASCHKE PRODUCT

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Abstract. We study the membership of derivatives of Blaschke products in Hardy and Bergman spaces, especially for interpolating Blaschke products and for those whose zeros lie in a Stolz domain. We obtain new and very simple proofs of some known results and prove new theorems that complement or extend the earlier works of Ahern, Clark, Cohn, Kim, Newman, Protas, Rudin, Vinogradov, and other authors.

Introduction

One of the central questions about Blaschke products is that of the membership of their derivatives in classical function spaces. This problem was studied by a number of authors in the 70's and 80's:

(1) for Hardy spaces H^p by Protas [30], Ahern - Clark [3], Ahern [1], and Cohn [10];

(2) for the related B^p spaces, the Banach envelopes of the Hardy spaces with exponents smaller than one, by Ahern and Clark [3], [4];

(3) for weighted Bergman spaces A^p_α by Ahern [1] and Kim [22].

The statements of many such results can be found in Colwell's monograph [11]. More recently, mean growth of the derivative of Blaschke products was investigated by Girela - González [18], Kutbi [23], and also by the first two authors of the present paper in [19]. A closely related question of the membership of Blaschke products in the Q_p spaces, which became an active area of study in the mid and late 90's, was investigated by Essén and Xiao [16] and Danikas and Mouratides [12], among others.

The Schwarz-Pick lemma readily implies that the derivative of any Blaschke product belongs to $\cap_{0<p<1} A^p$. Rudin [31] showed that there are Blaschke products whose derivative does not belong to A^1; see Piranian [29] for an explicit example. Ahern [2] found a necessary and sufficient condition for the membership of B' in the weighted space A^p_α. However, his criterion is expressed in terms of $|B(z)|$ and is therefore difficult to verify in practice. It is, thus, desirable to have criteria for membership of B' in, say, Hardy

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and Bergman spaces based just on some information on the zeros of B. The papers cited mainly considered conditions on the moduli of the zeros, as is the case of [30] and [22]. Many results obtained there appear to be sharp but the sharpness issue was not discussed.

In this paper we essentially determine the best possible exponents of integrability, either with respect to the area measure in the disk or with respect to the normalized arc length measure on the unit circle, of B' for those Blaschke products B which are either interpolating or have their zeros in a Stolz angle (or both). We obtain several new results and very simple proofs of some other, essentially known, ones. Besides our own new techniques we also employ those used earlier by Ahern, Clark, Naftalevich, and Vinogradov, among others.

Ahern and Clark [3] considered the condition that the zeros of B converge to 1 nontangentially (which is equivalent to requiring that the zeros belong to a Stolz angle) in Section 4 of *op. cit.*, again in conjunction with a condition on the moduli. By putting together various theorems from different sections of [3], one can readily deduce the following result: if the zeros of B belong to a Stolz angle, then $B' \in \cap_{0<p<1/2} H^p$ and the exponent one-half is sharp. While in our view the statement in this form deserves to be better known, we present it together with a new and very short proof in Section 1.

By a theorem of Hardy and Littlewood, $H^p \subset A^{2p}$ and the exponent $2p$ cannot be improved (see Theorem 5.6 of [13] and [24] or [34] for a simple proof). It is, thus, natural to ask whether the exponent one in the inclusion $B' \in \cap_{0<p<1} A^p$ is sharp if the zeros of B converge nontangentially to a point in the unit circle. However, a rather surprising phenomenon occurs here: $B' \in \cap_{0<p<3/2} A^p$, whenever the zeros of B lie in a Stolz angle. Though this can easily be deduced from Ahern's paper [1], we give a proof based only on the Schwarz lemma and the fact that $B' \in \cap_{0<p<1/2} H^p$. This is also part of Section 1. Our findings, thus, provide plenty of explicit examples of functions in a Hardy space that belong to a Bergman space with roughly the triple exponent of integrability.

We devote Section 2 to the interpolating Blaschke products. This is the term commonly used for those products whose zero sequences are uniformly separated (equivalently, interpolating for H^∞). For such products B and $1 < p < 2$ we show that $B' \in A^p$ if and only if $\sum_n (1 - |a_n|)^{2-p} < \infty$. This is a converse in this situation of a general result due to Kim [22].

In the same section, we prove the existence of an interpolating Blaschke product whose derivative does not belong to A^p for any $p > 1$. Nonetheless, for those interpolating Blaschke products whose zeros lie in a Stolz angle, we show that their derivative belongs to all H^p spaces with $p < 1$ and to all A^p spaces with $p < 2$, in accord with the usual pattern. Originally, we proved this statement using results of Danikas and Mouratides [12] and of Essén and Xiao [16] about the sequences of zeros of functions in Q_p spaces, $0 < p < 1$. Later on we found out that this can be deduced from the fact that an interpolating sequence in a Stolz angle is a finite union of exponential sequences, which is implicit in D.J. Newman's paper [26] (in the proof of his Theorem 3).

Throughout the paper the known lemmas and results that appear explicitly in other papers will be denoted by block capital letters. The results that seem new and those for which we do not have a specific reference will be enumerated.

1. Blaschke products with zeros in a Stolz angle

We consider the standard Hardy spaces H^p of analytic functions f in the unit disk $\mathbb{D}$ for which the Hardy-Littlewood integral means

$$M_p(r,f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p} \tag{1}$$

remain bounded as $r \to 1^-$. The reader is referred to [13] or [17] for the basic theory of these spaces. If a sequence of points $\{a_n\}$ in $\mathbb{D}$ satisfies the *Blaschke condition*: $\sum_{n=1}^\infty (1-|a_n|) < \infty$, the corresponding Blaschke product B is defined as

$$B(z) = \prod_{n=1}^\infty \frac{|a_n|}{a_n} \frac{a_n - z}{1-\overline{a_n}z}\,.$$

Such a product is analytic in $\mathbb{D}$, bounded by one, and with radial limits of modulus one almost everywhere on the unit circle.

1.1. **Membership of the derivative in Hardy spaces.**

There are several ways of saying that a point belongs to a Stolz angle. We will adopt the following one. Given $\xi \in \partial\mathbb{D}$ (the unit circle) and $\sigma \in [1,\infty)$ we set

$$\Omega_\sigma(\xi) = \{z \in \mathbb{D} : |1-\overline{\xi}z| \le \sigma(1-|z|)\}\,.$$

Any such domain $\Omega_\sigma(\xi)$, $1 \le \sigma < \infty$, will be called a *Stolz angle with vertex at* ξ. The domain $\Omega_\sigma(1)$ will be simply denoted by Ω_σ. The following observation which can be found in [33] will be useful in what follows:

$$\frac{1}{2+\sigma} \le \frac{|1-\overline{\lambda}z|}{|1-|\lambda||z||} \le 2+\sigma\,, \quad \text{whenever} \quad z \in \mathbb{D},\ \lambda \in \Omega_\sigma\,. \tag{2}$$

Lemma 1. *Given $\sigma \in [1,\infty)$ there exists a constant $K_\sigma > 0$ such that, if $a \in \Omega_\sigma$ then*

$$\left|\frac{1-z}{1-\overline{a}z}\right| \le K_\sigma\,, \quad \textit{for every } z \in \mathbb{D}\,.$$

Proof. We first consider the case when a is real and $0 < a < 1$. Set

$$S(z) = \frac{1-z}{1-az}\,, \quad z \in \mathbb{D}\,.$$

Bearing in mind that S is a Möbius transformation with real coefficients, we easily see that $S(\mathbb{D})$ is the disk of center $1/(1+a)$ and radius $1/(1+a)$ and is therefore contained in the disk of radius $2/(1+a)$ centered at the origin. Thus, we have

$$\left|\frac{1-z}{1-az}\right| \le \frac{2}{1+a} \le 2\,, \quad z \in \mathbb{D}\,, \quad 0 < a < 1\,.$$

Now for arbitrary a in Ω_σ, using (2) and the special case just considered, we obtain

$$\left|\frac{1-z}{1-\overline{a}z}\right| = \left|\frac{1-z}{1-|a|z}\right| \cdot \left|\frac{1-|a|z}{1-\overline{a}z}\right| \le 2(2+\sigma)\,.$$

Hence, we can choose $K_\sigma = 2(2+\sigma)$. □

Ahern and Clark ([3], Lemma 1, p. 121) found a very practical criterion for membership of the derivative of a Blaschke product in a Hardy space.

Lemma A. *If a Blaschke product B has zeros $a_n = r_n e^{it_n}$ then $B' \in H^p$ if and only if the function f defined on the unit circle by*

$$f(t) = \sum_{n=1}^{\infty} \frac{1-|a_n|}{(1-|a_n|)^2 + (t-t_n)^2}$$

belongs to $L^p(0,2\pi)$.

We now combine the above lemmas to give a simple proof of a theorem on integrability of the derivative of Blaschke products with zeros in a Stolz angle. The result is due to Ahern and Clark [3] although it was not stated there in unified way. Part (a) follows from their Theorem 12 in the case $\alpha = \gamma = 1$, with minor modifications of the domain $R(\delta,\gamma)$. Part (b) is contained in part (ii) of Theorem 8 in [3].

Our proof is based on a simple and direct idea. Namely, many complex analysts are familiar with the following exercise: if the zeros of a Blaschke product B lie on the radius $(0,1)$ and $f(z) = (1-z)^2 B(z)$, then f' is bounded in the unit disk (see, for example, Exercise 18, Chapter 15 of [32]). This gives the obvious information about the growth of B' as $z \to 1$ along the radius $[0,1)$. With the aid of Lemma 1 we can generalize the idea of this statement to a Stolz angle to obtain the proof of part (a) of the theorem below.

Theorem B. *(a) If the zeros of a Blaschke product B all lie in some Stolz angle, then $B' \in \cap_{0<p<1/2} H^p$.*

(b) The Blaschke product B with zeros $a_n = 1 - 1/(n\log^2 n)$, $n \ge 2$, has the property that $B' \notin H^{1/2}$.

Proof. (a) Without loss of generality we may assume that the Stolz angle has its vertex at $z = 1$. Since the function given by $f(z) = (1-z)^{-2}$ is a member of H^p whenever $p < 1/2$, it suffices to show that

$$(3) \qquad |B'(z)| \le \frac{C}{|1-z|^2}\,, \quad \text{whenever } z \in \mathbb{D}\,,$$

for some universal constant C.

Denoting by a_n the zeros of B, let us agree to write

$$(4) \qquad b_n(z) = \frac{|a_n|}{a_n}\frac{a_n - z}{1-\overline{a}_n z}\,, \quad B(z) = \prod_{n=1}^{\infty} b_n(z)\,, \quad B_n(z) = \frac{B(z)}{b_n(z)}\,.$$

Then

$$B'(z) = \sum_{n=1}^{\infty} b_n'(z) \cdot B_n(z)\,,$$

whence

$$|B'(z)| \leq \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2} |B_n(z)| \,.$$

Applying Lemma 1, we obtain

$$|(1-z)^2 B'(z)| \leq \sum_{n=1}^{\infty} (1-|a_n|^2) \left| \frac{1-z}{1-\overline{a}_n z} \right|^2 \leq 2K_\sigma^2 \sum_{n=1}^{\infty} (1-|a_n|) \,.$$

The statement follows since $\{a_n\}$ is a Blaschke sequence by assumption.

(b) We use Lemma A. In this case all $t_n = 0$, hence the function f becomes

$$f(t) = \sum_{n=2}^{\infty} \frac{1/(n\log^2 n)}{1/(n^2\log^4 n) + t^2} = \sum_{n=2}^{\infty} \frac{n\log^2 n}{1+t^2 n^2 \log^4 n} \,.$$

For $t>0$, let N_t be the unique number >1 with the property that

$$t N_t \log^2 N_t = 1 \,.$$

Then we have

$$f(t) \geq \sum_{2\leq n\leq N_t} n\log^2 n \asymp N_t^2 \log^2 N_t$$

by a standard argument involving summation by parts. Thus,

$$f(t)^{1/2} \geq C N_t \log N_t = \frac{C}{t\log N_t}$$

for a fixed positive constant C. Recalling the definition of N_t, we see that

$$\log N_t + 2\log\log N_t = \log\frac{1}{t}$$

and then it follows that $\log N_t \sim \log\frac{1}{t}$, as $t\to 0$. Consequently, there exist positive constants α and t_0 such that $f(t)^{1/2} \geq \alpha/(t\log\frac{1}{t})$ for all $t\in[0,t_0]$. Therefore

$$\int_0^{2\pi} f(t)^{1/2} dt \geq \alpha \int_0^{t_0} \frac{1}{t\log\frac{1}{t}}\, dt = \infty \,,$$

so $B' \notin H^{1/2}$ by Lemma A. $\Box$

It should be pointed out that the same exponent also seemed critical under other assumptions on the zeros considered in [30], [3], and [22].

1.2. **Membership of the derivative in Bergman spaces.** Denote by dA the normalized Lebesgue area measure on the disk: $dA(z) = \pi^{-1} r dr d\theta = \pi^{-1} dx dy$ and by $L^p(dA)$ the standard Lebesgue space equipped with the usual norm or metric, depending on whether $1\leq p<\infty$ or $0<p<1$. The Bergman space A^p is the closed subspace of $L^p(dA)$ consisting of analytic functions in $\mathbb{D}$. We refer the reader to [7], [21], and [14] for the theory of these spaces.

Our goal in this subsection is to obtain a result analogous to part (a) of Theorem B for Bergman spaces.

By a *holomorphic self-map* of the disk we mean an analytic functions φ in $\mathbb{D}$ for which $\varphi(\mathbb{D}) \subset \mathbb{D}$. The following simple statement based on the Schwarz-Pick lemma will be useful throughout the paper.

PROPOSITION 2. *Let φ be an arbitrary holomorphic self-map of the disk. If $\varphi' \in H^p$ then $\varphi' \in A^{p+1-\varepsilon}$ for any positive and sufficiently small ε.*

Proof. By the Schwarz-Pick lemma we have

$$\begin{aligned}
\int_{\mathbb{D}} |\varphi'(z)|^{p+1-\varepsilon} dA(z) &\leq \int_{\mathbb{D}} |\varphi'(z)|^p \left(\frac{1-|\varphi(z)|^2}{1-|z|^2}\right)^{1-\varepsilon} dA(z) \\
&\leq \int_{\mathbb{D}} |\varphi'(z)|^p \left(\frac{1}{1-|z|}\right)^{1-\varepsilon} dA(z) \\
&= 2\int_0^1 M_p^p(r,\varphi') \frac{r}{(1-r)^{1-\varepsilon}} dr \\
&\leq 2\|\varphi'\|_{H^p}^p \int_0^1 \frac{1}{(1-r)^{1-\varepsilon}} dr \\
&= \frac{2}{\varepsilon} \cdot \|\varphi'\|_{H^p}^p ,
\end{aligned}$$

and the statement follows. □

Theorem B and Proposition 2 immediately yield the following slightly surprising result.

THEOREM 3. *If the zeros of a Blaschke product B all lie in some Stolz angle, then $B' \in \cap_{0<p<3/2} A^p$.*

One usually encounters the following type of examples regarding the membership in Hardy and Bergman spaces: either (1) lacunary series which belong to many (or all) Bergman spaces, but lie in no Hardy space, or (2) functions that follow the usual pattern of belonging to H^p for $p < p_0$ and to A^q when $q < 2p_0$; for example, this happens with the appropriate negative powers of $1-z$ and with conformal maps onto angular domains.

Theorem B can also be deduced from Theorem 6.1 of Ahern's paper [1] and Theorem B by means of the Schwarz-Pick lemma and integration in θ, and the proof works even for arbitrary inner functions. In summary, our contribution here consists in giving an even simpler proof using Proposition 2. This explains why $3/2$ instead of simply 1 in the exponent. Yet another proof, inspired by ideas from Vinogradov's paper [33], can be found in the second author's thesis [28].

Is is natural to ask whether or not the exponent three-halves is sharp. Interpolating Blaschke products do not help here. Indeed, the following result shows that no counterexample (if any) for the exponent three-halves in Theorem 3 is possible with interpolating sequences.

THEOREM 4. *Let B be an interpolating Blaschke product whose zero sequence lies in a Stolz angle. Then*

(i) *$B' \in H^p$ for all $p \in (0,1)$.*
(ii) *$B' \in A^p$ for all $p \in (0,2)$.*

This is one of the results that we shall prove in the next section which will be mainly devoted to interpolating Blaschke products.

2. Derivatives of interpolating Blaschke products

In order to see where we stand in a general situation, let us recall two results due to Protas and Cohn respectively.

Protas [30] studied the membership of general Blaschke products in Hardy spaces in terms of their zero sequences and proved the following.

Theorem C. *Whenever $\{a_n\}$ is the zero set of a Blaschke product B and*

$$\sum_{n=1}^{\infty}(1-|a_n|)^{1-p}<\infty\,,\quad 1/2<p<1\,, \tag{5}$$

then $B'\in H^p$.

For a generalization, see Theorem 6.2 of Ahern [1]. Cohn [10] proved the converse of Theorem C for interpolating Blaschke products. This will be discussed in Subsection 2.2.

The following result is due to H.O. Kim [22].

Theorem D. *Let $\{a_n\}$ be the zero set of a Blaschke product B and suppose that also*

$$\sum_{n=1}^{\infty}(1-|a_n|)^{2-p}<\infty\,,\quad 1<p<2\,, \tag{6}$$

Then $B'\in A^p$.

The theorems above have the typical flavor of most results that appear in the literature when only the absolute values of the zeros are taken into account. Imposing additional conditions helps improve the exponent of integrability of the derivative in most situations; that is precisely the *leitmotiff* of this article (and many others already published on this topic).

We will show that Theorem D also has a converse for interpolating sequences.

2.1. **Background on interpolating sequences.** We first review some background on interpolating sequences. Throughout this section, ϱ will denote the pseudo-hyperbolic metric in the unit disk:

$$\varrho(z,w)=\left|\frac{z-w}{1-\overline{w}z}\right|\,,\quad z,w\in\mathbb{D}\,.$$

The Schwarz-Pick lemma tells us that $\varrho(f(z),f(w))\le\varrho(z,w)$ whenever f is a holomorphic self-map of $\mathbb{D}$ and z, $w\in\mathbb{D}$. Also, equality holds whenever f is a disk automorphism, *i.e.* ϱ is invariant under the conformal maps of the disk onto itself.

A *pseudohyperbolic disk* of (pseudohyperbolic) center a and radius r ($a\in\mathbb{D}$, $0<r<1$) is the set $\Delta(a,r)=\{z\in\mathbb{D}:\varrho(a,z)<r\}$. It coincides with the Euclidean disk whose (Euclidean) radius and center are (see [7]):

$$R=\frac{1-|a|^2}{1-r^2|a|^2}r\,,\qquad c=\frac{1-r^2}{1-r^2|a|^2}a\,. \tag{7}$$

A sequence $\{a_n\}$ of points in $\mathbb{D}$ is said to be *uniformly separated* if it satisfies the condition:

$$\inf_n \prod_{m\neq n} \varrho(a_m, a_n) = \inf_n B_n(a_n) \geq \delta > 0 \tag{8}$$

(using our earlier notation $B_n = B/b_n$ as in (4)). By a celebrated theorem of Carleson [8] (see also [13]), $\{a_n\}$ is an interpolating sequence for the space H^∞ of bounded analytic functions in the disk if and only if it satisfies (8). The Blaschke products whose zeros obey this condition are denominated *interpolating Blaschke products* and constitute an important and much studied class of Blaschke products.

2.2. **Interpolating Blaschke products.** Cohn [10] proved the following converse of Protas' Theorem C for interpolating sequences.

Theorem E. *If $\{a_n\}$ is the zero set of an interpolating Blaschke product B such that $B' \in H^p$, $1/2 < p < 1$, then*

$$\sum_{n=1}^{\infty} (1-|a_n|)^{1-p} < \infty\,. \tag{9}$$

The following statement shows that our Theorem D allows for a converse in the case of interpolating sequences. We use the notation $\gtrsim$ for signifying that one positive quantity is greater than another times a fixed constant and $\asymp$ for saying that the quotient of two positive quantities is bounded from above and away from zero.

Theorem 5. *If B is an interpolating Blaschke product with zeros $\{a_n\}$ then*

$$\int_{\mathbb{D}} |B'(z)|^p dA(z) \gtrsim \sum_{n=1}^{\infty} (1-|a_n|)^{2-p}\,.$$

In particular, if the series on the right diverges, then $B' \notin A^p$.

This theorem is a converse of Theorem 3.1 in Kim's paper [22]. The statement is non-trivial only when $1 < p < 2$. Of course, it adds nothing new when $0 < p \leq 1$ because we already have the Blaschke condition. On the other hand, we know that if B is an arbitrary infinite Blaschke product then $B' \notin A^p$ for any $p \geq 2$, see Theorem 1.1 of [22].

In order to prove Theorem 5 we need the following lemma.

Lemma 6. *Let $\{a_n\}_{n=1}^\infty$ be a uniformly separated sequence of points in the unit disk $\mathbb{D}$ with the constant δ as in (8), and let B be the Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^\infty$. Then there exist two positive constants α and β which depend only on δ such that the pseudohyperbolic disks $\{\Delta(a_n,\alpha)\}_{n=1}^\infty$ are pairwise disjoint and*

$$|B'(z)| \geq \frac{\beta}{1-|a_n|}\,, \quad \textit{for all} \;\; z \in \Delta(a_n,\alpha), \quad n = 1, 2, \ldots \tag{10}$$

Proof. Let b_n and $B_n = B/b_n$ be defined as in (4). Using (8), the definition of ϱ, the triangle inequality, and the Schwarz-Pick lemma applied to the self-map B_n of $\mathbb{D}$, we obtain

$$\begin{aligned}\delta &\leq |B_n(a_n)| \\ &= \varrho(B_n(a_n), 0) \\ &\leq \varrho(B_n(a_n), B_n(z)) + \varrho(B_n(z), 0) \\ &\leq \varrho(a_n, z) + |B_n(z)|\,,\end{aligned}$$

for all $z \in \mathbb{D}$. It follows that

$$|B_n(z)| \geq \frac{\delta}{2}\,, \quad \text{whenever} \quad z \in \Delta(a_n, \frac{\delta}{2})\,. \tag{11}$$

We remark that a very similar proof of the above formula can be found in a recent paper by Nolder ([27], p. 1800).

Continuing with the proof of the lemma, we have

$$\begin{aligned}|B'(z)| &= |b_n'(z)B_n(z) + b_n(z)B_n'(z)| \\ &\geq |b_n'(z)||B_n(z)| - |b_n(z)||B_n'(z)|\,. \qquad (12)\end{aligned}$$

Now,

$$|b_n'(z)| = \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2} = \frac{1-\varrho(z,a_n)^2}{1-|z|^2} \geq \frac{1-\varrho(z,a_n)}{1-|z|^2}\,. \tag{13}$$

Also, by the Schwarz-Pick lemma applied to B_n,

$$|B_n'(z)| \leq \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}\,.$$

Together with (12), (13), and (11), this gives

$$|B'(z)| \geq \frac{1}{1-|z|^2}\left(\left(1-\frac{\delta}{2}\right)\frac{\delta}{2} - \frac{\delta}{4}\right) \geq \frac{\delta(1-\delta)}{4(1-|z|^2)} \geq \frac{\delta(1-\delta)}{8(1-|z|)}\,, \tag{14}$$

whenever $z \in \Delta(a_n, \frac{\delta}{4})$.

Set $\alpha = \delta/4$. Bearing in mind that $\Delta(a_n, \alpha)$ is the Euclidean disk of center $\frac{1-\alpha^2}{1-\alpha^2|a_n|^2} \cdot a_n$ and radius $\frac{1-|a_n|^2}{1-\alpha^2|a_n|^2} \cdot \alpha$, an easy calculation gives that

$$1-|z| \leq \frac{4}{1-\alpha}\,(1-|a_n|), \quad z \in \Delta(a_n, \alpha)\,,$$

and then (14) implies

$$|B'(z)| \geq \frac{\delta(1-\delta)(1-\alpha)}{32} \cdot \frac{1}{1-|a_n|}, \quad z \in \Delta(a_n, \alpha)\,.$$

The sequence $\{a_n\}$ satisfies

$$\varrho(a_m, a_n) \geq \delta\,, \quad \text{whenever} \quad m \neq n\,.$$

Since $\alpha < \delta/2$, the triangle inequality implies that the pseudo-hyperbolic disks $\Delta(a_n, \alpha)$ are pairwise disjoint. Hence, the lemma is proved with $\alpha = \delta/4$ and $\beta = \delta\,(1-\delta)\,(1-\alpha)/32$. $\square$

Proof of Theorem 5. Let δ be the constant from the uniform separation condition (8). Then the pseudo-hyperbolic disks $\Delta(a_n, \alpha)$ chosen as in Lemma 6 are pairwise disjoint. Denote by R_k the Euclidean radius of such a disk. Then

$$R_k = \frac{1-|a_n|^2}{1-\alpha^2|a_n|^2}\alpha \asymp 1-|a_n| \quad \text{as} \quad n \to \infty\,. \tag{15}$$

Using the property that the disks $\Delta(a_n, \alpha)$ are pairwise disjoint, inequality (10), and asymptotic relation (15), we obtain

$$\begin{aligned}\int_{\mathbb{D}} |B'|^p dA \quad &\geq \quad \sum_{n=1}^{\infty} \int_{\Delta(a_n,\alpha)} |B'|^p dA \\ &\gtrsim \quad \sum_{n=1}^{\infty} \Big(\frac{\beta}{1-|a_n|}\Big)^p \cdot R_k^2 \\ &\asymp \quad \sum_{n=1}^{\infty} (1-|a_n|)^{2-p}\,,\end{aligned}$$

and we are done. □

¿From Kim's Theorem D and our Theorem 5 we easily derive the following characterization.

Corollary 7. *Let $\{a_n\}$ be an interpolating sequence, B the corresponding Blaschke product, and $1 < p < 2$. Then $B' \in A^p$ if and only if*

$$\sum_{n=1}^{\infty} (1-|a_n|)^{2-p} < \infty\,.$$

2.3. **Exponential zero sequences versus other interpolating sequences.** We wish to compare the results about integrability of the derivative of Blaschke products from three distinguished classes.

The first class is that of exponential sequences. Recall that $\{a_n\}$ is said to be an *exponential sequence* in $\mathbb{D}$ if

$$1 - |a_{n+1}| \leq q\,(1-|a_n|) \tag{16}$$

for some fixed q, $0 < q < 1$, and all n. As is well known, every exponential sequence is interpolating (*i.e.*, uniformly separated); see Chapter 9 of [13].

Kim [22] proved that if the zeros of B form an exponential sequence, then $B' \in \cap_{0<p<2} A^p$. Actually, using Theorem C we can deduce the stronger result that $B' \in \cap_{0<p<1} H^p$. This argument leads to a proof of Theorem 4.

Proof of Theorem 4. Let B be an interpolating Blaschke product whose sequence of zeros $\{a_n\}_{n=1}^{\infty}$ is contained in a Stolz angle. Using a result of Newman (see p. 506 of [26]), we deduce that $\{a_n\}_{n=1}^{\infty}$ is a finite union of exponential sequences and then it follows that $\sum_{n=1}^{\infty}(1-|a_n|)^p < \infty$, for all positive p. Then Protas' Theorem C yields that $B' \in \cap_{0<p<1} H^p$. This is (a). The inclusion $H^p \subset A^{2p}$ $(0 < p < \infty)$ yields (b). □

We should point out that the exponents one and two in Theorem 4 are both sharp. We have already seen this for A^p spaces; for H^p spaces this follows either from the inclusion $H^p \subset A^{2p}$ or from a classical theorem of Privalov (see Theorem 3.11 of [13]).

It is a matter of simple calculus to check that every exponential sequence satisfies the following condition:

$$\sum_{n=1}^{\infty} (1-|a_n|) \log \frac{1}{1-|a_n|} < \infty \,, \tag{17}$$

which has also been considered by various authors. For example, Rudin [31] proved that if the zeros of B satisfy condition (17) then $B' \in A^1$.

The following very useful theorem is due to Naftalevich [25]. Even though it is almost 50 years old and is mentioned in [11] and listed as an exercise in [17], it does not seem to be widely used. The original paper [25] is not easily accessible, but the reader can find a detailed proof in Cochran's paper [9].

Theorem F. *For any Blaschke sequence $\{a_n\}$, there exists an interpolating sequence $\{z_n\}$ such that $|z_n| = |a_n|$ for each n.*

Note that there are interpolating sequences which do not satisfy (17). This can be seen by appealing to Naftalevich's Theorem F: pick a sequence $\{a_n\}$ for which (17) does not hold but Blaschke condition does; then rotate its terms so as to get an interpolating sequence $\{b_n\}$ with $|b_n| = |a_n|$.

A sequence satisfying (17) need not be interpolating: for example, consider $a_n = 1 - n^{-2}$. Hence, the two classes are not comparable, but each of them contains the class of exponential sequences as a relatively small subclass.

How large can the exponent of integrability for B' be for an interpolating Blaschke product? We already mentioned in the introduction that the derivative of any Blaschke product necessarily belongs to all A^p, $0 < p < 1$. Combining Naftalevich's Theorem F with our Theorem 5 we obtain the following result.

Theorem 8. *There exists an interpolating Blaschke product B such that $B' \notin \cup_{1<p<\infty} A^p$.*

Proof. Note that the sequence $a_n = 1 - (n \log^2 n)^{-1}$ satisfies the Blaschke condition. By Naftalevich's theorem, there exists a sequence $\{b_n\}$ which is interpolating for H^∞ and such that $|b_n| = a_n$ for all n. Now whenever $p > 1$, we have

$$\sum_{n=1}^{\infty} (1-|b_n|)^{2-p} = \sum_{n=1}^{\infty} \frac{1}{n^{2-p} \log^{2(2-p)} n} = \infty \,,$$

and the claim follows from Theorem 5 by choosing the interpolating Blaschke product with zeros b_n. $\square$

To the best of our knowledge, it still seems an open question (and quite an interesting one) whether there exists an interpolating Blaschke product whose derivative is not in A^1. Note that this Bergman space coincides with the Banach space $B^{1/2}$ from the family of spaces studied by a number of authors (see [15], for example). This case seems to have been a stumbling block for a long time already.

2.4. **An approach via Q_p spaces and Carleson measures.** As we mentioned in the introduction, there is a nice connection between the membership of B' in Hardy and Bergman spaces on the one hand, and of B in Q_p spaces on the other hand.

When $0 < p < \infty$, an analytic function f in $\mathbb{D}$ belongs to the *space* Q_p if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^p \, dA(z) < \infty \,,$$

where g denotes the Green function for the disk given by

$$g(z,a) = \log \left| \frac{1-\overline{a}z}{a-z} \right| , \quad z, a \in \mathbb{D} , \quad z \neq a \,.$$

The spaces Q_p are conformally invariant. They have their origin in the papers [35] where it was shown that $Q_2 = \mathcal{B}$ (the Bloch space) and [5] where this result was extended by showing that $Q_p = \mathcal{B}$ for all $p > 1$. The space Q_1 coincides with $BMOA$. When $0 < p < 1$, Q_p is a proper subspace of $BMOA$ and has many interesting properties (see, [16], [6], or the recent detailed monograph [36]).

There are various characterizations of Q_p spaces. The one that will be useful for us is expressed in terms of p-Carleson measures. Denote by $|I|$ the length of an interval I on the unit circle $\mathbb{T}$. The *Carleson square* $S(I)$ is defined as

$$S(I) = \{re^{i\theta} : e^{i\theta} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\} \,.$$

Given a positive Borel measure μ in $\mathbb{D}$, we say that μ is a *p-Carleson measure* on $\mathbb{D}$ if there exists a positive constant C such that

$$\mu\big(S(I)\big) \leq C|I|^p, \qquad \text{for every interval } I \subset \mathbb{T}. \tag{18}$$

The special case $p = 1$ yields the classical Carleson measures (*cf*. [13]). The following characterization of Q_p spaces was obtained by Aulaskari, Stegenga and Xiao [6].

Theorem G. *Let $0 < p < \infty$. A function f holomorphic in $\mathbb{D}$ is a member of Q_p if and only if the measure μ on $\mathbb{D}$ defined by $d\mu_p(z) = (1-|z|^2)^p |f'(z)|^2 \, dA(z)$ is a p-Carleson measure.*

Essén and Xiao [16] used this result to characterize the inner functions that belong to Q_p spaces ($0 < p < 1$). In particular, their result can be stated for Blaschke products as follows.

Theorem H. *Let $p \in (0,1)$ and let B be a Blaschke product with zero sequence $\{a_n\}_{n=1}^\infty$. Then $B \in Q_p$ if and only if the measure μ_p defined as*

$$d\mu_p = \sum_{n=1}^\infty (1-|a_n|^2)^p \delta_{a_n}$$

is a p-Carleson measure. Here, as is usual, δ_{a_n} denotes the point mass at a_n.

The following concept was introduced in [12]: a sequence $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_n \geq 0$, $\alpha_n \geq \alpha_{n+1}$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \alpha_n = 0$, is said to be *asymptotically concentrated* if for each $k = 1, 2, \ldots$ there exists an increasing infinite subsequence n_p of positive integers that only depends on k, with the property that

$$\lim_{p\to\infty} \frac{\alpha_{n_p}}{\alpha_{n_p+k}} = 1\,. \tag{19}$$

Using this concept and Theorem H, Danikas and Mouratides [12] obtained a sufficient condition for membership of a Blaschke product in $\cap_{0<p<1} Q_p$ expressed only in terms of absolute values of its zeros:

Theorem I. *Let B a Blaschke product. Let its zeros be arranged in an increasing order of moduli: $|a_n| \le |a_{n+1}|$, $n \ge 1$. Also, for each n, let $\alpha_n = 1 - |a_n|$. If the sequence $\{\alpha_n\}$ is not asymptotically concentrated, then $B \in \cap_{0<p<1} Q_p$.*

Using the Lemma on p. 201 of [12], Lemma 1 on p. 150 of [13], Theorem I, and Theorem H, one can deduce the following result (see [28] for the details).

Theorem 9. *Let B be an interpolating Blaschke product whose zero sequence $\{a_n\}_{k=1}^{\infty}$ is contained in a Stolz angle. Then $B \in \bigcap_{0<p<1} Q_p$. Consequently, if $0 < p < 1$ then the measure μ_p given by $d\mu_p = \sum_{k=1}^{\infty} (1 - |a_n|^2)^p \delta_{a_n}$ is a p-Carleson measure (and, hence, a finite measure).*

This and Theorem C yield Theorem 4 as above, thus providing an alternative proof. We remark that, of course, Theorem 9 can also be deduced using the above mentioned result of Newman and Theorem H.

Added in proof

After the submission of this paper, the first two authors have proved that the exponent 3/2 in Theorem 3 is sharp: Theorem 1 of [20] asserts that the Blaschke product B with zeros $a_n = 1 - 1/(n \log^2 n)$, $n \ge 2$, has the property that $B' \notin A^{3/2}$.

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