Lecture 8: The consumption-saving decision

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Advanced Macroeconomics

8. The consumption-saving decision

- The economy is populated by millions of households (consumers or families) and each one takes economic decisions.
- Assumption: All agents have identical preferences (they want to be happy). Concept of representative agent.
- Assumption: Agents are optimizers (that is, they maximize a given objective function). That objective function is the felicity or utility function.
- Assumption: Discrete time.

- Instantaneous utility function:

$$U(C_t) \tag{1}$$

- Assumptions: function strictly increasing, strictly concave and two times differentiable:

$$U'(C_t) > 0 \qquad U''(C_t) < 0 \tag{2}$$

- Additional assumption: The utility function is additively separable in time (no consumption habits).

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- The household maximization problem can be defined as:

$$\max_{(C_t)} E_t \sum_{t=0}^{T} \beta^t U(C_t) \tag{3}$$

where E_t is the conditional expectation operator at time t, given the information available at that time and where β is the intertemporal discount factor, $\beta \in (0, 1)$, being:

$$\beta = \frac{1}{1+\theta} \tag{4}$$

where θ is the rate of time preference $(\theta > 0)$.

- The budget constraint is given by:

$$C_t + B_t = W_t + (1 + R_t)B_{t-1} \tag{5}$$

where B is saving, W is income and R is the interest rate.

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- Consumers choose an optimal consumption path for all their lifetime.
- Main result: The optimal consumption path chosen at time t is proportional to the discount value of total wealth.
- Current consumption is not restricted to be equal to current income. Households use saving to separate the consumption path from the income path.

- Household utility maximization problem with finite live and no uncertainty:

$$\max_{(C_t)} \sum_{t=0}^{T} \beta^t U(C_t) \tag{6}$$

subject to the budget constraint:

$$\begin{aligned} C_t + B_t &= W_t + (1 + R_t)B_{t-1} && (7) \\ B_{-1} &= 0 && (8) \\ B_T &= 0 && (9) \end{aligned}$$

Consumer problem	
Utility function	$U = U(C_t)$
Budget constraint	$C_t + B_t = (1 + R_{t-1})B_{t-1} + W_t$
Initial stock of assets	$B_{-1} = 0$
Final stock of assets	$B_T = 0$

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The auxiliary Lagrangian function to the above problem is given by:

$$\mathcal{L} = \beta^t \left\{ U(C_t) - \lambda_t (C_t + B_t - W_t - (1 + R_t) B_{t-1}) \right\} \tag{10}$$

where λ_t is the Lagrange multiplier (the shadow price of consumption). First order conditions for maximization are given by:

$$\frac{\partial \mathcal{L}}{\partial C_t} = U'(C_t) - \lambda_t = 0 \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1 + R_{t+1}) = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = C_t + B_t - Y_t - (1 + R_t) B_{t-1} = 0 \tag{13}$$

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When calculating the first order condition with respect to the stock of assets, we have to take into account that the stock of assets appears at time t and at time $t+1$:

$$
\begin{aligned}
&\ldots - \lambda_t \left[C_t + B_t - W_t - (1 + R_{t-1}) B_{t-1} \right] \\
&- \lambda_{t+1} \left[C_{t+1} + B_{t+1} - W_{t+1} - (1 + R_t) B_t \right] - \ldots
\end{aligned} \tag{14}
$$

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- From the first order condition, we obtain the value of the Lagrangian multiplier (the shadow price of consumption):

$$\lambda_t = U'(C_t) \tag{15}$$

By substituting in the second first order condition yields:

$$\beta^t U'(C_t) = \beta^{t+1} U'(C_{t+1})(1 + R_{t+1}) \tag{16}$$

and operating we obtain the final solution:

$$U'(C_t) = \beta U'(C_{t+1})(1 + R_{t+1}) \tag{17}$$

Consumer problem solution	
Optimal consumption path	$U_C(C_t) = \beta(1+R_t)U_C(C_{t+1})$
Financial assets dynamics	$B_t = (1+R_{t-1})B_{t-1} + W_t L_t - C_t$

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- Steady state. Solution in which the level of consumption is constant over time, $C_t = C_{t+1} = \overline{C}$. Thus, we have that $U_C(\overline{C}) = \beta U_C(\overline{C})(1 + R_t)$, and therefore the steady state implies that:

$$\overline{R} = \frac{1-\beta}{\beta} = \frac{1-\frac{1}{1+\theta}}{\frac{1}{1+\theta}} = \theta \tag{18}$$

that is, in steady state the real interest rate is equal to the intertemporal preference parameter.

- Example: Logarithmic utility function
- To solve the consumer problem numerically, first we need to assume a particular functional form for the utility function. The more simple function is the logarithmic shape.
- In this case, the consumer problem is given by:

$$\max_{\{C_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \ln C_t \tag{19}$$

subject to the budget constraint and the initial and final conditions.

- Lagrangian auxiliary function:

$$\mathcal{L} = \sum_{t=0}^{T} \left[\beta^t \ln C_t - \lambda_t (C_t + B_t - W_t - (1 + R_{t-1}) B_{t-1}) \right] \tag{20}$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_t} \quad : \quad \beta^t \frac{1}{C_t} - \lambda_t = 0 \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} \quad : \quad -\lambda_t + \lambda_{t+1}(1 + R_t) = 0 \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \quad : \quad C_t + B_t - W_t - (1 + R_{t-1}) B_{t-1} = 0 \tag{23}$$

- Solving for the Lagrange multiplier in the first first order condition and subtituting in the second order condition, we have:

$$\beta^t \frac{1}{C_t} = \beta^{t+1} \frac{1}{C_{t+1}} (1 + R_t) \tag{24}$$

and operating:

$$C_{t+1} = \beta(1 + R_t) C_t \tag{25}$$

indicating that the relationship between the level of consumption at time $t + 1$ with the level of consumption at time t, is driven by the preference discount factor β, and the real interest rate.

Consumer problem solution Logarithmic utility funciton	
Optimal consumption path	$C_{t+1} = \beta(1+R_t)C_t$
Financial assets dynamics	$B_t = (1+R_{t-1})B_{t-1} + W_t L_t - C_t$