Lecture 2: An example of dynamic system: The arms race model

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Advanced Macroeconomics

2. An example of dynamic system: The arms race model

System of equations (Discrete time):

$$\begin{bmatrix} \Delta x_{1,t} \\ \Delta x_{2,t} \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix}}_{A} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \underbrace{\begin{bmatrix} \theta & 0 \\ 0 & \eta \end{bmatrix}}_{B} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} \tag{1}$$

Partial equilibrium conditions:

$$\Delta x_{1,t} = -\alpha x_{1,t} + \beta x_{2,t} + \theta z_{1,t} \tag{2}$$

$$\Delta x_{2,t} = \gamma x_{1,t} - \delta x_{2,t} + \eta z_{2,t} \tag{3}$$

2. An example of dynamic system: The arms race model

Arms race model structure	
Country 1 reaction function	$\Delta x_{1,t} = -\alpha x_{1,t} + \beta x_{2,t} + \theta z_{1,t}$
Country 2 reaction function	$\Delta x_{2,t} = \gamma x_{1,t} - \delta x_{2,t} + \eta z_{2,t}$
Stock of armament change country 1	$\Delta x_{1,t} = x_{1,t+1} - x_{1,t}$
Stock of armament change country 2	$\Delta x_{2,t} = x_{2,t+1} - x_{2,t}$

2. An example of dynamic system: The arms race model

- Steady state definition:

$$\bar{\mathbf{x}} \Longrightarrow \dot{\mathbf{x}}_t = f(\mathbf{x}_t, \mathbf{z}_t) = 0 \Longrightarrow f(\bar{\mathbf{x}}, \mathbf{z}_t) = 0 \tag{4}$$

- To obtain the steady state for the model, we have to calculate the zero solution for the equations of the system. For that, the matrix A times the vector of endogenous variables must be equal to the negative of the matrix B times the vector of exgenous variables, such as:

$$\begin{bmatrix} \Delta x_{1,t} \\ \Delta x_{2,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow A \begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} = -Bz_t \tag{5}$$

- Solving for the vector of steady state variables, we arrive to the following vector indicating the value of each endogenous variable in steady state:

$$\begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} = -A^{-1}Bz_t \tag{6}$$

under the assumption that the matrix A is non singular, that is, $Det(A) \neq 0$, and thefore exists a matrix inverse (that is, we assume thatl $rank(A) = 2$).

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- Solving for the steady state: Given the model we are solving, we have:

$$A = \begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix}, \qquad B = \begin{bmatrix} \theta & 0 \\ 0 & \eta \end{bmatrix}, \qquad z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} \tag{7}$$

- Calculating the inverse of matrix A yields:

$$A^{-1} = \frac{1}{\alpha\delta - \gamma\beta} \begin{bmatrix} -\delta & -\beta \\ -\gamma & -\alpha \end{bmatrix} \tag{8}$$

and hence, the steady state, by using the definition given in (6), would be:

$$\begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} = -\frac{1}{\alpha\delta - \gamma\beta} \begin{bmatrix} -\delta & -\beta \\ -\gamma & -\alpha \end{bmatrix} \begin{bmatrix} \theta & 0 \\ 0 & \eta \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} \tag{9}$$

- Operating in the previous expression, we arrive to the steady state values for the endogenous variables, given by:

$$\bar{x}_{1,t} = \frac{\delta\theta}{\alpha\delta - \gamma\beta} z_{1,t} + \frac{\beta\eta}{\alpha\delta - \gamma\beta} z_{2,t} \tag{10}$$

$$\bar{x}_{2,t} = \frac{\gamma\theta}{\alpha\delta - \gamma\beta} z_{1,t} + \frac{\alpha\eta}{\alpha\delta - \gamma\beta} z_{2,t} \tag{11}$$

- For the model to have a steady state, the condition $\alpha\delta - \gamma\beta \neq 0$ must be hold. Otherwise the determinant of matrix A would be 0.

- Stability analisis. To obtain the eigenvalues associated to matrix A, we calculate:

$$Det\left[A-\lambda I\right]=0 \tag{12}$$

where I is the identity matrix and λ is the vector of eigenvalues, where:

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

- Therefore, returing to the dynamic system defined in (1), given the matrix A, we have,

$$Det\begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Det\begin{bmatrix} -\alpha-\lambda & \beta \\ \gamma & -\delta-\lambda \end{bmatrix} = 0 \tag{14}$$

- Computing the determinat, we arrive to the following second order equation:

$$\lambda^2 + \lambda(\alpha+\delta) + (\alpha\delta - \gamma\beta) = 0 \tag{15}$$

or alternatively

$$\lambda^2 - tr(A)\lambda + Det(A) = 0 \tag{16}$$

where $tr(A)$ is the trace of matrix A.

2. An example of dynamic system: The arms race model

- Solving the above second order equation, we have that the roots (the eigenvalues), are given by:

$$\lambda_1, \lambda_2 = \frac{-(\alpha+\delta) \pm \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta - \gamma\beta)}}{2} \tag{17}$$

- The stability of the system will depend on the values for the eigenvalues λ_1, λ_2.

2. An example of dynamic system: The arms race model

- The eigenvalues can be real numbers or imaginary numbers, depending on the sign of the term $(\alpha + \delta)^2 - 4(\alpha\delta - \gamma\beta)$.
- If its value is positive, then eigenvalues will be real numbers.
- If the value of that term is negative, then eigenvalues are imaginary number. However, for this particular model, eigenvalues will be always real number, given that $(\alpha + \delta)^2 - 4(\alpha\delta - \gamma\beta) = \alpha^2 + \delta^2 + 2\alpha\delta - 4\alpha\delta + 4\gamma\beta = (\alpha - \delta)^2 + 4\gamma\beta$, and this number is always larger than 0.

- When the roots are real numbers, the modulus (that is, the absolute value) is defined as:

$$Modulus(\lambda + 1) = |\lambda + 1| \tag{18}$$

- In the case in which the roots are imaginary numbers, that is, $\lambda = a \pm bi$, then the modulus is defined as:

$$Modulus(\lambda + 1) = \sqrt{(a+1)^2 + b^2} \tag{19}$$

2. An example of dynamic system: The arms race model

- The system will show global stability, that is, all trayectories are stable as they tend to the steady state, if $|\lambda_1 + 1| < 1$ and $|\lambda_2 + 1| < 1$, for the case eigenvalues are real numbers.
- If eigenvalues are imaginary numbers, $\lambda = a \pm bi$, the stability condition implies that $\sqrt{(a+1)^2 + b^2} < 1$.
- In the case where the modolus of one eigenvalue plus one is larger than one, for example, $|\lambda_1 + 1| < 1$ and $|\lambda_2 + 1| > 1$, then the solution is a saddle point, where some trajectories will tend to the steady state but other trajectories will be unstable.
- Finally, in the case where $|\lambda_1 + 1| > 1$ and $|\lambda_2 + 1| > 1$, for real numbers, of $\sqrt{(a+1)^2 + b^2} > 1$, for imaginary numbers, then the system is globally unstable, and all trajectories are explosive.

- Two possible solutions (we discard explosive solutions).
- A) Global stability, where:

$$|\lambda_1 + 1| < 1 \tag{20}$$
$$|\lambda_2 + 1| < 1 \tag{21}$$

- B) Saddle point solution, when:

$$|\lambda_1 + 1| < 1 \tag{22}$$
$$|\lambda_2 + 1| > 1 \tag{23}$$

When the solution is a saddle point, there is a unique stable trajectory to the steady state. That is called the "stable path".

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- For a saddle point solution dynamical system, we have to compute the stable path.
- A way to obtain the solution for a linear system of two difference equations is given by:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = v_1(\lambda_1 + 1)^t a_1 + v_2(\lambda_2 + 1)^t a_2 + \begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} \tag{24}$$

where v_1 and v_2 are the eigenvectors associated to the eigenvalues λ_1 and λ_2, respectively. Additionally, a_1 and a_2 are constants, depending on the initial conditions of the system.

2. An example of dynamic system: The arms race model

- Given the above expression, for any value of a_1 when $a_2 = 0$, we have

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \upsilon_1 (\lambda_1 + 1)^t a_1 + \begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} \tag{25}$$

where the variables will move to the steady state, a time goes on, give that the term $\upsilon_1 (\lambda_1 + 1)^t a_1$ will tend to zero. These are the only stable trajectories (for the different values of a_1) to the steady state, resulting in a stable path.

- On the other hand, when $a_1 = 0$ and for any value of a_2, we have that

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \upsilon_2(\lambda_2 + 1)^t a_2 + \begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} \tag{26}$$

where the variables will move away from the steady state (corner solution). In gneeral, when $a_2 \neq 0$, the trajectories will move the variables away from the steady state.

- Computing the stable path: When the solution of the system is a saddle point, there is a unique stable path, given by

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \upsilon_1(\lambda_1 + 1)^t a_1 + \begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} \quad (27)$$

where λ_1 is the eigenvalue for which the modulus of $\lambda_1 + 1$ is lower than 1, and υ_1 is the eigenvector associated to that eigenvalue.

- Computing the stable path: When the solution of the system is a saddle point, there is a unique stable path, given by

$$\left[\begin{array}{c} x_{1,t} \\ x_{2,t} \end{array}\right] = \upsilon_1(\lambda_1 + 1)^t a_1 + \left[\begin{array}{c} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{array}\right] \tag{28}$$

where λ_1 is the eigenvalue for which the modulus of $\lambda_1 + 1$ is lower than 1, and υ_1 is the eigenvector associated to that eigenvalue.

- Operating in the previous expression, in the next period, $t+1$, the solution of the system can be defined as:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \upsilon_1(\lambda_1+1)(\lambda_1+1)^t a_1 + \begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} \tag{29}$$

and substracting both expressions yields

$$\begin{bmatrix} \Delta x_{1,t} \\ \Delta x_{2,t} \end{bmatrix} = \lambda_1 \upsilon_1 (\lambda_1+1)^t a_1 \tag{30}$$

where $\Delta x_{1,t} = x_{1,t+1} - x_{1,t}$, and $\Delta x_{2,t} = x_{2,t+1} - x_{2,t}$.

- On the other hand, from expression (28), we have that:

$$v_1(\lambda_1 + 1)^t a_1 = \begin{bmatrix} x_{1,t} - \bar{x}_{1,t} \\ x_{2,t} - \bar{x}_{2,t} \end{bmatrix} \tag{31}$$

  and substituting in expression (30), we arrive to:

$$\begin{bmatrix} \Delta x_{1,t} \\ \Delta x_{2,t} \end{bmatrix} = \lambda_1 \begin{bmatrix} x_{1,t} - \bar{x}_{1,t} \\ x_{2,t} - \bar{x}_{2,t} \end{bmatrix} \tag{32}$$

- The eigenvalue λ_1 is known as the convergence rate, and it represents the rate at which the solution tends to the steady state along the stable path.

- Adjustment to the stable path. We assume that the jump variable is the endogenous variable 1. From expression (30), we have that the stable path is defined by the following trajectory:

$$\Delta x_{1,t} = \lambda_1 (x_{1,t} - \bar{x}_{1,t}) \tag{33}$$

- On the other hand, from the model, we have that the dynamic equation for this variable is given by:

$$\Delta x_{1,t} = -\alpha x_{1,t} + \beta x_{2,t} + \theta z_{1,t} \tag{34}$$

- Both equations must give the same value for $\Delta x_{1,t}$. Equating both equations we have that:

$$-\alpha x_{1,t} + \beta x_{2,t} + \theta z_{1,t} = \lambda_1 (x_{1,t} - \bar{x}_{1,t}) \tag{35}$$

and solving for $x_{1,1}$ yields,

$$x_{1,t} = \frac{\beta}{\alpha + \lambda_1} x_{2,t} + \frac{\theta}{\alpha + \lambda_1} z_{1,t} + \frac{\lambda_1}{\alpha + \lambda_1} \bar{x}_{1,t} \tag{36}$$

indicating the value of the endogenous variable to be in the stable path.