Macroeconomics

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Course outline

1. Session I: Introduction to DSGE modelling
2. Session II: Solving a DSGE model with Dynare
3. Session III: DSGE model estimation with Dynare

Textbook

- Introduction to Dynammic Macroeconomic General Equililbrium Models. José L. Torres. Verson Press, Series in Economic Methodology
- Other materials: http://webpersonal.uma.es/de/jtorres/jtorres.htm (Teaching section)

Session I: Introduction to DSGE modelling with Dynare

1. A brief introduction to DSGE modeling
2. A simple DSGE model
3. The steady state
4. How to solve a DSGE model?
5. A prototype Newton algorithm
6. Log-linearization of the model
7. Policy functions
8. Solution methods for DSGE models

I.1. A brief introduction to DSGE modeling

- DSGE: Dynamic Stochastic General Equilibrium modelling. This is the standard tool in modern macroeconomic analysis. This is a fully integrated framework for policy analysis.[1] Unified framework to study both business cycles and economic growth.
- **Dynamic**: Time is important. Notice that saving today implies consumption in the future. Forward-looking behavior of rational agents. Decisions today depend on expectations about the future.
- **Stochastic**: The model can be deterministic or stochastic. The difference is in the consideration of stochastic shocks (business cycles).
- **General Equilibrium**: Simultaneous determination of endogenous variables, markets clearing, and Walras law.

[1] Ramsey, F. (1928): A mathematical theory of saving. *Economic Journal*, 38(152), 543-559.

I.1. A brief introduction to DSGE modeling

- Steps in DSGE modelling:

1. Define the model economy environment (Agents, preferences, technology, institutions, etc.). This involves a small number of equations, both function and identities.
2. Parameterize functions (Consumers' utility function, production function, adjustment cost functions, etc.).
3. Obtain First Order Conditions. Together with feasibility conditions, technological restrictions and identities, we will arrive to a system of equations defining the model. (This is the model for Dynare)
4. Assign values to the parameters. Calibration and/or estimation.
5. Find the steady state.
6. (log)-linearize the equations.
7. Write the model in state space form.
8. Numerical solution of the DSGE model.
9. Simulation, policy analysis, shocks and forecasting.

I.1. A brief introduction to DSGE modeling

- DSGE models based on four key components:
  - **Preferences** (we want to consume as much as possible, we don't like to work,...).
  - **Endowments** (who is the owner of productive factors).
  - **Technology** (inputs, returns to scale, factor productivity, elasticity of substitution, ...).
  - **Institutional environment** (perfect or imperfect competition, information, government, nominal and real frictions,...)

I.1. A brief introduction to DSGE modeling

- Types of agents:
  - Households
  - Firms
  - The government
  - Capitalists
  - Central Bank
  - Financial Institutions
  - Foreign sector
  - International investors

I.1. A brief introduction to DSGE modeling

- DSGE models varieties: From the basic canonical RBC model (7-8 equations, 5-6 equations under the central planner solution in the deterministic and stochastic cases, respectively) to large-scale huge New Keynesian model (hundreds of equations: For instance, the QUEST III model has 100 equations).
- This framework is adequate to study the effects of a stochastic shock hitting the economy.
- It can also be used to study structural changes in the economy (for instance, a tax change or the introduction of a new tax).
- Shocks can hit the economy today or in any future time (in this case they are anticipated with perfect foresight).
- Shocks can be one shot or be long lasting.
- Standard procedure: A positive shock today and no other shock in the future.

I.1. A brief introduction to DSGE modeling

- DSGE models do not have explicit solutions, except for some simple cases (logarithmic utility functions and full depreciation of capital). DSGE models cannot be solved directly by hand.
- DSGE modelling requires the use of numerical and computational methods to obtain approximate solutions.
- DSGE models have two key characteristics:
  - Non-linear system of dynamic equations
  - Expectations about future endogenous variables

I.1. A brief introduction to DSGE modeling

- The key to solve a DSGE model consists in representing functional forms for the control variables (for instance, consumption) as function of lagged state variables (for instance, capital stock). One we have these functions, the system becomes **recursive** and then, given initial values for the state variables, the dynamic process for the control variables can be generated.
- This are the so called **decisions rules** or **policy functions**. The term decision rule or policy function refers to functional equations, that is, functions of functions, describing the dynamics of the forward-looking control variables.

I.1. A brief introduction to DSGE modeling

- At the beginning, it was very difficult to solve a DSGE model. Today is a much more easy task, thanks to the development of many techniques and to the existence of specific computer programs developed by clever people.
- Large variety of alternative numerical methods. Local methods (i.e. perturbation method) versus global methods (i.e. projection methods: dynamic programming, Chebyshev-polynominal method, finite-elements method, extended path method, parameterized expectations, Neural networks, etc.).
- Standard producure uses a local-approximation method around the steady state to solve DSGE models (perturbation method).

I.1. A brief introduction to DSGE modeling

- In general, a DSGE model can be defined as a stochastic non-linear forward-looking system with rational expectations. This system (mainly first order conditions) can be represented as follows:

$$\begin{aligned} E_t\left[f\left(y_{t+1}, y_t, y_{t-1}, \varepsilon_{t+1}, \Omega\right)\right] &= 0 \\ E_t\left(\varepsilon_{t+1}\right) &= 0 \\ E_t\left(\varepsilon_{t+1}, \varepsilon'_{t+1}\right) &= \Sigma_\varepsilon \end{aligned}$$

where E_t is the expectations operator, y is the vector of all endogenous variables (including state variables), ε is a vector of exogenous stochastic shocks (structural innovations) and Ω is the set of structural or deep parameters of the model.

I.1. A brief introduction to DSGE modeling

- If we distinguish between control, y_t, and state (pre-determined) variables, x_t, the system can be represented as follows:

$$\begin{aligned} E_t\left[f\left(y_{t+1}, y_t, y_{t-1}, x_t, x_{t-1}, \varepsilon_t, \Omega\right)\right] &= 0 \\ E_t\left(\varepsilon_t\right) &= 0 \\ E_t\left(\varepsilon_t, \varepsilon_t'\right) &= \Sigma_\varepsilon \end{aligned}$$

- In general, the solution of a DSGE model can be defined as:

$$E_t\left[f\left(g\left(x\right)\right)\right] = 0$$

where $f(y)$ is known and $g(x)$ is the unknown policy (or transition) function.

I.1. A brief introduction to DSGE modeling

- A solution to that system is a function in recursive form such as first order conditions and feasibility conditions are satisfies:

$$y_t = g(x_{t-1}, \varepsilon_t, \Omega)$$

  that is, endogenous variables are calculated as a function of their past values and the contemporaneous structural shocks.
- The function $g(x_{t-1}, \varepsilon_t, \Omega)$ represents the set of so-called policy (for the endogenous) and transition (for the states) functions. In general, it is not possible to obtain a closed form solution, so Dynare does a local approximation to the true solution.

I.1. A brief introduction to DSGE modeling

- Substituting policy and transitions functions into the model we get:

$$E_t[f(g(x_t, \varepsilon_{t+1}, \Omega), g(x_{t-1}, \varepsilon_t, \Omega), x_{t-1}, \varepsilon_t, \Omega] = 0$$

- Again, substituting for x_t we get:

$$E_t[f(g(g(x_{t-1}, \varepsilon_t, \Omega), \varepsilon_{t+1}, \Omega)), g(x_{t-1}, \varepsilon_t, \Omega), x_{t-1}, \varepsilon_t, \Omega] = 0$$

- Then, Dynare solves the above system of equations locally around the deterministic steady state, which only depends on the parameters of the model.

I.1. A brief introduction to DSGE modeling

- The steady sate can be defined as:

$$[f(\overline{y}, \overline{y}, \overline{y}, 0, \Omega) = 0$$

- The function $g(\cdot)$ must satisfy:

$$\overline{y} = g(\overline{y}, 0, \Omega) = 0$$

- Once the steady state had been calculated, we can compute the Jacobian matrix associated to $f(\cdot)$ to do approximations around the steady state.

I.1. A brief introduction to DSGE modeling

- Standard procedure: The non-linear system of equations is converted to a linear system of equations using a linear approximation for each equation in the neighborhood of the steady state (perturbation method).
- Models can be written in level (linearization) or in logs (log-linearization).
- The solution method implies to obtain approximations of the policy and transition functions that satisfy the first-order conditions. In the case of an approximation of first order (linear) we get:

$$y_t = \overline{y} + g_{y,x}(x_t - \overline{x})$$

where a bar over a variable indicates the steady state values, y is the set of control and state variables (all endogenous variables) and x is the set of state (predetermined and exogenous) variables.

I.1. A brief introduction to DSGE modeling

- Perturbation methods solve sequentially for the coefficients of the Taylor expansion of $g(y)$. Let $g(x_t)$ represent the policy function for the endogenous variables, y_t, for a vector of state variables, x_t, around the steady state, $\overline{x}$:

$$\begin{aligned} g(x_t) &= g(\overline{x}) + g'(\overline{x})(x_t - \overline{x}) + \frac{1}{2}g''(\overline{x})(x_t - \overline{x})^2 + ... \\ &= \overline{x} + g_1(x_t - \overline{x}) + \frac{g_2}{2}(x_t - \overline{x})^2 + ... \end{aligned}$$

- The goal is to find $\overline{x}, g_1, g_2, ...$

I.1. A brief introduction to DSGE modeling

- Solution can be done using first, second and third-order perturbation methods, depending on the order used for the Taylor expansion around the steady state.
- This implies the log-linearization of the equations of the model around the steady state. Next, the linear model is solved using a method for solving a forward-looking rational expectations system of equations.
- Several alternative solution method. The pioneer is the one proposed by Blanchard and Kahn (1980).[2] Other DSGE models solution methods are Binder and Pesaran (1995), Klein (2000), Sims (2001), undetermined coefficients, etc.

[2] Blanchard, O.J. and Khan C.M. (1980): The solution of linear difference models under rational expectations. *Econometrica*, 48, 1305-1312.

I.1. A brief introduction to DSGE modeling

- Dynare use the Klein method which is similar to the Blanchard-Kahn method but using a complex generalized Schur (QZ) decomposition. One reason why Dynare uses this method is that the computing solution algorithm is fast and easy to implement (and a key matrix has to be inverted).
- Important: Perturbation method is useful when the dynamics of the model consists in small deviations from the steady state values of the variables, i.e., the shocks represent small deviations from the steady state.

I.1. A brief introduction to DSGE modeling

- In summary, for DSGE modelling (to do for instance policy analysis) you need different things (you must be something similar to a Renaissance man):
  1. How to build a DSGE model (this is the easier part)
  2. How to solve a DSGE model (this is a much more difficult task than step 1)
  3. Programming skills (you must know how to write a computer program for computing step 2)

I.2. A simple DSGE model

- Basic RBC model. Very simple model economy setup:
  - Two type of economic agents: Households and firms.
  - Each agent maximizes a given objective function subject to a given restriction.
  - A lot of implicit assumptions: perfect capital markets, utility function additively separable in time, separability between consumption and leisure, saving as a state variable, perfect competition, constant return to scale, no externalities, etc. These implicit assumptions can be relaxed resulting in a large variety of DSGE models.

I.2. A simple DSGE model

- Households maximization problem (the heart of a DSGE model):

$$\max_{(C_t, O_t)} E_t \sum_{t=0}^{\infty} \beta^t U(C_t, O_t)$$

where β is the intertemporal discount factor, $\beta \in (0, 1)$, and where $E_t(\cdot)$ is the mathematical expectation operator of future variables at time t, C_t is consumption and O_t is leisure.

- Utility function can be extended to include more arguments:

$$U(C_t, C_{g,t}, O_t, M_t, H_t, S_t, X_t, ...)$$

where $C_{g,t}$ is consumption of goods and services provided by the government, M_t is money, H_t time devoted to home production (meals, child rearing, laundry, house cleaning, etc.), S_t, time devoted to skill activities, X_t pollution, ...

I.2. A simple DSGE model

- However, we cannot use numerical method with general functional forms. We must use a given specification for the utility function.
- We can use any particular specification that satisfy the following conditions:

$$U_C > 0$$

$$U_O > 0$$

$$U_{CC} < 0$$

$$U_{OO} < 0$$

I.2. A simple DSGE model

- We use the following specification:

$$U(C_t, O_t) = \gamma \log C_t + (1 - \gamma) \log(N_t \overline{H} - L_t),$$

- γ: weight for consumption over total income.
- N_t : Population (Working-age population between 16 and 65 years old).
- $\overline{H}$: Total discretionary available time in hours. Approximately 5.000 hours per year (16 hours per day x 6 days per week x 52 weeks per year=4,992 hours).
- L_t : Total working hours.
- Total available discretionary time is normalized to 1: $L_t + O_t = 1$.

I.2. A simple DSGE model

- More assumptions: Household are the owner of capital. Investment decisions are taken by the households and not by the firms.
- Saving is directly converted to investment at no cost.
- Output and investment measured in units of consumption.
- Firms rent production factors every period.

- Households budget constraint:

$$P_t^C C_t + P_t^I I_t = W_t L_t + R_t K_t$$

where I_t is investment (under the assumption that investment is equal to saving), W_t is the wage, R_t is the capital rate of return and K_t is the physical capital stock.

- Physical capital accumulation process:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

where δ $(0 < \delta < 1)$ is the capital depreciation rate.

- It is assumed that $P_t^C = P_t^I = 1$.

I.2. A simple DSGE model

- General problem for the households:

$$\max_{(C_t, I_t, O_t)} \mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left[\gamma \log C_t + (1-\gamma) \log(1-L_t)\right]$$

subject to:

$$C_t + I_t = W_t L_t + R_t K_t$$

$$K_{t+1} = (1-\delta) K_t + I_t$$

I.2. A simple DSGE model

- One simple solution is the use of the Lagrangian auxiliary function:

$$\max_{(C_t,K_t,O_t)} \beta^t \left\{ \begin{array}{c} \gamma \log C_t + (1-\gamma)\log(1-L_t) \\ -\lambda_t \left[C_t + K_{t+1} - W_t L_t - (R_t + 1 - \delta) K_t \right] \end{array} \right\}$$

Household first order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_t} &: \quad \frac{\gamma}{C_t} - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial L_t} &: \quad \frac{1-\gamma}{1-L_t} - \lambda_t W_t = 0 \\
\frac{\partial \mathcal{L}}{\partial K_t} &: \quad \beta^t \lambda_t \left[R_t + 1 - \delta \right] - \beta^{t-1} \lambda_{t-1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &: \quad C_t + K_{t+1} - (R_t + 1 - \delta) K_t - W_t L_t = 0
\end{aligned}$$

- Equilibrium condition that equals the marginal rate of substitution between consumption and leisure to the opportunity cost of an additional unit of leisure:

$$\frac{1-\gamma}{\gamma}\frac{C_t}{1-L_t} = W_t$$

- Equilibrium condition that equals the marginal rate of consumption and investment:

$$\frac{C_t}{C_{t-1}} = \beta\left[R_t + 1 - \delta\right]$$

I.2. A simple DSGE model

- The firms: Aggregate production function:

$$Y_t = A_t F(K_t, L_t)$$

  - Y_t: Output.
  - A_t: Total Factor Productivity. This can be considered as exogenously given (a constant) in the case of a deterministic model or an endogenous variable that follows a stochastic process in the case of a stochastic model.

- Technology must satisfy the following properties:

$$F_K > 0, F_L > 0$$

$$F_{KK} < 0, F_{LL} < 0$$

$$F_{KL} > 0$$

I.2. A simple DSGE model

- Problem for the firms: profit maximization:

$$\max \Pi_t = P_t Y_t - W_t L_t - R_t K_t$$

subject to:

$$Y_t = A_t F(K_t, L_t)$$

- Under the assumption of constant returns to scale and competitive markets: $\Pi_t = 0$.
- First order conditions:

$$A_t P_t F_K(K_t, L_t) - R_t = 0$$

$$A_t P_t F_L(K_t, L_t) - W_t = 0$$

I.2. A simple DSGE model

- Factor relative prices equals to their productivity:

$$A_t F_K(K_t, L_t) = \frac{R_t}{P_t}$$

$$A_t F_L(K_t, L_t) = \frac{W_t}{P_t}$$

- The price of the final good is normalized to 1 $(P_t = 1)$:

$$A_t F_K(K_t, L_t) = R_t$$

$$A_t F_L(K_t, L_t) = W_t$$

I.2. A simple DSGE model

- Standard specification: Cobb-Douglas production function:

$$A_t F(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- α : elasticity of capital relative to output.
- Main characteristic: Unitary elasticity of substitution between capital and labor. This is something between a Leontief technology and a perfect substitution among inputs technology.

- Therefore, first order conditions are:

$$\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0$$

$$(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} - W_t = 0$$

- Or:

$$R_t = \frac{\alpha A_t K_t^{\alpha} L_t^{1-\alpha}}{K_t} = \alpha \frac{Y_t}{K_t}$$

$$W_t = \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{1-\alpha}}{L_t} = (1-\alpha) \frac{Y_t}{L_t}$$

I.2. A simple DSGE model

- Equilibrium: Each type of economic agent takes its own decisions over the control variables. Their interaction determines the macroeconomic equilibrium.
- Households decide how much to consume, C_t, how much to invest (save), I_t and how much to work, L_t, with the objective of maximizing their happiness, taking as given the prices of the inputs.
- Firms produce a given amount of final goods, Y_t, depending on how much capital, K_t and labor L_t, they will hire, given the prices of the production factors.

I.2. A simple DSGE model

- The balanced path of the economy is composed of the following three sets of elements:
  - A pricing system for W and R.
  - A set of values assigned to Y, C, L and K.
  - A feasibility constraint of the economy, given by:

$$Y_t = C_t + I_t$$

- All markets (good market, labor market, capital market) are in equilibrium.

I.2. A simple DSGE model

- The competitive equilibrium for our economy is a sequence of consumption, leisure, and investment by consumers $\{C_t, L_t, I_t\}_{t=0}^{\infty}$ and a sequence of capital and labor hours used by firms $\{K_t, L_t\}_{t=0}^{\infty}$, such that given a sequence of prices $\{W_t, R_t\}_{t=0}^{\infty}$:
  - i) The consumers optimization problem is satisfied;
  - ii) Profit maximization FOCs for the firms hold;
  - and iii) The feasibility condition of the economy holds.

- By substituting the relative price of inputs in the households equilibrium conditions:

$$\frac{1-\gamma}{\gamma}\frac{C_t}{1-L_t} = (1-\alpha)K_t^{\alpha}L_t^{-\alpha}$$

$$\frac{C_t}{C_{t-1}} = \beta\left[\alpha K_t^{\alpha-1}L_t^{1-\alpha} + 1 - \delta\right]$$

- On the other hand, substituting the relative price of inputs in the households budget constraint we find that:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = C_t + K_{t+1} - (R_t + 1 - \delta)K_t - W_t L_t = 0$$

$$C_t + K_{t+1} - (\alpha K_t^{\alpha-1} L_t^{1-\alpha} + 1 - \delta)K_t - (1-\alpha)K_t^{\alpha} L_t^{-\alpha} L_t = 0$$

$$C_t + K_{t+1} - K_t - \alpha K_t^{\alpha} L_t^{1-\alpha} + \delta K_t - K_t^{\alpha} L_t^{1-\alpha} + \alpha K_t^{\alpha} L_t^{1-\alpha} = 0$$

$$C_t + K_{t+1} - (1-\delta)K_t - K_t^{\alpha} L_t^{1-\alpha} = 0$$

- In summary, we have to dynamic equations driving consumption and capital:

$$C_t = \beta \left[\alpha K_t^{\alpha-1} L_t^{1-\alpha} + 1 - \delta \right] C_{t-1}$$

$$K_{t+1} = (1-\delta) K_t + K_t^{\alpha} L_t^{1-\alpha} - C_t$$

plus and static equation for the labor supply:

$$\frac{1-\gamma}{\gamma} \frac{C_t}{1-L_t} = (1-\alpha) K_t^{\alpha} L_t^{-\alpha}$$

I.2. A simple DSGE model

Finally, the equations describing the model (deterministic) economy are:

$$\begin{aligned}
\frac{(1-\gamma)}{\gamma}\frac{C_t}{1-L_t} &= W_t \\
E_t\frac{C_{t+1}}{C_t} &= \beta E_t\left[R_{t+1}+1-\delta\right] \\
R_t &= \frac{\alpha A_t K_t^{\alpha} L_t^{1-\alpha}}{K_t} = \alpha\frac{Y_t}{K_t} \\
W_t &= \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{1-\alpha}}{L_t} = (1-\alpha)\frac{Y_t}{L_t} \\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
K_{t+1} &= (1-\delta)K_t + I_t \\
C_t + I_t &= Y_t
\end{aligned}$$

I.2. A simple DSGE model

- Moving to a stochastic model. In the previous system of equations, Total Factor Productivity, A_t, is not defined. If we consider TFP as an exogenous and constant variable, thus the model is deterministic.
- Alternatively we can consider TFP as an endogenous variable just by assuming that it is not a constant but follows a particular stochastic process. In this case the model will be stochastic.
- We can consider a large number of stochastic shocks in a DSGE model. For instance, in this simple DSGE model we can introduce up to six stochastic shocks (an aggregate productivity shock, an aggregate shock to the utility function, a consumption shock, a labor supply shock, an investment-specific technological shock, and a labor-augmented shock).

I.2. A simple DSGE model

- We assume that the aggregate productivity shock follows a first order autoregressive process, such that:

$$\ln A_t = (1-\rho_A)\ln \overline{A} + \rho_A \ln A_{t-1} + \varepsilon_t^A, \qquad \varepsilon_t^A \sim N(0, \sigma_A^2)$$

where $|\rho_A| < 1$ for the process to be stationary and where $\overline{A}$ is the steady state value for A_t.

I.2. A simple DSGE model

- As an example, if we are interesting is study stochastic shocks in the households utility function, we can consider the following problem:

$$\max_{(C_t,K_t,L_t)} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t B_t \left[\gamma D_t \log C_t + (1-\gamma) H_t \log(1-L_t)\right]$$

where B_t is a disturbance that reflects a preference shock that affects the consumer's intertemporal substitution, D_t represents a consumption shock and H_t represents a labor supply shock.

I.2. A simple DSGE model

- If two or more structural shocks are present in the model, then we can consider correlation among them.
- Let's assume that the process followed by these three disturbance is the following:

$$\begin{bmatrix} \ln B_t \\ \ln D_t \\ \ln H_t \end{bmatrix} = \begin{bmatrix} \rho_B & v_{BD} & v_{BH} \\ v_{DB} & \rho_D & v_{DH} \\ v_{HB} & v_{HD} & \rho_H \end{bmatrix} \begin{bmatrix} \ln B_{t-1} \\ \ln D_{t-1} \\ \ln H_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^B \\ \varepsilon_t^D \\ \varepsilon_t^H \end{bmatrix}$$

where $|\rho_i \pm v_{ij}| < 1$, $i \neq j$, $i, j = B, D, H$, in order to ensure stationarity, where $E(\varepsilon_t^i) = 0$ and $E(\varepsilon_t^i \varepsilon_t^i) = \sigma_i^2$, $\forall i$.

I.2. A simple DSGE model

- Finally, the equations describing the model (stochastic) economy are:

$$
\begin{aligned}
\frac{(1-\gamma)}{\gamma}\frac{C_t}{1-L_t} &= W_t \\
E_t\frac{C_{t+1}}{C_t} &= \beta E_t\left[R_{t+1}+1-\delta\right] \\
R_t &= \frac{\alpha A_t K_t^{\alpha} L_t^{1-\alpha}}{K_t} = \alpha\frac{Y_t}{K_t} \\
W_t &= \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{1-\alpha}}{L_t} = (1-\alpha)\frac{Y_t}{L_t} \\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
K_{t+1} &= (1-\delta)K_t + I_t \\
C_t + I_t &= Y_t \\
\ln A_t &= (1-\rho_A)\ln\overline{A} + \rho_A \ln A_{t-1} + \varepsilon_t^A, \qquad \varepsilon_t^A \sim N(0,\sigma_A^2)
\end{aligned}
$$

I.3. The steady state

- **Steady state**:
- Computing the steady state:

$$f(\overline{y}, \overline{y}, \overline{y}, \overline{x}, 0) = 0$$

$$\overline{y} = g(\overline{y}, \overline{x}, 0)$$

- Using of the Taylor expansion function on the structural model in order to compute the coefficient of the unknown decision function.

I.3. The steady state

- Steady State: To compute the steady state, first, eliminate time. Second, assume that $\sigma_A = 0$. This means, for instance, that we would have an equilibrium value for consumption such as $... = C_{t-1} = C_t = C_{t+1} = ... = \overline{C}$. Thus, the equations of the model can be written as:

$$\frac{(1-\gamma)}{\gamma}\frac{\overline{C}}{1-\overline{L}} = (1-\alpha)\overline{AK}^{\alpha}\overline{L}^{-\alpha} \tag{1}$$

$$1 = \beta\left[\overline{R} + 1 - \delta\right] \tag{2}$$

$$\overline{Y} = \overline{AK}^{\alpha}\overline{L}^{1-\alpha} \tag{3}$$

$$\overline{I} = \delta\overline{K} \tag{4}$$

$$\overline{C} + \overline{I} = \overline{Y} \tag{5}$$

I.3. The steady state

- From equation (2) it can be obtained the steady state value for the rental rate of capital:

$$\overline{R} = \frac{1}{\beta} + \delta - 1$$

- Note that:

$$\overline{R} = \alpha \frac{\overline{Y}}{\overline{K}}$$

I.3. The steady state

- The easy thing it first express all the steady state values in terms of the steady state value for output. Expression (2) can be written as:

$$1 = \beta \left[\alpha \frac{\overline{Y}}{\overline{K}} + 1 - \delta \right]$$

- Solving for $\overline{K}$ results:

$$\overline{K} = \frac{\alpha\beta}{1 - \beta + \beta\delta} \overline{Y} \tag{8}$$

I.3. The steady state

- Second, using (4) and given (8) the investment steady state value is given by:

$$\overline{I} = \frac{\alpha\beta\delta}{1-\beta+\beta\delta}\overline{Y} \tag{9}$$

- Third, using expressions (5) and (9) we reach the steady state value for consumption:

$$\overline{C} = \frac{1-\beta+(1-\alpha)\beta\delta}{1-\beta+\beta\delta}\overline{Y} \tag{10}$$

I.3. The steady state

- Next, using (1) we obtain:

$$\overline{L} = \frac{\gamma(1-\alpha)(1-\beta+\beta\delta)}{(1-\gamma)\left(1-\beta+(1-\alpha)\beta\delta\right)+\gamma(1-\alpha)\left(1-\beta+\beta\delta\right)} \tag{11}$$

- Finally, substituting (8) and (11) in (3) we reach the steady state value of output:

$$\overline{Y} = \overline{A}^{\frac{1}{1-\alpha}}\left[\frac{\alpha\beta}{1-\beta+\beta\delta}\right]^{\frac{\alpha}{1-\alpha}} \tag{12}$$

$$\left[\frac{\gamma(1-\alpha)(1-\beta+\beta\delta)}{(1-\gamma)\left(1-\beta+(1-\alpha)\beta\delta\right)+\gamma(1-\alpha)\left(1-\beta+\beta\delta\right)}\right] \tag{13}$$

I.4. How to solve a DSGE model

- Solving a DSGE model implies the transformation of a **non-linear** system of equations with **forward-looking** variables into a **linear in-difference** system of equations once rational expectations have been solved.
- Standard approach uses a local approximation method: the perturbation method. Perturbation methods involve three steps:
  - First, compute the steady state of the model.
  - Second, (log)-linearization of the equations of the model.
  - Third, implementation of a solution method (policy and transition functions or decision rules).

I.4. How to solve a DSGE model

- Solving for the steady state is a very difficult task. Even for simple models, we can have lot difficulties in finding the steady state.
- Several algorithm can be used for solving a system of equations.
- We need very good initial conditions. The guess values must be sufficiently close to the true state steady values.
- If the model is complex enough, the best strategy is to start with a simpler model and then add new variables one by one.

I.4. How to solve a DSGE model

- Another alternative consists in the linearization of the model. In this case, the variable are defined in terms of percent deviation from the steady state. However, this not solve completely the problem, given that in the case that equations involve sums (this is the case for instance of the feasibility condition), the linearized equations will include ratios of the steady state values, which in any case must be computed.

I.5. A prototype Newton algorithm

- Standard procedure uses a Newton type algorithm to solve both static and dynamic system of equations. In general a DSGE model is composed by a set of first order conditions and feasibility constraints. This is a non-linear system.
- In general we will have n equations with n unknowns: $F : R^n \longrightarrow R^n$.
- The solution for this problem consists in to find a vector $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ of R^n such as $F : R^n \longrightarrow R^n$ should be $F(\hat{x}) = 0$.

I.5. A prototype Newton algorithm

- To run the Newton algorithm we need a guess about the solution, x_0. This is the so-called the "seed" of the algorithm. To build the seed we must use all the information about F.
- Another ingredient of the algorithm is the tolerance criterion. This is the minimum value for the error in the approximated solution to be considered as acceptable.

I.5. A prototype Newton algorithm

- The basic of the Newton algorithm is the following. The aim of the algorithm is to find a solution, x, such as $F(x)$ will be zero. To do that, we start from a Taylor expansion function around an initial value x_n:

$$F(x) = F(x_n) + F'(x_n)(x - x_n) + \frac{F''(x_n)}{2!}(x - x_n)^2 + ...$$

- The procedure is iterative and consists in computing the above expression for the different approximations, step by step. This expression is then evaluated for different values, starting from the initial guess value, such as:

$$F(x_{n+1}) = F(x_n) + F'(x_n)(x_{n+1} - x_n) + \frac{F''(x_n)}{2!}(x_{n+1} - x_n)^2 + ...$$

I.5. A prototype Newton algorithm

- The solution is given for the value that makes the Taylor expansion equals zero:

$$F(x_n) + F'(x_n)(x_{n+1} - x_n) + \frac{F''(x_n)}{2!}(x_{n+1} - x_n)^2 + ... = 0$$

I.5. A prototype Newton algorithm

- If the Taylor expansion series is truncated in the second term (first order approximation), we have that:

$$F(x_n) + F'(x_n)(x_{n+1} - x_n) \simeq 0$$

- The above approximation is closer to zero when the value would be closer to the true solution. Operating we obtain:

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

and this is the so-called Newton-Raphson formulae.

I.5. A prototype Newton algorithm

- The structure of the algorithm is as follows. We star from an initial value, say x_0. The solution from the computation of the Newton-Raphson formulae would be:

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

- If $F(x_1)$ is (enough) close to zero, then the algorithm stops.
- If $F(x_1)$ is different from zero, then we proceed to the second iteration and compute:

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

I.5. A prototype Newton algorithm

- Again, if $F(x_2)$ is close to zero, the algorithm stops. Otherwise, we proceed to the third iteration and compute:

$$x_3 = x_2 - \frac{F(x_2)}{F'(x_2)}$$

This process will continue until we find a value for the function close to zero.

I.5. A prototype Newton algorithm

- Tolerance criterion. This criterion determines how close to (absolute) zero must be the function value to be consider as a solution. In each iteration, we can compute the error relative to zero of the solution, given by:

$$|\varepsilon_a| = \left(\frac{x_{n+1} - x_n}{x_n} \right) \times 100$$

- If $|\varepsilon_a|$ is larger that a predetermined value, ε (the tolerance), then the algorithm proceeds to the next iteration. In the case the relative error in absolute terms would be lower than the tolerance value, then the algorithm stops and the value of the last iteration is considered the final solution.

I.5. A prototype Newton algorithm

- Example: In the case in which the equation to be solved is linear, the Newton-Raphson algorithm finds the solution directly, whatsoever the guess solution, given that in this case the approximation error is zero. Lets assume that we want to find the solution $F(x) = 0$ for the following linear function:

$$F(x) = x - 2$$

- Obviously, the solution is simply $x = 2$. Imagine that we do not know the solution and we believe that the solution should be 5, $(x_0 = 5)$.

I.5. A prototype Newton algorithm

- If we apply the Newton-Raphson algorithm then we find that:

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)} = 5 - \frac{3}{1} = 2$$

- Evaluating the function for such solution we obtain:

$$F(x_1) = x_1 - 2 = 2 - 2 = 0$$

I.5. A prototype Newton algorithm

- Another example: Solve for the square root of a number y. Given that $x = \sqrt{y}$, we can write the following function $x^2 = y$. This means that solving a square root is equivalent to find the zero for the following function:

$$F(x) = x^2 - y$$

- Using the Newton-Raphson formulae we obtain:

$$x_{i+1} = x_i - \frac{x_i^2 - y}{2x_i} = \frac{2x_i^2 - x_i^2 + y}{2x_i} = \frac{1}{2}\left(x_i + \frac{y}{x_i}\right)$$

I.5. A prototype Newton algorithm

- For instance, assume that $y = 20$. The square root solution is $x = \sqrt{20} = 4.4721$. Imagine we do not known the final solution and we guess a value of $x_0 = 3$.
- Computing the Newton-Raphson algorithm we find that:

$$x_1 = x_0 - \frac{x_0^2 - y}{2x_0} = \frac{1}{2}\left(x_0 + \frac{y}{x_0}\right) = \frac{1}{2}\left(3 + \frac{20}{3}\right) = 4.8333$$

- In the next iteration we obtain:

$$x_2 = \frac{1}{2}\left(x_1 + \frac{y}{x_1}\right) = \frac{1}{2}\left(4.83 + \frac{20}{4.83}\right) = 4.4856$$

- In the next iteration we obtain:

$$x_3 = \frac{1}{2}\left(x_2 + \frac{y}{x_2}\right) = \frac{1}{2}\left(4.48 + \frac{20}{4.48}\right) = 4.4722$$

I.5. A prototype Newton algorithm

- In the case of a system of equations, the first order Taylor expansion of the function F:

$$F(x) \approx F(\overline{x}) + J(\overline{x})(x - \overline{x})$$

where $J(\overline{x})$ is the Jacobian matrix of F evaluated at $\overline{x}$:

$$J(\overline{x}) = \begin{bmatrix} F_{11}(\overline{x}) & F_{12}(\overline{x}) & \dots & F_{1n}(\overline{x}) \\ F_{21}(\overline{x}) & F_{22}(\overline{x}) & \dots & F_{2n}(\overline{x}) \\ \dots & \dots & \dots & \dots \\ F_{n1}(\overline{x}) & F_{n1}(\overline{x}) & \dots & F_{nn}(\overline{x}) \end{bmatrix}$$

and where $F_{ij}(\overline{x}) = \frac{\partial F_i(\overline{x})}{\partial x_j}$.

I.5. A prototype Newton algorithm

- Taylor theorem: As $\overline{x}$ goes closed to the value x, the higher order terms go to zero. Given that we are looking for the zero value for equation $F(x)$, the above expression can be evaluated at $\hat{x}$ and thus, be written as:

$$\hat{x} \approx \overline{x} - J(\overline{x})^{-1} F(\overline{x})$$

I.5. A prototype Newton algorithm

- The Newton-Raphson algorithm works as follows:

1. The algorithm starts from the proposed seed, x_0, and evaluates the functions $F(x_0)$ and $J^{-1}(x_0)$, to calculate:

$$x_1 = x_0 - J(x_0)^{-1}F(x_0)$$

2. Given the tolerance, ε, the distance between x_0 and x_1 is calculated. If that difference is lower than ε, then we keep the value x_0 as the approximated solution. Otherwise, step 1 is repeated but starting with the value x_1, and evaluating the functions $F(x_1)$ and $J^{-1}(x_1)$, to compute:

$$x_2 = x_1 - J(x_1)^{-1}F(x_1)$$

3. Repeat step 2. Again, if the difference is lower than the tolerance rate, then step 1 is repeated with the value x_2, evaluating $F(x_2)$ and $J^{-1}(x_2)$, to compute:

$$x_3 = x_2 - J(x_2)^{-1}F(x_2)$$

I.5. A prototype Newton algorithm

- Example: Solve the following system of equations:

$$\begin{aligned} F_1(x,z) &= x - 3z + 1 \\ F_2(x,z) &= 2x + z - 5 \end{aligned}$$

- The corresponding Jacobian to that system is:

$$J(x,z) = \begin{bmatrix} \frac{\partial F_1(x,z)}{\partial x} & \frac{\partial F_1(x,z)}{\partial z} \\ \frac{\partial F_2(x,z)}{\partial z} & \frac{\partial F_2(x,z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$$

I.5. A prototype Newton algorithm

- Our guess solution is $x = 1$, $z = 2$:
- Calculating both equations for that seed we have

$$\begin{aligned} F_1(x,z) &= x - 3z + 1 = 1 - 6 + 1 = -4 \\ F_2(x,z) &= 2x + z - 5 = 2 + 2 - 5 = -1 \end{aligned}$$

which are differerent from zero. Applying the Newton-Raphson algorithm we have that:

$$\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Again, calculating both equations for the new guess solution we have:

$$\begin{aligned} F_1(x,z) &= x - 3z + 1 = 2 - 3 + 1 = 0 \\ F_2(x,z) &= 2x + z - 5 = 4 + 1 - 5 = 0 \end{aligned}$$

- Thefore, the solution is $\widehat{x} = 2$, $\widehat{z} = 1$.

I.5. A prototype Newton algorithm

- The Newton-Raphson algorithm, although powerful, is not free of problems:

1. Is the seed is bad (or not so good), the algorithm can have problems of convergence or find a wrong solution in the case of multiple equilibria. We can start from any seed when we solve a linear equation. But for a non-linear system, we must be careful with the guess solution.
2. It is necessary to obtain analytical expressions for all partial derivatives of F. (This problem had been solved with symbolic calculus). In previou examples, the Jacobian was easy to obtain. But for more complex system of equations the Jacobian can be a very complicated task. This can be solved using numerical derivatives. Or we can use programs to calculate the Jacobian.

I.5. A prototype Newton algorithm

- Using the concept of derivative:

$$
\begin{aligned}
J_1(x) &= \lim_{h_1 \longrightarrow o} \frac{F(x) - F(x_1 + h_1, x_2, ..., x_n)}{h_1} \\
J_2(x) &= \lim_{h_1 \longrightarrow o} \frac{F(x) - F(x_1, x_2 + h_2, ..., x_n)}{h_2} \\
&... \\
J_n(x) &= \lim_{h_n \longrightarrow o} \frac{F(x) - F(x_1, x_2, ..., x_n + h_n)}{h_n}
\end{aligned}
$$

where $J_i(x)$ is a column vector with the n partial derivatives with respect to x_i.

I.5. A prototype Newton algorithm

- Numerically, we can compute approximate derivatives, using an increment h enough small to write:

$$
\begin{aligned}
J_1(x) &\approx \frac{F(x) - F(x_1 + h_1, x_2, ..., x_n)}{h_1} \\
J_2(x) &\approx \frac{F(x) - F(x_1, x_2 + h_2, ..., x_n)}{h_2} \\
&... \\
J_n(x) &\approx \frac{F(x) - F(x_1, x_2, ..., x_n + h_n)}{h_n}
\end{aligned}
$$

I.6. Log-linearization of the model

- Once the steady state of the model economy have been computed, the next step is the linearization of the model.
- The model can be linearized or log-linearized (the later is the more standard procedure).
- The log-linearization of the model consists in expressing the variables as log-linear deviations with respect to their steady state values. The log-linear deviation of a variable u around its steady state, $\overline{u}$, is denoted as $\widehat{u}$, where $\widehat{u}_t = \ln u_t - \ln \overline{u}$. That is

$$u_t = \overline{u} e^{\widehat{u}_t} \approx \overline{u}(1 + \widehat{u}_t)$$

I.6. Log-linearization of the model

- In constructing the log-linear deviations we follow two basic rules. First, for the case of two variables u_t and z_t, we have:

$$u_t z_t \approx \overline{u}(1+\widehat{u}_t)\overline{z}(1+\widehat{z}_t) \approx \overline{uz}(1+\widehat{u}_t+\widehat{z}_t)$$

  that is, we assume that the product of the two deviations, i.e., $\widehat{u}_t\widehat{z}_t$, is approximately equal to zero, as they are small numbers.
- Second, we assume the following approximation:

$$u_t^a \approx \overline{u}^a(1+\widehat{u}_t)^a \approx \overline{u}^a(1+a\widehat{u}_t)$$

I.6. Log-linearization of the model

- Taking into account the above definitions, we can proceed to the log-linearization of our model. Let start from the production function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

In steady state, the production function can be written as:

$$\overline{Y} = \overline{AK}^{\alpha} \overline{L}^{1-\alpha}$$

Therefore, using the above basic rules, we can write:

$$\overline{Y}(1+\widehat{y}_t) = \overline{AK}^{\alpha} \overline{L}^{1-\alpha}(1+\widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{l}_t)$$

Substituting, we obtain the log-linear equation for the production function:

$$\widehat{y}_t = \widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{l}_t$$

I.6. Log-linearization of the model

- This procedure must be applied to the other equations of the model. For instance, the second equation we consider is:

$$C_t = Y_t - I_t$$

By calculating the deviation with respect to the steady state we obtain:

$$\widehat{c}_t = \frac{\overline{Y}}{\overline{C}}\widehat{y}_t - \frac{\overline{I}}{\overline{C}}\widehat{i}_t$$

Substituting the steady state values in the feasibility condition of the economy, we obtain:

$$[1 - \beta + (1 - \alpha)\beta\delta]\,\widehat{c}_t = (1 - \beta + \beta\delta)\widehat{y}_t - \alpha\beta\delta\widehat{i}_t$$

I.6. Log-linearization of the model

The log-linear version of the capital stock accumulation equation is given by:

$$\widehat{k}_{t+1} = (1-\delta)\widehat{k}_t + \delta\widehat{i}_t$$

I.6. Log-linearization of the model

- Next equation of the model is the labor supply equilibrium condition, given by:

$$\frac{1-\gamma}{\gamma}\frac{C_t}{1-L_t} = (1-\alpha)\frac{Y_t}{L_t}$$

- Again, substituting the steady state values previously computed, we obtain the following expression:

$$\left[1+\frac{\gamma(1-\alpha)}{(1-\gamma)}\frac{1-\beta+\beta\delta}{1-\beta+(1-\alpha)\beta\delta}\right]\widehat{l}_t = \widehat{y}_t - \widehat{c}_t$$

I.6. Log-linearization of the model

- The next equation is:

$$\frac{E_t C_{t+1}}{C_t} = E_t \beta \left[1 + \left(\alpha \frac{Y_{t+1}}{K_{t+1}} - \delta \right) \right]$$

and applying the same procedure, we obtain the following expression:

$$E_t \widehat{c}_{t+1} - \widehat{c}_t = (1 - \beta + \beta\delta) E_t \widehat{y}_{t+1} - (1 - \beta + \beta\delta) E_t \widehat{k}_{t+1}$$

I.6. Log-linearization of the model

- Finally, given our assumption that the TFP follows an AR(1) process, the log-deviation with respect to the steady state is given by:

$$\widehat{a}_t = \rho \widehat{a}_{t-1} + \varepsilon_t$$

I.6. Log-linearization of the model

- The log-linearized model is:

$$
\begin{aligned}
\widehat{y}_t &= \widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{l}_t \\
[1-\beta+(1-\alpha)\beta\delta]\,\widehat{c}_t &= (1-\beta+\beta\delta)\widehat{y}_t - \alpha\beta\delta\widehat{i}_t \\
\widehat{k}_{t+1} &= (1-\delta)\widehat{k}_t + \delta\widehat{i}_t \\
\widehat{y}_t - \widehat{c}_t &= \left[1 + \frac{\gamma(1-\alpha)}{(1-\gamma)}\frac{1-\beta+\beta\delta}{1-\beta+(1-\alpha)\beta\delta}\right]\widehat{l}_t \\
E_t\widehat{c}_{t+1} - \widehat{c}_t &= (1-\beta+\beta\delta)\,E_t\widehat{y}_{t+1} - (1-\beta+\beta\delta)\,E_t\widehat{k}_{t+1} \\
\widehat{a}_t &= \rho\widehat{a}_{t-1} + \varepsilon_t
\end{aligned}
$$

I.7. Policy functions

Solving the model implies the finding of a set of policy functions relating recursively the endogenous variables with respect to the state variables, i.e.:

$$
\begin{aligned}
\widehat{y}_t &= g_{yk}\widehat{k}_{t-1} + g_{ya}a_t \\
\widehat{c}_t &= g_{ck}\widehat{k}_{t-1} + g_{ca}a_t \\
\widehat{i}_t &= g_{ik}\widehat{k}_{t-1} + g_{ia}a_t \\
\widehat{l}_t &= g_{lk}\widehat{k}_{t-1} + g_{la}a_t \\
\widehat{k}_t &= g_{kk}\widehat{k}_{t-1} + g_{ka}a_t \\
a_t &= \rho a_{t-1} + \varepsilon_t
\end{aligned}
$$

I.7. The policy functions

- Solving a DSGE model implies to find unknown functions for the endogenous variables that satisfies the first order conditions. Typically, first order conditions take the form:

$$E_t f(y, x, \Omega) = 0$$

  where y are the endogenous variables and x the state (exogenous and pre-determined) variables.
- Let $y = g(x, \Omega)$ be the unknown policy function.
- Previous step: Steady state.

I.7. The policy functions

- What to do is to find the coefficients of the linear approximation to the function $g(x, \Omega)$, such as:

$$g(x, \Omega) = g_0(\Omega) + g_1(\Omega)(x - \overline{x})$$

and

$$E_t(f(g(x, \Omega)) = 0$$

I.7. The policy functions

Linearized solution:

$$Y_t = \overline{Y} + g_{Y,K}(K_{t-1} - \overline{K}) + g_{Y,A}(a_t - \overline{a})$$

$$C_t = \overline{C} + g_{C,K}(K_{t-1} - \overline{K}) + g_{C,A}(a_t - \overline{a})$$

$$I_t = \overline{I} + g_{I,K}(K_{t-1} - \overline{K}) + g_{I,A}(a_t - \overline{a})$$

$$L_t = \overline{L} + g_{L,K}(K_{t-1} - \overline{K}) + g_{L,A}(a_t - \overline{a})$$

$$W_t = \overline{W} + g_{W,K}(K_{t-1} - \overline{K}) + g_{W,A}(a_t - \overline{a})$$

$$R_t = \overline{R} + g_{R,K}(K_{t-1} - \overline{K}) + g_{R,A}(a_t - \overline{a})$$

$$K = \overline{K} + g_{K,K}(K_{t-1} - \overline{K}) + g_{K,A}(a_t - \overline{a})$$

$$a_t = \overline{a} + \rho_A a_{t-1} + \varepsilon_t$$

I.7. The policy functions

- In the case of a second order approximation, the policy function takes the form:

$$\begin{aligned} Y_t &= \overline{Y} + 0.5\Delta^2 + g_{Y,K}(K_{t-1} - \overline{K}) + g_{Y,A}\rho_A(A_{t-1} - \overline{A}) + g_{Y,\varepsilon}\varepsilon_t + \\ &\quad 0.5g_{\varepsilon,\varepsilon}(\varepsilon_t \otimes \varepsilon_t) + g_{Y\varepsilon}[(Y_{t-1} - \overline{Y}) \otimes \varepsilon_t] \end{aligned}$$

  where Δ^2 is the variance-covariance matrix of the innovations in the shocks.

- The main difference between first and second order approximation is that in the latter the standard deviation of the shock Δ has effects on the level of the variables.

I.8. Solution methods for DSGE models

- **Policy and transition functions**: The solution for a DSGE model consists in the description of the optimal plan by finding a rule that tells what the control should be, given any possible value of the states. The rules determining the controls as a function of the states is what is called a policy function. The rules determining the states as a function of the states is what is called a transition function.
  - y_t: control
  - x_t: state
  - a_t: exogenous stochastic process

$$x_t = Px_{t-1} + Qa_t$$

$$y_t = Rx_{t-1} + Sa_t$$

where P, Q, R and S are matrices such that the computed equilibrium is stable.

- This is the so-called Blanchard and Kahn condition.[3] This is the first solution method proposed for DSGE models.
- Number of eigenvalues outside the unitary circle must be equals to the number of forward-looking variables.
- If the rank condition is not satisfied, then the steady state does not exist or there are infinity paths to the steady state.

[3] Blanchard and Kahn (1980).

I.8. Solution methods for DSGE models

- Briefly, the rank condition can be defined as follows. First, we can write the following systems:

$$Ay_t = Bx_t + C\widehat{a}_t \tag{14}$$

$$DE_t x_{t+1} + FE_t y_{t+1} = Gx_t + Hy_t \tag{15}$$

- The matrix model is closed by incorporating the expected deviation of total factor productivity:

$$E_t \widehat{a}_{t+j} = \rho_A^j \widehat{a}_t$$

where y_t is the vector of deviations from the steady state for the endogenous variables (not predetermined or jumps) and x_t is the vector of deviations from the steady state for the predetermined and jumps variables.

I.8. Solution methods for DSGE models

- The system (14) can be written as:

$$y_t = A^{-1}Bx_t + A^{-1}C\widehat{a}_t$$

- Taking one period ahead, the above system should be:

$$E_t y_{t+1} = A^{-1}BE_t x_{t+1} + A^{-1}C\rho_A \widehat{a}_t$$

- Substituting in the system (15) we find that:

$$(D + FA^{-1}B)E_t x_{t+1} = (G + HA^{-1}B)x_t + (HA^{-1}C - FA^{-1}C\rho_A)\widehat{a}_t$$

- Solving for the matrices, the final system would be:

$$E_t x_{t+1} = J x_t + M \widehat{a}_t$$

where:

$$J = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

$$M = (D + FA^{-1}B)^{-1}(HA^{-1}C - FA^{-1}C\rho_A)$$

I.8. Solution methods for DSGE models

- Using the Jordan decomposition, the matrix J can be decomposed such as:

$$J = O^{-1} N O$$

where:

$$N = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix}$$

and where:

$$O = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}$$

- The elements of the diagonal of N are the eigenvalues of the matrix J. In order the solution to be unique, the value of N_{11} must be inside the unit circle and the value of N_{22} outside the unit circle. This is the so-called the Blanchard-Kahn rank condition. If the rank condition does not hold, then the equilibrium is not unique. The columns of O^{-1} are the eigenvectors of the matrix J.

I.8. Solution methods for DSGE models

- Too many eigenvalues larger than one: No stable solution.
- Not enough eigenvalues larger than one: Multiple equilibria.
- Number of eigenvalues larger than one equals to the number of forward-looking variables: Uniqueness.

I.8. Solution methods for DSGE models

Using the Jordan decomposition, the matrix J can be decomposed such as:

$$J = O^{-1} N O$$

where:

$$N = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix}$$

and where:

$$O = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}$$

Notice that the elements of the diagonal of N are the eigenvalues of the matrix J. In order the solution to be unique, the value of N_{11} must be inside the unit circle and the value of N_{22} outside the unit circle. This is the so-called the Blanchard-Kahn rank condition. If the rank condition does not hold, then the equilibrium is not unique. The columns of O^{-1} are the eigenvectors of the matrix J.

- Therefore, the system can be written as:

$$E_t s_{t+1} = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix} \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} s_t + \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \widehat{a}_t$$

$$\begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} E_t s_{t+1} = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix} \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} s_t + \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}^{-1} \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \widehat{a}_t$$

Alternatively, we write the following expectations:

$$E_t s_{1,t+1} = N_{11} s^1_{1,t} + Q_{11} \widehat{a}_t$$

$$E_t s_{2,t+1} = N_{22} s^1_{2,t} + Q_{21} \widehat{a}_t$$

where:

$$s^1_{1,t} = O_{11} \widehat{k}_t + O_{12} \widehat{c}_t$$

$$s^1_{2,t} = O_{21} \widehat{k}_t + O_{22} \widehat{c}_t$$

and where:

$$Q = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = O^{-1} M$$

Given that the value of N_{22} is outside the unit circle, we can solve $s^1_{2,t}$ ahead:

$$s^1_{2,t} = \frac{1}{N_{22}} E_t s_{2,t+1} - \frac{Q_{21}}{N_{22}} \widehat{a}_t$$

resulting:

$$\begin{aligned} s^1_{2,t} &= -\frac{Q_{21}}{N_{22}} \sum_{j=0}^{\infty} \left(\frac{1}{N_{22}} \right)^j E_t \widehat{a}_{t+j} \\ &= -\frac{Q_{21}}{N_{22}} \sum_{j=0}^{\infty} \left(\frac{\rho_A}{N_{22}} \right)^j \widehat{a}_t = \frac{Q_{21}}{\rho_A - N_{22}} \widehat{a}_t \end{aligned}$$

Solving for $\widehat{c}_t$ we obtain:

$$\frac{Q_{21}}{\rho_A - N_{22}} \widehat{a}_t = O_{21} \widehat{k}_t + O_{22} \widehat{c}_t$$

Thus, the log-deviation of consumption is (policy function):

$$\widehat{c}_t = -\frac{O_{21}}{O_{22}}\widehat{k}_t + \frac{Q_{21}}{O_{22}(\rho_A - N_{22})}\widehat{a}_t$$

or alternatively:

$$\widehat{c}_t = S_1\widehat{k}_t + S_2\widehat{a}_t$$

being

$$S_1 = -\frac{O_{21}}{O_{22}}$$

$$S_2 = \frac{Q_{21}}{O_{22}(\rho_A - N_{22})}$$

In the case of the vector $s^1_{1,t}$ we find that:

$$s^1_{1,t} = (O_{11} + O_{12}S_1)\widehat{k}_t + O_{12}S_2\widehat{a}_t$$

and substituting we obtain:

$$E_t s^1_{1,t+1} = N_{11}s^1_{1,t} + Q_{11}\widehat{a}_t$$

$$E_t s^1_{1,t+1} = N_{11}\left[(O_{11} + O_{12}S_1)\widehat{k}_t + O_{12}S_2\widehat{a}_t\right] + Q_{11}\widehat{a}_t$$

$$(O_{11} + O_{12}S_1)\widehat{k}_{t+1} = N_{11}(O_{11} + O_{12}S_1)\widehat{k}_t + (Q_{11} + O_{12}S_2(1-\rho_A))\widehat{a}_t$$

or alternatively (transition function for capital stock):

$$\widehat{k}_{t+1} = S_3 \widehat{k}_t + S_4 \widehat{a}_t$$

where:

$$S_3 = N_{11}$$

$$S_4 = \frac{Q_{11} + N_{11} O_{12} S_2 - O_{12} S_2 \rho_A}{O_{11} + O_{12} S_1}$$

Finally, returning to the initial system:

$$x_t = A^{-1}Bs_t + A^{-1}C\widehat{a}_t$$

$$x_t = A^{-1}B\begin{bmatrix} \widehat{k}_t \\ \widehat{c}_t \end{bmatrix} + A^{-1}C\widehat{a}_t$$

$$x_t = A^{-1}B\begin{bmatrix} 1 \\ S_1 \end{bmatrix}\widehat{k}_t + \left[A^{-1}C + A^{-1}B\begin{bmatrix} 0 \\ S_2 \end{bmatrix}\right]\widehat{a}_t$$

or:

$$x_t = S_5 s_t + S_6 \widehat{a}_t$$

where:

$$S_5 = A^{-1} B \begin{bmatrix} 1 \\ S_1 \end{bmatrix}$$

$$S_6 = A^{-1} C + A^{-1} B \begin{bmatrix} 0 \\ S_2 \end{bmatrix}$$

I.8. Solution methods for DSGE models

Having completed all these computations, the solution of the model can be obtained. Collecting terms, the solution of the model is given by:

$$\begin{bmatrix} \widehat{k}_{t+1} \\ \widehat{a}_{t+1} \end{bmatrix} = \begin{bmatrix} S_3 & S_4 \\ 0 & \rho_A \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{a}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix}$$

and

$$\begin{bmatrix} \widehat{y}_t \\ \widehat{i}_t \\ \widehat{l}_t \\ \widehat{c}_t \end{bmatrix} = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{a}_t \end{bmatrix}$$

that is, the solution implies that the vector of log-deviation of control variables is a function of the vector of the state variables, and where the matrices S_5 and S_6 are function on the parameters of the model (α, β, γ, δ, ρ_A, σ_A).

I.8. Solution methods for DSGE models

- Therefore, the resolution of the model involves the calibration or estimation of the above matrices, i.e., the structural parameters of the model, linking the dynamic of the control variables with the state variables, where the state variables follow an autoregressive vector of order 1.
- Given the process for the state variables, we can predict its future value, so using the latter system, we can obtain projections for the future value of control variables.

I.8. Solutions methods for DSGE models

- The method proposed by Klein (2000) is a hybrid of those of Blanchard and Khan (1980) and Sims (2001). Taking one period ahead in the first system of equations:

$$Ax_{t+1} = Bs_{t+1} + C\rho\widehat{a}_t$$

- By substituting into the second system of equations, we arrive to the same expression than in the Blanchard-Kahn method:

$$\begin{aligned} & DE_t s_{t+1} + (FA^{-1}B)E_t s_{t+1} + (FA^{-1}C)\rho\widehat{a}_t \\ = \quad & Gs_t + (HA^{-1}B)s_t + (HA^{-1}C + I)\widehat{a}_t \end{aligned}$$

I.8. Solutions methods for DSGE models

$$(D + FA^{-1}B)E_t s_{t+1} = (G + HA^{-1}B)s_t + (HA^{-1}C + I - FA^{-1}C\rho)\widehat{a}_t$$

$$JE_t s_{t+1} = Ks_t + L\widehat{a}_t$$

where:

$$\begin{aligned} J &= D + FA^{-1}B \\ K &= G + HA^{-1}B \\ L &= HA^{-1}C + I - FA^{-1}C\rho \end{aligned}$$

I.8. Solutions methods for DSGE models

Klein method use a complex generalized Schur decomposition. The Schur decomposition of J and K are given by:

$$\begin{aligned} QJZ &= S \\ QKZ &= T \end{aligned}$$

whereQ, Z are unitary and S, T are upper triangular matrices with diagonal elements containing the generalized eigenvalues of J and K:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

I.8. Solutions methods for DSGE models

- Partitioning Z as follows:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$z_t = Z^H s_t$$

where Z^H is the Hermitian transpose.

- Given that:

$$\begin{aligned} J &= Q' S Z^H \\ K &= Q' T Z^H \end{aligned}$$

the initial system can be written as:

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \begin{bmatrix} z_{1,t+1} \\ z_{2,t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Xi \widehat{a}_t$$

I.8. Solutions methods for DSGE models

- The unstable component of the system is given by:

$$S_{22} E_t z_{2,t+1} = T_{22} z_{2,t} + Q_2 \Xi \widehat{a}_t$$

- Forward iteration over the unstable components of the system yields:

$$z_{2,t} = M \widehat{a}_t$$

$$vec(M) = [(\rho^T \otimes S_{22}) - I_z \otimes T_{22}]^{-1} vec(Q_2 \Xi)$$

$$M = [\rho^T S_{22} - T_{22}]^{-1} Q_2 \Xi$$

- Therefore:

$$z_{2,t} = [\rho^T S_{22} - T_{22}]^{-1} Q_2 \Xi \widehat{a}_t$$

I.8. Solutions methods for DSGE models

- The stable component of the system is given by:

$$S_{11}E_t z_{1,t+1} + S_{12}E_t z_{2,t+1} = T_{11}z_{1,t} + T_{12}z_{2,t} + Q_1\Xi\widehat{a}_t$$

Then, the solution for the unstable component is used to solve the stable component:

$$z_{1,t+1} = S_{11}^{-1}T_{11}z_{1,t} + S_{11}^{-1}(T_{12}M - S_{12}M\rho + Q_1\Xi)\widehat{a}_t - Z_{11}^{-1}Z_{12}M\varepsilon_{t+1}$$

In terms of the original variables of the model, the solution can be expressed as:

$$\widehat{c}_t = Z_{21}Z_{11}^{-1}\widehat{k}_t + (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})M\widehat{a}_t$$

$$\begin{aligned}\widehat{k}_{t+1} &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}\widehat{c}_t + (Z_{11}S_{11}^{-1}[T_{12}M - S_{12}M\rho + Q_1\Xi] \\ &\quad - Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{12}M + Z_{12}M\rho)\widehat{a}_t\end{aligned}$$

- Finally, substituting we obtain the solution as a policy function:

$$\widehat{k}_{t+1} = N\widehat{k}_t + O\widehat{a}_t$$

where:

$$\begin{aligned} N &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{21}Z_{11}^{-1} \\ O &= (Z_{11}S_{11}^{-1}[T_{12}M - S_{12}M\rho + Q_1\Xi] \\ &\quad -Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{12}M + Z_{12}M\rho) \\ &\quad +Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})M \end{aligned}$$

- Policy function for the other variable is given by:

$$\widehat{c}_t = P\widehat{k}_{t-1} + Q\widehat{a}_{t-1}$$

where

$$\begin{aligned} P &= Z_{21}Z_{11}^{-1}Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{21}Z_{11}^{-1} \\ Q &= Z_{21}Z_{11}^{-1}\left[\begin{array}{c}(Z_{11}S_{11}^{-1}[T_{12}M - S_{12}M\rho + Q_1\Xi] - \\ Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}Z_{12}M + Z_{12}M\rho) \\ +Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})M\end{array}\right] \\ &\quad +(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})M\rho \end{aligned}$$