# GROWTH PROPERTIES AND SEQUENCES OF ZEROS OF ANALYTIC FUNCTIONS IN SPACES OF DIRICHLET TYPE

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**Abstract**

For $0 < p < \infty$ we let $\mathcal{D}^p_{p-1}$ denote the space of those functions $f$ which are analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy $\int_\Delta (1-|z|)^{p-1}|f'(z)|^p\,dx\,dy < \infty$. The spaces $\mathcal{D}^p_{p-1}$ are closely related to Hardy spaces. We have, $\mathcal{D}^p_{p-1} \subset H^p$, if $0 < p \le 2$, and $H^p \subset \mathcal{D}^p_{p-1}$, if $2 \le p < \infty$. In this paper we obtain a number of results about the Taylor coefficients of $\mathcal{D}^p_{p-1}$-functions and sharp estimates on the growth of the integral means and the radial growth of these functions as well as information on their zero sets.

*Keywords and phrases*: Spaces of Dirichlet type, Hardy spaces, Bergman spaces, integral means, radial growth, sequences of zeros.

## 1. Introduction and main results

We denote by $\Delta$ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. If $f$ is a function which is analytic in $\Delta$ and $0 < r < 1$, we set

$$\begin{aligned} M_p(r,f) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p\,dt\right)^{1/p}, && 0 < p < \infty, \\ I_p(r,f) &= M_p^p(r,f), && 0 < p < \infty, \\ M_\infty(r,f) &= \sup_{|z|=r} |f(z)|. \end{aligned}$$

For $0 < p \le \infty$, the *Hardy space* $H^p$ consists of all analytic functions $f$ in the disc for which

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

We refer the reader to [10] and [13] for the theory of Hardy spaces.

If $0 < p < \infty$ and $\alpha > -1$, we let $A^p_\alpha$ denote the (standard) *weighted Bergman space*, that is, the set of analytic functions $f$ in $\Delta$ such that

$$\int_\Delta (1-|z|)^\alpha |f(z)|^p \, dA(z) < \infty.$$

Here, $dA(z) = \frac{1}{\pi} dx\, dy$ denotes the normalized Lebesgue area measure in $\Delta$. The standard unweighted Bergman space $A^p_0$ is simply denoted by $A^p$. We mention the [11] and [18] as general references for the theory of Bergman sapces.

The space $\mathcal{D}^p_\alpha$ ($p > 0$, $\alpha > -1$) consists of all functions $f$ which are analytic in $\Delta$ such that $f' \in A^p_\alpha$. The space $\mathcal{D}^2_0$ is the classical Dirichlet space $\mathcal{D}$. For other values of $p$ and $\alpha$ the spaces $\mathcal{D}^p_\alpha$ have been extensively in a number papers such as [27, 28, 30, 33] (for $p = 2$) and [3, 8, 34, 36] for other values of $p$. If $p < \alpha + 1$, it is well known that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$ with equivalence of norms (see Theorem 6 of [12]). For $\alpha = p-2$, the space $\mathcal{D}^p_\alpha$ is the Besov space $B^p$ (cf. [2]).

The space $\mathcal{D}^p_\alpha$ is said to be a Dirichlet space if $p \geq \alpha + 1$. In this paper we shall be primarily interested in the "limit case" $p = \alpha + 1$, that is, in the spaces $\mathcal{D}^p_{p-1}$, $0 < p < \infty$, which are closely related to Hardy spaces. Indeed, a classical result of Littlewood and Paley [19] (see also [20]) asserts that

$$H^p \subset \mathcal{D}^p_{p-1}, \quad 2 \leq p < \infty. \tag{1}$$

On the other hand, we have

$$\mathcal{D}^p_{p-1} \subset H^p, \quad 0 < p \leq 2, \tag{2}$$

(see Lemma 1.4 of [34]). Notice that, in particular, we have $\mathcal{D}^2_1 = H^2$. However, we remark that if $p \neq 2$ then

$$H^p \neq \mathcal{D}^p_{p-1}. \tag{3}$$

This can be seen using the characterization of power series with Hadamard gaps which belong to the spaces $\mathcal{D}^p_{p-1}$.

Proposition A. If $f$ is an analytic function in $\Delta$ which is given by a power series with Hadamard gaps,

$$f(z) = \sum_{k=1}^\infty a_k z^{n_k} \ (z \in \Delta) \text{ with } n_{k+1} \geq \lambda n_k \text{ for all } k \ (\lambda > 1),$$

then, for every $p \in (0, \infty)$,

$$f \in \mathcal{D}^p_{p-1} \iff \sum_{k=1}^\infty |a_k|^p < \infty.$$

Since for Hadamard gap series as above we have, for $0 < p < \infty$,

$$f \in H^p \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} |a_k|^2 < \infty,$$

we immediately deduce that $\mathcal{D}^p_{p-1} \neq H^p$ if $p \neq 2$. We remark that Proposition A follows from Proposition 2.1 of [7]. In section 2 we shall see that Proposition A can also be deduced from the following theorem which gives a condition on the Taylor coefficients of a function $f$, analytic in $\Delta$, which implies that $f \in \mathcal{D}^p_{p-1}$.

THEOREM 1.1. *Let $f$ be an analytic function in $\Delta$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \Delta$).*

*(i) If $0 < p < \infty$ and*

$$\sum_{n=0}^{\infty} \left( \sum_{k \in I(n)} |a_k| \right)^p < \infty, \tag{4}$$

*then $f \in \mathcal{D}^p_{p-1}$.*

*(ii) If $0 < p \leq 2$ and*

$$\sum_{n=1}^{\infty} \left( \sum_{k \in I(n)} |a_k|^2 \right)^{p/2} < \infty, \tag{5}$$

*then $f \in \mathcal{D}^p_{p-1}$.*

Here and throughout the paper, for $n = 0, 1, \dots,$ $I(n)$ is the set of the integers $k$ such that $2^n \leq k < 2^{n+1}$.

Notice that, if $0 < p \leq 2$, then (4) $\Rightarrow$ (5). Hence, for $p \in (0, 2]$, (ii) is stronger than (i). We remark also that if $0 < p \leq 2$ then the condition $\sum_{n=0}^{\infty} |a_n|^p < \infty$ implies (5). Consequently, (ii) improves Lemma 1.5 of [34].

In Theorem 1.2 we give a condition on the Taylor coefficients of an analytic function $f$ which is necessary for its membership in $\mathcal{D}^p_{p-1}$ if $2 \leq p < \infty$.

THEOREM 1.2. *Let $f$ be an analytic function in $\Delta$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \Delta$). If $2 \leq p < \infty$ and $f \in \mathcal{D}^p_{p-1}$ then*

$$\sum_{n=1}^{\infty} \left( \sum_{k \in I(n)} |a_k|^2 \right)^{p/2} < \infty. \tag{6}$$

If $0 < p < 2$ then (3) can be seen in some other ways. Rudin proved in [29] that there exists a Blaschke product $B$ which does not belong to $\mathcal{D}_0^1$ (see also [24]). Vinogradov [34] extended this result showing that for every $p \in (0, 2)$ there exist Blaschke products $B$ which do not belong to $\mathcal{D}_{p-1}^p$. This clearly gives that $\mathcal{D}_{p-1}^p \neq H^p$ if $0 < p < 2$, a fact which can be also deduced from the results of [9] and of [14]. In contrast with what happens for $0 < p < 2$, it is not easy to give examples of functions $f \in \mathcal{D}_{p-1}^p \setminus H^p$ for a certain $p \in (2, \infty)$ which are not given by power series by Hadamard gaps. Since $H^p \subset \mathcal{D}_{p-1}^p$ if $p \geq 2$, any Blaschke product belongs to $\cap_{2 \leq p < \infty} \mathcal{D}_{p-1}^p$. Also, for a number of classes $\mathcal{F}$ of analytic functions in $\Delta$ we have $\mathcal{F} \cap \mathcal{D}_{p-1}^p = \mathcal{F} \cap H^p$ $(0 < p < \infty)$. For example, it is very easy to prove the following Lemma.

Lemma 1.3. *(i) If $\alpha > 0$, $0 < p < \infty$ and $f(z) = 1/(1-z)^\alpha$, $(z \in \Delta)$, then*

$$f \in H^p \quad \Leftrightarrow \quad f \in \mathcal{D}_{p-1}^p \quad \Leftrightarrow \quad \alpha p < 1.$$

*(ii) If $\alpha, \beta > 0$, $p \in (0, \infty)$ and $f(z) = \frac{1}{(1-z)^\alpha (\log \frac{2}{1-z})^\beta}$, $(z \in \Delta)$, then*

$$f \in H^p \;\Leftrightarrow\; f \in \mathcal{D}_{p-1}^p \;\Leftrightarrow\; \alpha p < 1 \text{ and } \beta > 0 \text{ or } \alpha p = 1 \text{ and } \beta p > 1.$$

A much deeper result is stated in Theorem 1 of [6] which asserts that, if $\mathcal{U}$ denotes the class of all univalent (holomorphic and one-to-one) functions in $\Delta$, then $\mathcal{U} \cap H^p = \mathcal{U} \cap \mathcal{D}_{p-1}^p$ for all $p > 0$ (see also [25] for the case $p = 1$).

In spite of these facts we shall prove that, for every $p \in (2, \infty)$, there are a lot of differences between the space $H^p$ and the space $\mathcal{D}_{p-1}^p$. In section 3 we shall be mainly concerned in obtaining sharp estimates on the growth of the integral means of $\mathcal{D}_{p-1}^p$-functions. If $0 < p \leq 2$ and $f \in \mathcal{D}_{p-1}^p$ then $f \in H^p$ and, hence, the integral means $M_p(r, f)$ are bounded. This is no longer true for $p > 2$. Our main results in section 3 are stated in the following two theorems.

Theorem 1.4. *If $2 < p < \infty$ and $f \in \mathcal{D}_{p-1}^p$, then*

*(i)*

$$M_p(r, f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)\right), \quad \text{as } r \to 1. \tag{7}$$

*(ii)*

$$M_2(r, f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2} - \frac{1}{p}}\right), \quad \text{as } r \to 1. \tag{8}$$

Theorem 1.5. *If $2 < p < \infty$ and $0 < \beta < \frac{1}{2} - \frac{1}{p}$, then there exists a function $f \in \mathcal{D}^p_{p-1}$ such that*

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{it})|\,dt\right) \neq \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad \textit{as } r \to 1^-. \tag{9}$$

Notice that since $\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{it})|\,dt\right) \leq M_2(r,f)$, Theorem 1.5 shows that part (ii) of Theorem 1.4 is sharp in a very strong sense.

Remark. Using Theorem 1.4 we can obtain an upper bound on the integral means $M_q(r,f)$, $2 < q < p$, of a function $f \in \mathcal{D}^p_{p-1}$. Indeed, if $q \in (2,p)$ then $q = p\lambda + 2(1-\lambda)$, where $\lambda = \frac{q-2}{p-2} \in (0,1)$. Consequently, using Theorem 3 and Hölder's inequality with exponents $\frac{1}{\lambda}$ and $\frac{1}{1-\lambda}$ we see that, if $f \in \mathcal{D}^p_{p-1}$ and $2 < q < p$, then

$$M_q(r,f) = \left(\left(\log\frac{1}{1-r}\right)^{\eta}\right), \quad \text{as } r \to 1,$$

where $\eta = \eta(p,q) = \frac{p}{q}\lambda + \frac{p-2}{pq}(1-\lambda)$ and $\lambda = \frac{q-2}{p-2}$.

In section 4 we shall study properties of the sequences of zeros of non trivial $\mathcal{D}^p_{p-1}$-functions. If $0 < p \leq 2$ then $\mathcal{D}^p_{p-1} \subset H^p$ and, hence, the sequence of zeros of a non-identically zero $\mathcal{D}^p_{p-1}$-function satisfies the Blaschke condition. This does not remain true for $p > 2$. Our main results about the sequences of zeros of functions $f$ in the space $\mathcal{D}^p_{p-1}$, $2 < p < \infty$, are stated in Theorem 1.6 and Theorem 1.7

Theorem 1.6. *Suppose that $2 < p < \infty$ and let $f$ be a function which belongs to the space $\mathcal{D}^p_{p-1}$ with $f(0) \neq 0$. Let $\{z_k\}_{k=1}^{\infty}$ be the sequence zeros of $f$ ordered so that $|z_k| \leq |z_{k+1}|$, for all $k$. Then*

$$\prod_{k=1}^{N} \frac{1}{|z_k|} = \mathrm{o}\left(\left(\log N\right)^{\frac{1}{2}-\frac{1}{p}}\right), \quad \textit{as } N \to \infty. \tag{10}$$

From now on, if $f$ is a non-identically zero analytic function of zeros and $\{z_k\}_{k=1}^{\infty}$ is the sequence zeros of $f$ ordered so that $|z_k| \leq |z_{k+1}|$, for all $k$, we shall say that $\{z_k\}_{k=1}^{\infty}$ is the sequence of ordered zeros of $f$. Theorem 1.7 asserts that Theorem 1.6 is best possible.

THEOREM 1.7. *If* $2 < p < \infty$ *and* $0 < \beta < \frac{1}{2} - \frac{1}{p}$*, then there exists a function* $f \in \mathcal{D}^p_{p-1}$ *with* $f(0) \neq 0$ *such that if* $\{z_k\}_{k=1}^{\infty}$ *is the sequence of ordered zeros of* $f$*, then*

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \neq \mathrm{o}\left(\left(\log N\right)^{\beta}\right), \quad \text{as } N \to \infty. \tag{11}$$

As a consequence of Theorem 1.6 and Theorem 1.7 we obtain the following result.

COROLLARY 1.8. *If* $2 \leq p < q < \infty$ *then there exists a sequence* $\{z_k\} \subset \Delta$ *which is the sequence of zeros of a* $\mathcal{D}^q_{q-1}$*-function but is not the sequence of zeros of any* $\mathcal{D}^p_{p-1}$*-function.*

Hence the situation in this setting is similar to that in the setting of Bergman spaces (see Theorem 1 of [17]).

Next we shall get into the proofs of these and some other results but, before doing so, let us remark that, as usual, we shall be using the convention that $C_{p,\alpha,\dots}$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, \dots$ but not necessarily the same at different occurrences.

## 2. Taylor coefficients of $\mathcal{D}^p_{p-1}$ functions.

We start recalling the following useful result due to Mateljevic and Pavlovic [21] (see also Lemma 3 of [5] and [22]) which will be basic in the proofs of Theorem 1.1 and Theorem 1.2.

LEMMA B. Let $\alpha > 0$ and $p > 0$. There exists a constant $K$ which depends only on p and $\alpha$ such that, if $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-negative numbers, $t_n = \sum_{k \in I(n)} a_n$ $(n \geq 0)$ and $f(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$ $(x \in (0,1))$, then

$$K^{-1} \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \leq \int_0^1 (1-x)^{\alpha-1} f(x)^p \, dx \leq K \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p.$$

PROOF. Take $p \in (0, \infty)$ and let $f$ be analytic in $\Delta$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \Delta. \tag{12}$$

Suppose that (4) holds. Using Lemma B and (4) we see that

$$\int_\Delta |f'(z)|^p (1-|z|^2)^{p-1}\,dA(z) \le C_p \int_0^1 (1-r)^{p-1} \left(\sum_{n=1}^\infty n|a_n| r^{n-1}\right)^p dr$$

$$\le C_p \sum_{n=0}^\infty 2^{-np} \left(\sum_{k\in I(n)} k|a_k|\right)^p \le C_p \sum_{n=0}^\infty 2^{-np} 2^{(n+1)p} \left(\sum_{k\in I(n)} |a_k|\right)^p$$

$$\le C_p \sum_{n=0}^\infty \left(\sum_{k\in I(n)} |a_k|\right)^p < \infty.$$

Hence, $f \in \mathcal{D}^p_{p-1}$ and the proof of (i) is finished.

Suppose now that $0 < p \le 2$, $f$ is as in (12) and satisfies (5). Using that $M_p(r,f') \le M_2(r,f')$ for all $r \in (0,1)$, making the change of variable $r^2 = s$ and using Lemma B, we obtain

$$\int_\Delta |f'(z)|^p (1-|z|^2)^{p-1}\,dA(z) = 2\int_0^1 r(1-r^2)^{p-1} M_p(r,f')^p\,dr$$

$$\le 2\int_0^1 r(1-r^2)^{p-1} M_2(r,f')^p\,dr = 2\int_0^1 r(1-r^2)^{p-1} \left(\sum_{n=1}^\infty n^2|a_n|^2 r^{2n-2}\right)^{p/2} dr$$

$$\le C\int_0^1 (1-s)^{p-1} \left(\sum_{n=1}^\infty n^2|a_n|^2 s^{n-1}\right)^{p/2} ds \le C_p \sum_{n=0}^\infty 2^{-np} \left(\sum_{k\in I(n)} k^2|a_k|^2\right)^{p/2}$$

$$\le C_p \sum_{n=0}^\infty \left(\sum_{k\in I(n)} |a_k|^2\right)^{p/2} < \infty.$$

Hence, $f \in \mathcal{D}^p_{p-1}$. This finishes the proof of (ii). □

Next we shall see that Proposition A can be deduced from Theorem 1.1 as announced.

Proof of Proposition A. Let $f$ be an analytic function in $\Delta$ given by a power series with Hadamard gaps

$$f(z) = \sum_{j=1}^\infty a_j z^{n_j} \quad \text{with } \frac{n_{j+1}}{n_j} \ge \lambda > 1 \text{ for all j}, \tag{13}$$

and suppose that $\sum_{j=1}^\infty |a_j|^p < \infty$. Using the gap condition, we see that there are at most $C_\lambda = \log_\lambda 2 + 1$ of the $n_j's$ in the set $I(n)$. Then there exists a constant $C_{\lambda,p} > 0$ such that

$$\sum_{n=0}^\infty \left(\sum_{j\in I(n)} |a_j|\right)^p \le C_{\lambda,p} \sum_{j=1}^\infty |a_j|^p < \infty,$$

consequently, using Theorem 1.1, we deduce that $f \in \mathcal{D}^p_{p-1}$.

To prove the other implication suppose that $f$ is as in (13) and $f \in \mathcal{D}^p_{p-1}$ for a certain $p > 0$. It is well known (see Chapter $V$ of Vol. $I$ of [38]) that there exist constants $A(\lambda, p)$ and $B(\lambda, p)$ such that

$$A(\lambda,p)M_2^p(r,f') \le M_p^p(r,f') \le B(\lambda,p)M_2^p(r,f'), \quad 0 < r < 1.$$

This and Lemma B give

$$\begin{aligned}
\infty > \int_\Delta |f'(z)|^p(1-|z|^2)^{p-1}\,dA(z) &= \int_0^1 r(1-r^2)^{p-1}M_p^p(r,f')\,dr \\
&\ge A(\lambda,p)\int_0^1 r(1-r^2)^{p-1}M_2^p(r,f')\,dr \\
&\ge A(\lambda,p)\int_0^1 r(1-r^2)^{p-1}\left(\sum_{j=1}^\infty {n_j}^2|a_j|^2 r^{2n_j-2}\right)^{\frac{p}{2}} dr \\
&\ge A(\lambda,p)\int_0^1 t(1-t)^{p-1}\left(\sum_{j=1}^\infty {n_j}^2|a_j|^2 t^{j-1}\right)^{\frac{p}{2}} dt \\
&\ge C_pA(\lambda,p)\sum_{n=0}^\infty 2^{-np}\left(\sum_{n_j\in I(n)} {n_j}^2|a_j|^2\right)^{\frac{p}{2}} \\
&\ge C_pA(\lambda,p)\sum_{n=0}^\infty 2^{-np}2^{np}\left(\sum_{n_j\in I(n)} |a_j|\right)^{p} \ge C_{\lambda,p}A(\lambda,p)\sum_{j=0}^\infty |a_j|^p.
\end{aligned}$$

We remark that the last inequality is obvious if $p \ge 1$ and, in the case $0 < p < 1$, follows using again the fact that there are at most $C_\lambda = \log_\lambda 2+1$ of the $n_j's$ in the set $I(n)$. Thus, we have $\sum_{j=0}^\infty |a_j|^p < \infty$. This finishes the proof. □

Proof of Theorem 1.2. Suppose that $2 \le p < \infty$ and $f \in \mathcal{D}^p_{p-1}$,

$$f(z) = \sum_{n=0}^\infty a_n z^n, \quad z \in \Delta.$$

Using Lemma B, bearing in mind that $k \asymp 2^n$ if $k \in I(n)$, making a change of variable and using that, since $p \geq 2$, $M_2(r,f') \leq M_p(r,f')$, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\sum_{k\in I(n)}|a_k|^2\right)^{p/2} &\leq \sum_{n=1}^{\infty}2^{-np}\left(\sum_{k\in I(n)}k^2|a_k|^2\right)^{p/2}\\
\leq & C_p\int_0^1(1-t)^{p-1}\left(\sum_{n=1}^{\infty}n^2|a_n|^2t^{n-1}\right)^{p/2}\,dt\\
\leq & C_p\int_0^1(1-r^2)^{p-1}\left(\sum_{n=1}^{\infty}n^2|a_n|^2r^{2n-2}\right)^{p/2}\,dt\\
\leq & C_p\int_0^1(1-r)^{p-1}M_p(r,f')^p<\infty.
\end{aligned}$$

□

## 3. Growth properties of $\mathcal{D}^p_{p-1}$-functions

In this section we shall be mainly interested in obtaining sharp estimates on the growth of functions $f$ in the spaces $\mathcal{D}^p_{p-1}$ $(2<p<\infty)$.

**3.1. Integral means estimates** Let us start with estimates on the growth of the maximum modulus $M_\infty(r,f)$. We can prove the following result.

THEOREM 3.1. *Let $f$ be an analytic function in $\Delta$. If $f\in\mathcal{D}^p_{p-1}$, $0<p<\infty$ then*

$$M_\infty(r,f)=\mathrm{o}\left(\frac{1}{(1-r)^{\frac{1}{p}}}\right),\quad \text{as } r\to 1^-. \tag{14}$$

PROOF. Let $f\in\mathcal{D}^p_{p-1}$ and $z\in\Delta$. Let $D(z)$ denote the open disc

$$\left\{w\in\mathbb{C}:|z-w|<\frac{1-|z|}{2}\right\}.$$

Clearly, $D(z)\subset\Delta$. Since the function $z\to|f'(z)|^p$ is subharmonic in $\Delta$, we have

$$|f'(z)|^p\leq\frac{C}{|D(z)|}\int_{D(z)}|f'(\omega)|^p\,dA(\omega)\leq\frac{C}{(1-|z|^2)^2}\int_{D(z)}|f'(\omega)|^p\,dA(\omega). \tag{15}$$

It is clear that

$$(1-|z|^2)\asymp(1-|\omega|^2),\quad \omega\in D(z),\quad z\in\Delta.$$

Using this and (15) we obtain

$$\begin{aligned}|f'(z)|^p \leq & \frac{C_p}{(1-|z|^2)^2}\int_{D(z)}\left[\frac{1-|\omega|}{1-|z|}\right]^{p-1}|f'(\omega)|^p\,dA(\omega)\\ = & \frac{C_p}{(1-|z|^2)^{p+1}}\int_{D(z)}(1-|\omega|)^{p-1}|f'(\omega)|^p\,dA(\omega).\end{aligned} \tag{16}$$

On the other hand, since $f \in \mathcal{D}^p_{p-1}$, it follows that

$$\int_{D(z)}(1-|\omega|)^{p-1}|f'(\omega)|^p\,dA(\omega) = \mathrm{o}(1), \quad \text{as } |z| \to 1^-,$$

which, with (16), implies

$$M_\infty(r, f') = \mathrm{o}\left(\frac{1}{(1-r)^{1+\frac{1}{p}}}\right), \quad \text{as } r \to 1^-, \tag{17}$$

and (14) follows by integration. □

Remark. We observe that for any $p \in (0,\infty)$, the exponent $1/p$ in (14) is the best possible. Even more, if we take

$$f_{p,\beta}(z) = (1-z)^{-1/p}\left(\log\frac{2}{1-z}\right)^{-\beta}, \quad z \in \Delta,$$

with $\beta > \frac{1}{p}$ then, as we noticed in Lemma 1.3, $f_{p,\beta} \in \mathcal{D}^p_{p-1}$ and it is easy to see that

$$M_\infty(r, f) \approx (1-r)^{-1/p}\left(\log\frac{1}{1-r}\right)^{-\beta}, \quad 0 < r < 1.$$

So condition (14) in Theorem 3.1 cannot be substituted by the condition

$$M_\infty(r, f) = \mathrm{o}\left(\frac{1}{(1-r)^{\frac{1}{p}}\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}+\varepsilon}}\right), \qquad \text{as } r \to 1^-,$$

for any $\varepsilon > 0$.

Now we turn to prove Theorem 1.4 and Theorem 1.5.
Proof of Theorem 1.4. Suppose that $2 < p < \infty$ and $f \in \mathcal{D}^p_{p-1}$. Then

$$\lim_{r\to 1^-}\int_r^1 (1-s)^{p-1}M_p^p(s, f')\,ds = 0. \tag{18}$$

Since $M_p(s,f')$ is an increasing function of $s$

$$\int_r^1 (1-s)^{p-1} M_p^p(s,f')\,ds \geq M_p^p(r,f') \int_r^1 (1-s)^{p-1}\,ds \geq C_p M_p^p(r,f')(1-r)^p,$$

which, together with (18), yields

$$M_p(r,f') = \mathrm{o}\left(\frac{1}{1-r}\right), \quad \text{as } r \to 1^-, \tag{19}$$

which, using Minkowski's integral inequality, implies (7).

Using (19) and the fact that for any fixed $r$ with $0<r<1$ the integral means $M_p(r,f')$ increase with $p$, we deduce that

$$I_2(r,f') = \mathrm{o}\left(\frac{1}{(1-r)^2}\right), \quad \text{as } r \to 1^-.$$

and then using the well known inequality (see [26] pp. 125-126)

$$\frac{d^2}{dr^2}\big(I_2(r,f)\big) \leq 4 I_2(r,f'), \quad 0<r<1,$$

we obtain

$$\frac{d^2}{dr^2}\big(I_2(r,f)\big) = \mathrm{o}\left(\frac{1}{(1-r)^2}\right) \quad \text{as } r \to 1^-,$$

which, integrating twice, gives $M_2(r,f) = \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right)$, as $r \to 1$. This is worse than (8). To obtain this we shall use Theorem 1.2.

Say that $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $(z \in \Delta)$. Suppose, without loss of generality that $a_0 = 0$. Using Hölder's inequality with the exponents $p/2$ and $p/(p-2)$ and Theorem 1.2, we obtain

$$\begin{aligned}
M_2(r,f)^2 =& \sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} \sum_{k\in I(n)} |a_k|^2 r^{2k} \\
\leq & \sum_{n=0}^{\infty} r^{2^{n+1}} \left(\sum_{k\in I(n)} |a_k|^2\right) \\
\leq & \left[\sum_{n=0}^{\infty} \left(\sum_{k\in I(n)} |a_k|^2\right)^{p/2}\right]^{2/p} \left[\sum_{n=0}^{\infty} r^{2^{n+1}\frac{p}{p-2}}\right]^{1-\frac{2}{p}} \\
\leq & C_{f,p} \left(\log\frac{1}{1-r}\right)^{1-\frac{2}{p}}.
\end{aligned}$$

□

Since

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,d\theta\right)\le M_2(r,f),\quad 0<r<1,$$

we trivially have the following result.

Corollary 3.2. *If $2<p<\infty$ and $f\in\mathcal{D}^p_{p-1}$ then*

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,d\theta\right)=\mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}}\right),\quad \textit{as } r\to 1.$$

Theorem 9 shows that Corollary 3.2 and the estimate (8) are sharp in very strong sense. The following lemma, whose proof is simple and will be omitted, will be used in the proof of Theorem 9.

Lemma 3.3. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an analytic function in $\Delta$. If $0<\beta\le 1$ and*

$$\sum_{k=0}^{N}|a_k|^2\approx\big(\log N\big)^{\beta},\quad \textit{as } N\to\infty,$$

*then*

$$I_2(r,f)\approx\left(\log\frac{1}{1-r}\right)^{\beta}\quad \textit{as } r\to 1^-.$$

We shall also make use of the technique introduced by D. Ullrich in [32]. Let start introducing some notation:

Let $\omega=[0,1]^{\mathbb{N}}$ and let $\omega_1,\omega_2,\dots$ be "the coordinate functions" $\omega_j:\Omega\longrightarrow[0,1]$. Let $d\omega$ denote the product measure $\Omega$ derived from Lebesgue measure on $[0,1]$. Now $\omega_1,\omega_2,\dots$ are the Steinhaus variables (i. i. d. random variables uniformly distributed on $[0,1]$). Note that $\{e^{2\pi i\omega_j}\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(\Omega)$, hence, if $\sum_{j=1}^{\infty}|a_j|^2<\infty$, then $\sum_{j=1}^{\infty}a_je^{2\pi i\omega_j}$ is a well defined element of $L^2(\Omega)$ with $L^2$-norm $\left(\sum_{j=1}^{\infty}|a_j|^2\right)^{1/2}$. The following theorem is Theorem 1 of [32].

Theorem C. There exists $C>0$ such that for any sequence of complex numbers $\{a_j\}_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty}|a_j|^2<\infty$, we have

$$\exp\left[\int_{\Omega}\log\left|\sum_{j=1}^{\infty}a_je^{2\pi i\omega_j}\right|d\omega\right]\ge C\left(\sum_{j=1}^{\infty}|a_j|^2\right)^{\frac{1}{2}}.$$

Proof of Theorem 1.5. Suppose that $2 < p < \infty$ and $0 < \beta < \frac{1}{2} - \frac{1}{p}$. Set $\varepsilon = \frac{1}{2} - \frac{1}{p} - \beta$, hence, $\varepsilon > 0$. We define the sequence $\{b_j\}_{j=1}^{\infty}$ as follows

$$b_j = \frac{1}{j^{\frac{1}{p}+\varepsilon}} \quad j = 1, 2, \dots.$$

Now, for every $\omega \in \Omega$ we define

$$f_\omega(z) = \sum_{j=1}^{\infty} b_j e^{2\pi i \omega_j} z^{2^j} = \sum_{k=1}^{\infty} a_{k,\omega} z^k, \quad z \in \Delta. \tag{20}$$

Since $\sum_{j=1}^{\infty} |b_j|^p < \infty$, using Proposition A we deduce that $f_\omega \in \mathcal{D}^p_{p-1}$ for every $\omega \in \Omega$.

We are going to see that for a.e. $\omega \in \Omega$

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f_\omega(re^{it})|\,dt\right) \neq \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad \text{as } r \to 1^-. \tag{21}$$

This will finish the proof.

Suppose that (21) is false. Then there exists a measurable set $E \subset \Omega$ with positive measure and such that for all $\omega \in E$

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f_\omega(re^{it})|\,dt\right) = \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad \text{as } r \to 1^-. \tag{22}$$

This is equivalent to saying that

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\frac{|f_\omega(re^{it})|}{\left(\log\frac{1}{1-r}\right)^{\beta}}\right] dt = -\infty, \quad \omega \in E. \tag{23}$$

On the other hand,

$$\left(\sum_{j=1}^{N} |b_j|^2\right)^{1/2} = \left(\sum_{j=1}^{N} \frac{1}{j^{\frac{2}{p}+2\varepsilon}}\right)^{1/2}$$
$$\sim \left(\int_1^N \frac{1}{x^{\frac{2}{p}+2\varepsilon}}\,dx\right)^{1/2} \sim N^{\frac{1}{2}-\frac{1}{p}-\varepsilon}, \quad \text{as } N \to \infty.$$

Thus, there exist $C > 0$ and $N_0 > 0$ such that

$$\left(\sum_{k=1}^{N} |a_{k,\omega}|^2\right)^{1/2} \leq C\big(\log N\big)^{\frac{1}{2}-\frac{1}{p}-\varepsilon}, \quad N \geq N_0. \tag{24}$$

Using (24) and Lemma 3.3 we deduce that

$$M_2(r,f_\omega)=I_2(r,f_\omega)^{\frac{1}{2}}\le C\left[\log\frac{1}{1-r}\right]^{\frac{1}{2}-\frac{1}{p}-\varepsilon},\quad 0<r<1,\quad \omega\in\Omega,$$

which implies that

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f_\omega(re^{it})|\,dt\right)\le C\left[\log\frac{1}{1-r}\right]^{\frac{1}{2}-\frac{1}{p}-\varepsilon},\quad 0<r<1,\quad \omega\in\Omega. \tag{25}$$

From this we deduce as in (23), that there exists $C>0$ such that

$$\int_{-\pi}^{\pi}\log\left[\frac{|f_\omega(re^{it})|}{\left(\log\frac{1}{1-r}\right)^{\beta}}\right]dt\le C,\quad 0<r<1,\quad \omega\in\Omega. \tag{26}$$

Bearing in mind that $E$ has positive measure, (26) and (23) imply

$$\lim_{r\to1^-}\int_\Omega\left[\int_{-\pi}^{\pi}\log\frac{|f_\omega(re^{it})|}{\left(\log\frac{1}{1-r}\right)^{\beta}}\,dt\right]d\omega=-\infty. \tag{27}$$

For $N=1,2,\dots$, let $\Omega_N=[0,1]^N$, and let $m_N$ be the Lebesgue measure on $\Omega_N$. Observe now that, for any $N$, we have

$$\begin{aligned}&\int_{\Omega_N}\log|f_\omega(re^{it})|\,dm_N(\omega)\\&=\int_0^1\dots\int_0^1\log|\sum_{j=1}^{N}b_jr^{2^j}e^{i[2\pi\omega_j+2^jt]}+\sum_{j=N+1}^{\infty}b_jr^{2^j}e^{i[2\pi\omega_j+2^jt]}|\,d\omega_1d\omega_2\dots d\omega_N\\&=\int_0^1\dots\int_0^1\log|\sum_{j=1}^{N}b_jr^{2^j}e^{2\pi i\omega_j}+\sum_{j=N+1}^{\infty}b_jr^{2^j}e^{i[2\pi\omega_j+2^jt]}|\,d\omega_1d\omega_2\dots d\omega_N.\end{aligned}$$

Letting $N$ tend to $\infty$, we deduce that $\int_\Omega\log|f_\omega(re^{it})|\,d\omega$ is indepedent of $t$, then using (27) and Fubini's Theorem we obtain

$$\lim_{r\to1^-}\int_\Omega\log\frac{|f_\omega(r)|}{\left(\log\frac{1}{1-r}\right)^{\beta}}\,d\omega=-\infty. \tag{28}$$

But, if we set

$$r_N=1-\frac{1}{2^N}\quad N=1,2,\dots$$

by Theorem C and the inequality

$$e^{-1}\le r_N^{2^N}\le r_N^{2^j}\quad 1\le j\le N,$$

we deduce that

$$\begin{aligned}
&\exp\left[\int_\Omega \log|f_\omega(r_N)|\,d\omega\right] = \exp\left[\int_\Omega \log\left|\sum_{j=1}^\infty b_j e^{2\pi i\omega_j} r_N^{2^j}\right|\right]\\
\geq &C\left(\sum_{j=1}^\infty |b_j|^2 (r_N^{2^j})^2\right)^{1/2} \geq C\left(\sum_{j=1}^N |b_j|^2\right)^{1/2} = C\left(\sum_{j=1}^N \frac{1}{j^{\frac{2}{p}+2\varepsilon}}\right)^{1/2}\\
\geq &C\frac{1}{N^{\frac{1}{p}+\varepsilon-\frac{1}{2}}} \geq C\left(\log\frac{1}{1-r_N}\right)^{\frac{1}{2}-\frac{1}{p}-\varepsilon} = C\left(\log\frac{1}{1-r_N}\right)^{\beta},
\end{aligned}$$

which implies

$$\int_\Omega \log\frac{|f_\omega(r_N)|}{\left(\log\frac{1}{1-r_N}\right)^\beta}\,d\omega \geq \log C,$$

for all $N$, which contradicts (28). Consequently, (21) is true and the proof is finished. □

**3.2. Radial growth of $\mathcal{D}^p_{p-1}$-functions** In this section we are going to obtain some estimates on the radial growth of $\mathcal{D}^p_{p-1}$-functions. If $0 < p \leq 2$ and $f \in \mathcal{D}^p_{p-1}$, then $f \in H^p$ and so $f$ has nontangential limit a.e. $\mathbb{T}$. Thererefore, we have:

*If $0 < p \leq 2$ and $f \in \mathcal{D}^p_{p-1}$, then*

$$|f(re^{i\theta})| = \mathrm{O}(1), \qquad \textit{as } r \to 1^- \textit{ for a. e. } e^{it} \in \partial\Delta.$$

Zygmund proved in [37] that if $f$ is an analytic function in $\Delta$ then

$$\int_0^r |f'(\rho e^{it})|\,d\rho = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1/2}\right], \qquad \text{as } r \to 1^-. \tag{29}$$

for almost every point $e^{it}$ in the Fatou set of $f$, $F_f$, which consists of those $e^{it} \in \mathbb{T}$ such that $f$ has finite nontangential limit at $e^{it}$. Obviously, (29) implies

$$|f(re^{it})| = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1/2}\right], \qquad \text{as } r \to 1^-, \tag{30}$$

If $2 < p < \infty$ there are functions $f \in \mathcal{D}^p_{p-1}$ such that $F_f$ has Lebesgue measure equal to zero. Indeed, an analytic function $f$ given by a power series with Hadamard gaps whose sequence of Taylor coefficients $\{a_k\}$ belongs to $l^p \backslash l^2$, is a $\mathcal{D}^p_{p-1}$-function by Proposition A and $F_f$ has null Lebesgue measure (see Chapter V of [38]). In spite of this, we can prove that the following result for $\mathcal{D}^p_{p-1}$-functions.

THEOREM 3.4. *If* $2 < p < \infty$ *and* $f \in \mathcal{D}^p_{p-1}$ *then*

$$|f(re^{it})| = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1-\frac{1}{p}}\right], \qquad \textit{as } r \to 1^- \textit{ for a. e. } e^{it} \in \partial\Delta. \tag{31}$$

Note that this is better that the a. e. estimate which can be deduced from (17).

PROOF OF THEOREM 3.4. Let $p$ and $f$ be as the statement of the theorem. Then

$$\int_{-\pi}^{\pi}\left(\int_0^1 (1-r)^{p-1}|f'(re^{it})|^p\,dt\right)dr < \infty,$$

and it follows that the set $A$ of points $e^{it} \in \partial\Delta$ for which

$$\int_0^1 (1-r)^{p-1}|f'(re^{it})|^p\,dt < \infty,$$

has Lebesgue measure equal to $2\pi$.

Take and fix $e^{it} \in A$. Take also $\varepsilon > 0$, then there exists $r_\varepsilon \in (0,1)$ such that

$$\int_{r_\varepsilon}^1 (1-s)^{p-1}|f'(se^{it})|^p\,ds < \varepsilon. \tag{32}$$

Using (32) and Hölder's inequality with exponents $p$ and $\frac{p}{p-1}$, we obtain for $r_\varepsilon < r < 1$,

$$\begin{aligned}
\int_0^r |f'(se^{it})|\,ds &= \int_0^{r_\varepsilon} |f'(se^{it})|\,ds + \int_{r_\varepsilon}^r |f'(se^{it})|\,ds \\
&\le C_{f,\varepsilon} + \int_{r_\varepsilon}^r \frac{(1-s)^{1-\frac{1}{p}}}{(1-s)^{1-\frac{1}{p}}}|f'(se^{it})|\,ds \\
&\le C_{f,\varepsilon} + \left[\int_{r_\varepsilon}^r (1-s)^{p-1}|f'(se^{it})|^p\,ds\right]^{1/p}\left[\int_{r_\varepsilon}^r \frac{1}{(1-s)}\,ds\right]^{1-\frac{1}{p}} \\
&\le C_{f,\varepsilon} + \varepsilon\left(\log\frac{1}{1-r}\right)^{1-\frac{1}{p}}.
\end{aligned} \tag{33}$$

Consequently, we have proved that

$$\limsup_{r\to1}\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}-1}\int_0^r |f'(se^{it})|\,ds \le \varepsilon.$$

Since $\varepsilon > 0$ and $e^{it} \in A$ are arbitrary, we have

$$\int_0^r |f'(se^{it})|\,ds = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1-\frac{1}{p}}\right], \qquad \text{as } r \to 1^-,$$

for all $e^{it} \in A$. This implies that (31) holds for all $e^{it} \in A$, which has Lebesgue measure equals to $2\pi$. This finishes the proof. $\square$

We do not know whether or not the exponent $1 - \frac{1}{p}$ in Theorem 3.4 is sharp but we know that it cannot be substitutes by any exponent smaller than $\frac{1}{2} - \frac{1}{p}$. Indeed, we can prove the following result.

THEOREM 3.5. *If $2 < p < \infty$, then there exists a function $f \in \mathcal{D}^p_{p-1}$ such that*

$$\lim_{r \to 1^-} \frac{|f(re^{it})|}{\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\log\log \frac{1}{1-r}\right)^{-1}} = \infty, \quad \textit{for a.e. } e^{it} \in \partial\Delta. \tag{34}$$

PROOF. Take $p > 2$. Define

$$a_k = \frac{1}{k^{1/p} \log 2k}, \quad k = 1, 2, \dots$$

and

$$f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}, \quad z \in \Delta.$$

Since $\sum_{k=1}^{\infty} |a_k|^p < \infty$, by Proposition A, we have that $f \in \mathcal{D}^p_{p-1}$.

On the other hand,

$$\left(\sum_{k=1}^{N} |a_k|^2\right)^{1/2} = \left(\sum_{k=1}^{N} \frac{1}{k^{2/p} \log^2 2k}\right)^{1/2}$$
$$\sim \left(\int_1^N \frac{1}{x^{2/p} \log^2 2x}\, dx\right)^{1/2} \sim \frac{N^{\frac{1}{2}-\frac{1}{p}}}{\log N}, \quad \text{as } N \to \infty,$$

and then it is easy to see that

$$M_2(r,f) = I_2(r,f)^{\frac{1}{2}} \sim \frac{\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}}}{\log\log \frac{1}{1-r}}, \quad \text{as } r \to 1^-. \tag{35}$$

Now, by the law of the iterated logarithm for lacunary series, see [35], we have that

$$\lim_{r \to 1^-} \frac{|f(re^{it})|}{\left[I_2(r,f) \log\log\log I_2(r,f)\right]^{\frac{1}{2}}} = 1, \quad \text{for a. e. } e^{it} \in \partial\Delta. \tag{36}$$

Now we observe that (36) and (35) imply (34). This finishes the proof. $\square$

## 4. Zeros of $\mathcal{D}^p_{p-1}$ functions

**4.1. Products of the zeros of $\mathcal{D}^p_{p-1}$ functions** We start recalling the the following result due to Horowitz, (see p. 65 of [17]).

LEMMA D. Let $f$ be an analytic function in $\Delta$ with $f(0) \neq 0$ and let $\{z_k\}$ be the sequence of ordered zeros of $f$. If $0 < p < \infty$, $0 \leq r < 1$ and $N$ is a positive integer, then

$$|f(0)|^p \prod_{k=1}^{N} \frac{r^p}{|z_k|^p} \leq M_p(r, f)^p. \tag{37}$$

This lemma and the estimates for the integral means of $\mathcal{D}^p_{p-1}$-functions obtained in section 3.1 are the basic ingredients in the proofs of Theorem 1.6 and Theorem 1.7. This method was use by C. Horowitz in [17] for the Bergman spaces and later on by the first author of this paper, M. Nowak and P. Waniurski in [15] for the Bloch space $\mathcal{B}$ and some other related spaces.

PROOF OF THEOREM 1.6. Let $p$, $f$ and $\{z_k\}_{k=1}^{\infty}$ be as in the statement of Theorem 1.6. Using Theorem 1.4, we see that $f$ satifies (8) and then using Lemma D with $p = 2$, we deduce that

$$\prod_{k=1}^{N} \frac{r}{|z_k|} \leq C M_2(r, f) \leq C\Big(\log \frac{1}{1-r}\Big)^{\frac{1}{2}-\frac{1}{p}}, \quad \text{if } r \text{ is close enough to 1.} \tag{38}$$

Now, taking $r = 1 - \frac{1}{N}$ with $N$ big enough in (38) and bearing in mind that $\Big(1 - \frac{1}{N}\Big)^N > \frac{1}{2e}$, we deduce that

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \leq C(\log N)^{\frac{1}{2}-\frac{1}{p}}. \tag{39}$$

This finishes the proof. $\square$

Our next objective is to prove Theorem 1.7 which asserts that Theorem 1.6 is sharp. We start recalling some notation and facts from Nevalinna theory (see [16], [23] or [31]) which will be needed in our proof.

Let $f$ be a non-constant analytic function in $\Delta$. For any $a \in \mathbb{C}$ and $0 < r < 1$, we denote by $n(r, a, f)$ the number of zeros $f - a$ in the disc $\{|z| \leq r\}$, where each zero is counted according to its multiplicity. We define also

$$N(r, a, f) \stackrel{\text{def}}{=} \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t}\, dt + n(0, a, f) \log r, \quad 0 < r < 1. \tag{40}$$

For simplicity, we shall write

$$n(r, f) = n(r, 0, f) : \qquad N(r, f) = N(r, 0, f).$$

The Nevanlinna characteristic function $T(r,f)$ is defined by

$$T(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta, \quad 0 < r < 1.$$

The proximity function $m(r,a,f)$ is given by

$$m(r,a,f) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \frac{1}{|f(re^{it}) - a|}\, dt, \quad 0 < r < 1.$$

Now we can state the *First Fundamental Theorem of Nevanlinna.*

Theorem E. Let $f$ be a non-constant analytic function in $\Delta$. Then

$$m(r,a,f) + N(r,a,f) = T(r,f) + O(1), \qquad \text{as } r \to 1^-.$$

for every $a \in \mathbb{C}$.

Now we can prove the following result.

Proposition 4.1. *If $2 < p < \infty$ and $f$ is a non-constant $\mathcal{D}^p_{p-1}$-function, then*

$$n(r,a,f) = \mathrm{O}\left(\frac{1}{1-r}\log\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1^-, \text{ for all } a \in \mathbb{C}. \tag{41}$$

Proof. Using the arithmetic-geometric mean inequality we obtain

$$\begin{aligned} &T(r,f) \le \frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left(|f(re^{it})|^2 + 1\right) dt \\ \le &\frac{1}{2}\log\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|f(re^{it})|^2 + 1\right) dt\right) \le \frac{1}{2}\log\Big(I_2(r,f) + 1\Big), \end{aligned}$$

which, with part (ii) of Theorem 1.4, gives

$$T(r,f) = \mathrm{O}\left(\log\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1^-. \tag{42}$$

Using Theorem E, we deduce that

$$N(r,a,f) = \mathrm{O}\left(\log\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1^-, \text{ for all } a \in \mathbb{C}. \tag{43}$$

Now, it is well known (see p. 22 of [4]) that this implies (41). $\square$

Now, we can proceed with the proof of Theorem 1.7.

Proof of Theorem 1.7. Take $p$ and $\beta$ with $2 < p < \infty$ and $0 < \beta < \frac{1}{2} - \frac{1}{p}$. Take $f \in \mathcal{D}^p_{p-1}$ with $f(0) \neq 0$ and

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{it})|\,dt\right) \neq \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad \text{as } r \to 1^-, \tag{44}$$

such a function exists by Theorem 1.5. Using (44) we see that there exist a sequence $\{r_j\}_{j=1}^{\infty} \subset (0,1)$ with $r_j \uparrow 1$ and a positive constant $C$ (indepedent of $j$), such that

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r_je^{it})|\,dt\right) \geq C\left(\log\frac{1}{1-r_j}\right)^{\beta}, \quad j = 1, 2 \ldots. \tag{45}$$

We shall write n(r) instead of $n(r, f)$ for simplicity. Using Jensen's formula (see [1], p. 206) and (45) we deduce that

$$|f(0)| \prod_{k=1}^{n(r_j)} \frac{r_j}{|z_k|} \geq C\left(\log\frac{1}{1-r_j}\right)^{\beta}, \quad j = 1, 2 \ldots, \tag{46}$$

which implies that

$$n(r_j) \to \infty, \quad \text{as } j \to \infty. \tag{47}$$

On the other hand, Proposition 4.1 implies that there exists $C > 0$ such that

$$n(r) \leq C\frac{1}{1-r}\log\log\frac{1}{1-r}, \quad \text{if } r \text{ is sufficiently close to 1.}$$

This implies that

$$\log n(r) \leq C\log\frac{1}{1-r}, \quad \text{if } r \text{ is sufficiently close to 1,}$$

which, together with (46), shows that there exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$

$$|f(0)| \prod_{k=1}^{n(r_j)} \frac{r_j}{|z_k|} \geq C\Big[\log n(r_j)\Big]^{\beta}.$$

This finishes the proof. □

**4.2. A substitute of Blaschke condition** If $2 < p < \infty$ the sequence $\{z_k\}$ of ordered zeros of a non trivial $\mathcal{D}^p_{p-1}$ function need not satisfy the Blaschke condition. Indeed, the Blaschke condition is equivalent to saying that $\prod_{n=1}^{N} \frac{1}{|z_n|} = \mathrm{O}(1)$ and we have seen that this is not always true. Using Theorem 1.6 and arguing exactly as in the proof of Theorem 5 of [15] we can prove the following result.

THEOREM 4.2. *Let $2 < p < \infty$ and let $f \in \mathcal{D}^p_{p-1}$ with $f \not\equiv 0$. Let $\{z_k\}_{k=1}^{\infty}$ be the sequence of zeros of $f$. Then*

$$\sum_{|z_k|>1-\frac{1}{e}} (1-|z_k|)\left(\log\log\frac{1}{1-|z_k|}\right)^{-\alpha} < \infty \tag{48}$$

*for all $\alpha > 1$.*

Next, we shall prove that the condition $\alpha > 1$ is needed in Theorem 4.2.

THEOREM 4.3. *Let $2 < p < \infty$. Then there exists a function $f \in \mathcal{D}^p_{p-1}$ wirh $f \not\equiv 0$, whose sequence of zeros $\{z_k\}_{k=1}^{\infty}$ satisfies*

$$\sum_{|z_k|>1-\frac{1}{e}} (1-|z_k|)\left(\log\log\frac{1}{1-|z_k|}\right)^{-1} = \infty. \tag{49}$$

PROOF. Set

$$g(z) = \sum_{k=1}^{\infty} k^{-\frac{p+2}{4p}} z^{2^k}, \quad z \in \Delta.$$

Since $g$ is given by a power series with Hadamard gaps and

$$\sum_{k=1}^{\infty} k^{-\frac{p+2}{4}} < \infty,$$

it follows that $g \in \mathcal{D}^p_{p-1}$.

We shall follow the argument of the proof of Theorem 6 of [15] Set

$$r_n = 1 - \frac{1}{2^n} \quad n = 1, 2, 3, \dots \tag{50}$$

It is easy to see that, for all sufficiently large $n$, $I_2(r_n, g) \geq Cn^{\frac{1}{2}-\frac{1}{p}}$, which, since $\log\frac{1}{1-r_n} = n\log 2$, implies that

$$I_2(r_n, g) \geq C\left(\log\frac{1}{1-r_n}\right)^{\frac{1}{2}-\frac{1}{p}} \quad \text{if } n \text{ is sufficiently large.} \tag{51}$$

Now, since $\log\frac{1}{1-r_n} \sim \log\frac{1}{1-r_{n+1}}$, as $n \to \infty$, and since $I_2(r, g)$ and $\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}}$ are increasing functions of $r$, we deduce

$$I_2(r, g) \geq C\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}} \tag{52}$$

if $r$ is sufficiently close to 1.

Using this and arguing as in p. 126 of [15] we deduce that there exist a complex number $a$ with $g(0) \neq a$, a positive constant $\beta$ and a number $r_0 \in (0,1)$ such that

$$N(r,a,g) \geq \beta \log\log \frac{1}{1-r} \quad r \in (r_0, 1). \tag{53}$$

Take such an $a \in \mathbb{C}$ and set

$$f(z) = g(z) - a, \quad z \in \Delta.$$

Then $f \in \mathcal{D}^p_{p-1}$ and $f(0) \neq 0$. Also (53) can be written as

$$N(r,f) \geq \beta \log\log \frac{1}{1-r} \quad r \in (r_0, 1). \tag{54}$$

Let $\{z_n\}$ be the sequence of zeros of $f$. Using Proposition 4.1 and arging as in p. 127 of [15], we obtain (49). $\square$

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