ON THE MEMBERSHIP IN BERGMAN SPACES OF THE DERIVATIVE OF A BLASCHKE PRODUCT WITH ZEROS IN A STOLZ DOMAIN

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ABSTRACT. It is known that the derivative of a Blaschke product whose zero sequence lies in a Stolz angle belongs to all the Bergman spaces A^p with $0 < p < 3/2$. The question of whether this result is best possible remained open. In this paper, for a large class of Blaschke products B with zeros in a Stolz angle, we obtain a number of conditions which are equivalent to the membership of B' in the space A^p $(p > 1)$. As a consequence, we prove that there exists a Blaschke product B with zeros on a radius such that $B' \notin A^{3/2}$.

1. Introduction. We denote by $\mathbb{D}$ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and by H^p $(0 < p \leq \infty)$ the classical Hardy spaces of analytic functions in $\mathbb{D}$ (see [3]). The Bergman space A^p $(0 < p < \infty)$ consists of all functions f analytic in $\mathbb{D}$ which belong to $L^p(\mathbb{D}, dA)$, where $dA(z) = \frac{1}{\pi} dx\, dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We mention [4] and [6] as general references for the theory of Bergman spaces.

A sequence $\{a_n\}$ of points in $\mathbb{D}$ is said to be a Blaschke sequence if $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. The corresponding Blaschke product B is defined as $B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$.

If $\xi \in \partial\mathbb{D}$ and $\sigma \in (1, \infty)$, we set $\Omega_\sigma(\xi) = \{z \in \mathbb{D} : |1 - \overline{\xi} z| \leq \sigma(1 - |z|)\}$. The domains $\Omega_\sigma(\xi)$ $(1 < \sigma < \infty)$ are called Stolz angles with vertex at ξ. The domain $\Omega_\sigma(1)$ will be simply denoted by Ω_σ.

If a Blaschke product B has zeros $a_n = r_n e^{it_n}$, we define

$$f_B(t) = \sum_{a_n \neq 0} \frac{1 - |a_n|}{(1 - |a_n|)^2 + (t - t_n)^2}, \quad t \in (-\pi, \pi).$$

Ahern and Clark ([2], Lemma 1, p. 121) proved that

$$B' \in H^p \;\Leftrightarrow\; f_B \in L^p(-\pi, \pi), \quad 0 < p < \infty. \tag{1}$$

Using this criterion we can deduce:

(i) If the zeros of a Blaschke product B all lie in some Stolz angle, then $B' \in \cap_{0<p<1/2} H^p$.

(ii) If B is the Blaschke product with zeros $a_n = 1 - 1/(n \log^2 n)$, $n \geq 2$, then $B' \notin H^{1/2}$.

2. The main results. Even though we do not have a Bergman space analogue of (1), using Theorem 6.1 of [1] (see also Theorem 3 of [5]), it follows that if the zeros of a Blaschke product B all lie in some Stolz angle, then $B' \in A^p$ for all $p \in (0, 3/2)$. We shall prove that the exponent $3/2$ is sharp in this result even for Blaschke products with zeros on a radius.

THEOREM 1. *The Blaschke product B with zeros $a_n = 1 - 1/(n \log^2 n)$, $n \geq 2$, has the property that $B' \notin A^{3/2}$.*

For a large class of Blaschke products B with zeros in a Stolz angle, we shall obtain a number of conditions which are equivalent to the membership of B' in the space A^p $(1 < p < \infty)$. Theorem 1 will follow from these results. We remark that if B is an arbitrary

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infinite Blaschke product, $B' \notin A^p$ for any $p \geq 2$ (see Theorem 1.1 of [7]). Hence, our coming results are really significant only for $3/2 \leq p < 2$.

Following Vinogradov [9], if B is the Blaschke product with zeros $\{a_n\}_{n=1}^{\infty}$, we define

$$\varphi_B(\theta) = \sum_{a_n \neq 0} \frac{1-|a_n|}{[\theta + (1-|a_n|)]^2}, \quad \theta \in (0,\infty). \tag{2}$$

We shall prove the following result.

THEOREM 2. *Let B be a Blaschke product whose sequence of zeros lies in a Stolz angle. If there exist a positive constant C and $\theta_0 \in (0,\pi)$ such that*

$$\theta\varphi_B(\theta) \geq C \quad \text{for all} \quad \theta \in (0,\theta_0), \tag{3}$$

then, for any given $p \in (1,\infty)$, we have that $B' \in A^p$ if and only if $\varphi_B \in L^{p-1}(0,1)$.

Theorem 1 can be deduced from Corollary 2 below but here we give a direct proof using Theorem 2.

Proof of Theorem 1. If B is the Blaschke product considered in Theorem 1 then

$$\varphi_B(\theta) = \sum_{n=2}^{\infty} \frac{1-|a_n|}{[\theta + (1-|a_n|)]^2} = \sum_{n=2}^{\infty} \frac{n\log^2 n}{[1+\theta n \log^2 n]^2}. \quad \theta > 0.$$

For $0 < \theta < 1$, let N_θ be the unique number greater than 1 such that $\theta N_\theta \log^2 N_\theta = 1$. By a standard argument involving summation by parts, we have

$$\varphi_B(\theta) \geq \frac{1}{4} \sum_{2 \leq n \leq N_\theta} n \log^2 n \asymp N_\theta^2 \log^2 N_\theta = \frac{N_\theta}{\theta} = \frac{1}{\theta^2 \log^2 N_\theta}.$$

Now, the definition of N_θ easily implies that $\log N_\theta \sim \log \frac{1}{\theta}$, as $\theta \to 0$. Then it follows that there exist a positive constant C and $\theta_0 \in (0,1)$ such that

$$\varphi_B(\theta) \geq C \frac{1}{\theta^2 \log^2 \frac{1}{\theta}}, \quad 0 < \theta < \theta_0.$$

This implies that $\varphi_B \notin L^{1/2}(0,1)$. Then using Theorem 2 we deduce that $B' \notin A^{3/2}$. $\square$

Theorem 2 follows immediately from Theorem 3.

THEOREM 3. *Suppose that $1 \leq p < \infty$ and $\sigma > 1$, and let B be a Blaschke product whose zeros lie in a Stolz angle. Then there exist $C_1 > 0$, $C_2 > 0$, $M > 0$ and $\theta_0 \in (0,\pi)$ such that*

$$C_1 \int_0^{2\pi} \varphi_B^{p-1}(\theta)d\theta \geq \int_{\mathbb{D}} |B'(z)|^p dA(z) \geq C_2 \int_0^{\theta_0} \varphi_B^{p-1}(\theta) \left(1 - e^{(-M\theta\varphi_B(\theta))}\right) d\theta. \tag{4}$$

A number of results which will be needed to prove Theorem 3. The pseudo-hyperbolic metric in the unit disc will be denoted by ϱ: $\varrho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|$, $z,w \in \mathbb{D}$. The following result, which is due to Marshall and Sarason, is proved in Proposition 4 of [8].

PROPOSITION A. *Let K be a closed convex subset of $\overline{\mathbb{D}}$ with $0 \in K$. Let B be a Blaschke product whose zeros $\{a_n\}$ are all contained in K. If $z \in \mathbb{D} \setminus K$ and $\varepsilon = \varrho(z,K)$, then*

$$|B'(z)| \geq \frac{2\varepsilon}{1+\varepsilon^2} \frac{|B(z)|}{1-|z|^2} \sum_{n=1}^{\infty} \left(1 - \varrho^2(z,a_n)\right)$$

The following lemma can be proved using simple geometric arguments.

LEMMA 1. *Given $\sigma > 1$ and $0 < \delta < 1$ there exists $\overline{\sigma} > \sigma$ such that $\rho(z,\Omega_\sigma) \geq \delta$ for every $z \in \mathbb{D} \setminus \Omega_{\overline{\sigma}}$.*

LEMMA 2. *Let B be the Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^\infty$ and let $\delta \in (0,1)$. If $z \in \mathbb{D}$ satisfies that $\varrho(z, a_n) \geq \delta$, for all n, then*

$$|B(z)| \geq \exp\left(-\frac{1}{2\delta^2} \sum_{n=1}^{\infty} \left(1 - \varrho^2(z, a_n)\right) \right). \tag{5}$$

Proof. Take $z \in \mathbb{D}$ such that $\varrho(z, a_n) \geq \delta$ for all $n = 1, 2, \dots$, then using the elementary inequality $\log x \leq x - 1$, for $x \geq 1$, we deduce that

$$\log \frac{1}{|B(z)|} = \frac{1}{2} \sum_{n=1}^{\infty} \log \frac{1}{\varrho^2(z, a_n)} \leq \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{\varrho^2(z, a_n)} - 1 \right) \leq \frac{1}{2\delta^2} \sum_{n=1}^{\infty} \left(1 - \varrho^2(z, a_n)\right),$$

which implies (5). □

We shall use also the two following elementary lemmas.

LEMMA 3. *Given $R \in (0,1)$, there exists $C_R \in (0,1)$ such that*

$$C_R\big[(1-r) + (1-\varrho) + |t|\big] \leq |1 - \varrho r e^{it}| \leq (1-r) + (1-\varrho) + |t| \ , r, \varrho \in [R, 1) \ t \in [-\pi, \pi]. \tag{6}$$

LEMMA 4. *If $\sigma > 1$ then $\frac{1}{2+\sigma} \leq \frac{|1-\overline{\lambda}z|}{|1-|\lambda|z|} \leq 2 + \sigma$, whenever $z \in \mathbb{D}$ and $\lambda \in \Omega_\sigma$.*

Proof of Theorem 3. Take $p \geq 1$ and assume, without loss of generality, that B is a Blaschke product with $B(0) \neq 0$ whose sequence of zeros $\{a_n\}_{n=1}^\infty$ lies in the Stolz angle Ω_σ ($\sigma > 1$). Write φ for φ_B.

There exists $R \in (0,1)$ such that $|a_n| \geq R$, for all n. Let C_R be the constant associated to R by Lemma 3. Fix a number $\delta \in (0,1)$. Using Lemma 1, we can take $\overline{\sigma} > \sigma$ such that $\varrho(z, \Omega_\sigma) \geq \delta$, for all $z \in \mathbb{D} \setminus \Omega_{\overline{\sigma}}$. Using Proposition A with $K = \overline{\Omega_\sigma}$ and bearing in mind that the function $x \mapsto \frac{2x}{1+x^2}$ is increasing in $(0,1)$, we obtain that, for every $z \in \mathbb{D} \setminus \Omega_{\overline{\sigma}}$,

$$|B'(z)| \geq \frac{2\varrho(z, \Omega_\sigma)}{1 + \varrho(z, \Omega_\sigma)^2} \frac{|B(z)|}{1 - |z|^2} \sum_{n=1}^{\infty} \left(1 - \varrho^2(z, a_n)\right) \geq \frac{2\delta}{1+\delta^2} |B(z)| \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2}.$$

If $z \in \mathbb{D} \setminus \Omega_{\overline{\sigma}}$, then $\varrho(z, a_n) \geq \delta$ for all n. Lemma 2 and the above inequality yield

$$|B'(z)| \geq \frac{2\delta}{1+\delta^2} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2} \exp\left(-\frac{1}{2\delta^2} \sum_{n=1}^{\infty} \frac{(1 - |z|^2)(1 - |a_n|^2)}{|1 - \overline{a_n} z|^2} \right), \ z \in \mathbb{D} \setminus \Omega_{\overline{\sigma}}. \tag{7}$$

Using (7), Lemma 4 and Lemma 3, we see that if $z = re^{it} \in \{z \in \mathbb{D} : |z| \geq R\} \setminus \Omega_{\overline{\sigma}}$

$$\begin{aligned}
&|B'(z)| \geq \tfrac{2\delta}{1+\delta^2} \textstyle\sum_{n=1}^\infty \tfrac{1-|a_n|^2}{|1-\overline{a_n}z|^2} \exp\left(-\tfrac{1}{2\delta^2} \textstyle\sum_{n=1}^\infty \tfrac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a_n}z|^2} \right) \\
&\geq \tfrac{2\delta}{(1+\delta^2)(2+\sigma)^2} \textstyle\sum_{n=1}^\infty \tfrac{1-|a_n|^2}{|1-|a_n|z|^2} \exp\left(-\tfrac{(2+\sigma)^2}{2\delta^2} \textstyle\sum_{n=1}^\infty \tfrac{(1-|z|^2)(1-|a_n|^2)}{|1-|a_n|z|^2} \right) \\
&\geq \tfrac{2\delta}{(1+\delta^2)(2+\sigma)^2} \textstyle\sum_{n=1}^\infty \left(\tfrac{1-|a_n|^2}{[(1-r)+(1-|a_n|)+|t|]^2} \right) \cdot \\
&\quad \cdot \exp\left(-\tfrac{(2+\sigma)^2}{2\delta^2} \textstyle\sum_{n=1}^\infty \tfrac{(1-|z|^2)(1-|a_n|^2)}{C_R^2[(1-r)+(1-|a_n|)+|t|]^2} \right) \\
&\geq \tfrac{2\delta}{(1+\delta^2)(2+\sigma)^2} \varphi\left((1-r)+|t|\right) \cdot \exp\left(-\tfrac{4(2+\sigma)^2}{2\delta^2} (1-r) \varphi\left((1-r)+|t|\right) \right) \\
&= A\varphi\left((1-r)+|t|\right) \exp\left(-K(1-r)\varphi\left((1-r)+|t|\right) \right),
\end{aligned} \tag{8}$$

where A and K are two positive constants. Observe that there exists a positive constant β such that

$$|t| \geq \beta(1-r), \quad \text{for } z = re^{it} \in \{z \in \mathbb{D} : |z| \geq R\} \setminus \Omega_{\overline{\sigma}}. \tag{9}$$

Take $R_0 \geq R$ such that $(\beta + 1)(1 - R_0) \leq \pi$. Using (8), making three consecutive changes of variable: $\theta = \theta(t) = 1 - r + t$, $u = u(r) = 1 - r$, $x = x(u) = u\varphi(\theta)$ and using Fubini's theorem, we obtain

$$
(10)\quad \begin{aligned}
&\int_{\mathbb{D}} |B'(z)|^p \, dA(z) \ge \int_{\{z\in\mathbb{D}:|z|\ge R_0\}\setminus\Omega_\sigma} |B'(z)|^p \, dA(z)\\
&\ge 2A^p \int_{R_0}^1 \int_{\beta(1-r)}^{\pi} \varphi^p\left((1-r)+t\right)\exp\left(-Kp(1-r)\varphi\left((1-r)+t\right)\right) dt dr\\
&\ge 2A^p \int_0^{1-R_0} \int_{(\beta+1)u}^{(\beta+1)(1-R_0)} \varphi^p\left(\theta\right)\exp\left(-Kpu\varphi\left(\theta\right)\right) d\theta du\\
&= 2A^p \int_0^{(\beta+1)(1-R_0)} \int_0^{\frac{\theta}{(\beta+1)}} \varphi^p\left(\theta\right)\exp\left(-Kpu\varphi\left(\theta\right)\right) du d\theta\\
&= 2A^p \int_0^{(\beta+1)(1-R_0)} \varphi^{p-1}\left(\theta\right) \int_0^{\frac{\theta\varphi(\theta)}{(\beta+1)}} \exp\left(-Kpx\right) dx d\theta\\
&= \frac{2A^p}{kp} \int_0^{(\beta+1)(1-R_0)} \varphi^{p-1}\left(\theta\right)\left[1-\exp\left(\frac{-kp\theta\varphi(\theta)}{(\beta+1)}\right)\right] d\theta.
\end{aligned}
$$

This proves the second inequality of (4) with $C_2 = \frac{2A^p}{kp}$, $\theta_0 = (\beta+1)(1-R_0)$ and $M = kp/(\beta+1)$.

Now we turn to prove the other inequality. Write $b_n(z) = \frac{|a_n|}{a_n}\frac{a_n-z}{1-\overline{a}_n z}$ and $B_n(z) = \frac{B(z)}{b_n(z)}$, $n = 1, 2, \dots$. We have,

$$
(11)\qquad |B'(z)| = \left|\sum_{n=1}^{\infty} b_n'(z)\cdot B_n(z)\right| \le \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2}|B_n(z)|.
$$

The elementary inequality $\log(1-x) \le -x$, $0 < x < 1$, yields

$$
(12)\qquad \log|b_n(z)| = \frac{1}{2}\log\left(1-(1-|b_n(z)|^2)\right) \le -\frac{1}{2}(1-|b_n(z)|^2),\ z\in\mathbb{D}.
$$

Summing up over all $j \ne n$ and using the well known identity $1-|b_j(z)|^2 = \frac{(1-|z|^2)(1-|a_j|^2)}{|1-\overline{a}_j z|^2}$, we get from (12) that $\log|B_n(z)| \le -\frac{1}{2}\sum_{j\ne n}\frac{(1-|z|^2)(1-|a_j|^2)}{|1-\overline{a}_j z|^2}$ which, together with (11), Lemma 4 and Lemma 3, implies that, whenever $r\in[R,1)$ and $t\in[-\pi,\pi]$,

$$
\begin{aligned}
&|B'(re^{it})| \le \sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|1-\overline{a}_n re^{it}|^2}\exp\left(-\frac{1}{2}\sum_{j\ne n}\frac{(1-r^2)(1-|a_j|^2)}{|1-\overline{a}_j re^{it}|^2}\right)\\
&\le e^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|1-\overline{a}_n re^{it}|^2}\cdot\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(1-r^2)(1-|a_n|^2)}{|1-\overline{a}_n re^{it}|^2}\right)\\
&\le e^{\frac{1}{2}}(2+\sigma)^2\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|1-|a_n|re^{it}|^2}\cdot\exp\left(-\frac{1}{2(2+\sigma)^2}\sum_{n=1}^{\infty}\frac{(1-r^2)(1-|a_n|^2)}{|1-|a_n|re^{it}|^2}\right)\\
&\le A\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{\left[(1-|a_n|)+(1-r)+|t|\right]^2}\cdot\exp\left(-K\sum_{n=1}^{\infty}\frac{(1-r^2)(1-|a_n|^2)}{\left[(1-|a_n|)+(1-r)+|t|\right]^2}\right)\\
&\le A\varphi\big((1-r)+|t|\big)\exp\left(-K(1-r)\varphi\big((1-r)+|t|\big)\right),
\end{aligned}
$$

where, A and K depend only on σ and R. After three changes of variable: $\theta = \theta(t) = 1-r+t$, $u = u(r) = 1-r$ and $x = x(u) = u\varphi(\theta)$, some obvious estimates, and using Fubini's theorem, we obtain

$$
\begin{aligned}
&\int_{R\le|z|<1}|B'(z)|^p dA(z) \le 2A\int_R^1\int_0^{\pi}\varphi^p\big((1-r)+t\big)\exp[-Kp(1-r)\varphi((1-r)+t)]dt\,dr\\
&\le 2A\int_R^1\int_0^{2\pi}\varphi^p(\theta)\exp[-Kp(1-r)\varphi(\theta)]\,d\theta\,dr \le 2A\int_0^{2\pi}\int_0^1\varphi^p(\theta)\exp[-Kpu\varphi(\theta)]\,du\,d\theta\\
&\le 2A\big(\int_0^{2\pi}\varphi^{p-1}(\theta)\,d\theta\big)\big(\int_0^{\varphi(\theta)}\exp(-Kp\,x)\,dx\big) \le \frac{2A}{Kp}\int_0^{2\pi}\varphi^{p-1}(\theta)\,d\theta.
\end{aligned}
$$

Since $\int_0^{2\pi}|B'(re^{it})|^p dt$ increases with r, this implies the first inequality of (4). □

Corollary 1. *Suppose that $1 < p < \infty$ and B is a Blaschke product whose zeros lie in a Stolz angle and with the property that there exist $C > 0$ and $\theta_0 \in (0,\pi)$ such that (3) holds. Then the following conditions are equivalent:*

(a) $B' \in A^p$. (b) $\varphi_B \in L^{p-1}(0,\pi)$. (c) $B' \in H^{p-1}$. (d) $f_B \in L^{p-1}(-\pi,\pi)$.

Proof. Theorem 2 shows that $(a) \Leftrightarrow (b)$. The equivalence $(c) \Leftrightarrow (d)$ follows from Lemma 1 and, the implication $(c) \Rightarrow (a)$ follows from Theorem 6.1 of [1].

To prove that $(a) \Rightarrow (c)$, suppose that B is a Blaschke product with $B(0) \neq 0$, $B' \in A^p$ and such that its zeros $\{a_n\}$ lie in Ω_σ for a certain $\sigma > 1$. Write $a_n = |a_n|e^{i\theta_n}$ with $|\theta_n| \leq \pi$. Since $\{a_n\} \subset \Omega_\sigma$, there exists a positive constant λ such that $|\theta_n| \leq \lambda(1-|a_n|)$, $n = 1, 2, \ldots$. We have $[(1-|a_n|)+|\theta|]^2 \leq 2[(1-|a_n|)^2+\theta^2]$, and $\theta^2 \leq 2\left((\theta-\theta_n)^2+\theta_n^2\right) \leq 2\left((\theta-\theta_n)^2+\lambda^2(1-|a_n|)^2\right)$ whenever $n \geq 1$ and $\theta \in [-\pi, \pi]$. Then it follows that there exists a constant $C > 0$ such that

$$[(1-|a_n|)+|\theta|]^2 \leq C\left((1-|a_n|)^2+(\theta-\theta_n)^2\right), \quad n \geq 1, \quad \theta \in [-\pi, \pi]. \tag{13}$$

Since (a) $\Leftrightarrow$ (b), $\varphi_B \in L^{p-1}(0,1)$. Then (13) gives $f_B \in L^{p-1}(-\pi, \pi)$ and $B' \in H^{p-1}$. □

Condition (3) is not a simple one. Next we find a simple condition which implies it.

Corollary 2. *If the zeros $\{a_n\}$ of Blaschke product B lie in a Stolz angle and there exist $\lambda > 0$ and $n_0 \geq 1$ such that $1-|a_{n+1}| \geq \lambda(1-|a_n|)$, if $n \geq n_0$, then there exist $C > 0$ and $\theta_0 \in (0, \pi)$ such that (3) holds. Hence, $B' \in A^p \Leftrightarrow \varphi_B \in L^{p-1}(0, \pi)$ $(p > 1)$.*

Proof. Given $\theta \in (0, 1-|a_{n_0}|)$ take $n \geq n_0$ such that $1-|a_{n+1}| < \theta \leq 1-|a_n|$. Then

$$\theta\varphi_B(\theta) \geq \frac{\theta(1-|a_n|)}{(\theta+((1-|a_n|))^2} \geq \frac{(1-|a_{n+1}|)(1-|a_n|)}{4(1-|a_n|)^2} = \frac{(1-|a_{n+1}|)}{4(1-|a_n|)} \geq \frac{\lambda}{4}.$$

Hence, we have proved (3) with $C = \lambda/4$ and $\theta_0 = 1-|a_{n_0}|$. □

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