

Lecture 18: Numerical solution of the Ramsey model

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Advanced Macroeconomics

18. Numerical solution of the Ramsey model

Log-linearization rules:

$$x_t \approx \bar{x}_t \exp(\hat{x}_t) \approx \bar{x}_t(1 + \hat{x}_t) \quad (1)$$

$$x_t z_t \approx \bar{x}_t(1 + \hat{x}_t)\bar{z}_t(1 + \hat{z}_t) \approx \bar{x}_t \bar{z}_t(1 + \hat{x}_t + \hat{z}_t) \quad (2)$$

$$x_t^a \approx \bar{x}_t^a(1 + \hat{x}_t)^a \approx \bar{x}_t^a(1 + a\hat{x}_t) \quad (3)$$

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Output:

$$y_t = Ak_t^\alpha \quad (4)$$

Log-linearization rules:

$$\bar{y}(1 + \hat{y}_t) = A\bar{k}^\alpha(1 + \hat{k}_t) \quad (5)$$

or:

$$\bar{y} + \bar{y}\hat{y}_t = A\bar{k}^\alpha + \alpha A\bar{k}^\alpha \hat{k}_t \quad (6)$$

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In steady state $\bar{y} = \bar{A}\bar{k}^\alpha$, and then:

$$\bar{y}\hat{y}_t = \alpha\bar{A}\bar{k}^\alpha\hat{k}_t \quad (7)$$

and operating:

$$\hat{y}_t = \alpha\hat{k}_t \quad (8)$$

where $\hat{y}_t = \ln Y_t - \ln \bar{Y}$.

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Log-linearization of the capital stock dynamic equation:

$$c_t + k_{t+1}(1+n) = (1-\delta)k_t + y_t \quad (9)$$

Using log-linearization rules:

$$\bar{c}(1+\hat{c}_t) + \bar{k}(1+n)(1+\hat{k}_{t+1}) = (1-\delta)\bar{k}(1+\hat{k}_t) + \bar{y}(1+\hat{y}_t) \quad (10)$$

Using steady state:

$$\bar{c} + \bar{k}(1+n) = (1-\delta)\bar{k} + \bar{y} \quad (11)$$

and operating:

$$\bar{c}\hat{c}_t + \bar{k}(1+n)\hat{k}_{t+1} = (1-\delta)\bar{k}\hat{k}_t + \bar{y}\hat{y}_t \quad (12)$$

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Operating:

$$\frac{\bar{c}}{k} \hat{c}_t + (1+n) \hat{k}_{t+1} = (1-\delta) \hat{k}_t + \frac{\bar{y}}{k} \hat{y}_t \quad (13)$$

Using steady state values:

$$\frac{\bar{c}}{k} = \frac{A \left(\frac{1-\beta+\beta\delta}{\alpha A \beta} \right)^{\frac{\alpha}{\alpha-1}} - (n+\delta) \left(\frac{1-\beta+\beta\delta}{\alpha A \beta} \right)^{\frac{1}{\alpha-1}}}{\left(\frac{1-\beta+\beta\delta}{\alpha A \beta} \right)^{\frac{1}{\alpha-1}}} \quad (14)$$

and

$$\frac{\bar{y}}{k} = \left(\frac{1-\beta+\beta\delta}{\alpha \beta} \right) \quad (15)$$

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Operating:

$$\left[\frac{1 - \beta + \beta\delta}{\alpha\beta} - (n + \delta) \right] \hat{c}_t + (1 + n)\hat{k}_{t+1} = (1 - \delta)\hat{k}_t + \left(\frac{1 - \beta + \beta\delta}{\beta} \right) \hat{k}_t \quad (16)$$

or:

$$\left[\frac{1 - \beta + \beta\delta}{\alpha\beta} - (n + \delta) \right] \hat{c}_t + (1 + n)\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t \quad (17)$$

or:

$$(1 + n)\hat{k}_{t+1} = - \left[\frac{1 - \beta + \beta\delta}{\alpha\beta} - (n + \delta) \right] \hat{c}_t + \frac{1}{\beta}\hat{k}_t \quad (18)$$

or

$$\hat{k}_{t+1} = - \left[\frac{1 - \beta + \beta\delta}{\alpha\beta(1 + n)} - \frac{(\delta + n)}{(1 + n)} \right] \hat{c}_t + \frac{1}{\beta(1 + n)}\hat{k}_t \quad (19)$$

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Adding and subtracting \hat{k}_t we have:

$$\hat{k}_{t+1} - \hat{k}_t + \hat{k}_t = - \left[\frac{1 - \beta + \beta\delta - \alpha\beta(\delta + n)}{\alpha\beta(1 + n)} \right] \hat{c}_t + \frac{1}{\beta(1 + n)} \hat{k}_t \quad (20)$$

and given that $\Delta \hat{k}_t = \hat{k}_{t+1} - \hat{k}_t$, we get:

$$\Delta \hat{k}_t = - \left[\frac{1 - \beta + \beta\delta - \alpha\beta(\delta + n)}{\alpha\beta(1 + n)} \right] \hat{c}_t + \left[\frac{1 - \beta(1 + n)}{\beta(1 + n)} \right] \hat{k}_t \quad (21)$$

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Log-linearization of the dynamic equation for consumption:

$$c_{t+1} = \beta(\alpha A k_{t+1}^{\alpha-1} + 1 - \delta) c_t \quad (22)$$

Using log-linearization rules:

$$\bar{c}(1 + \hat{c}_{t+1}) = \beta(1 - \delta)\bar{c}(1 + \hat{c}_t) + \beta\alpha A \bar{k}^{\alpha-1} \bar{c}(1 + (\alpha - 1)\hat{k}_{t+1} + \hat{c}_t) \quad (23)$$

Operating:

$$\begin{aligned} \bar{c} + \bar{c}\hat{c}_{t+1} &= \beta(1 - \delta)\bar{c} + \beta(1 - \delta)\bar{c}\hat{c}_t + \beta\alpha A \bar{k}^{\alpha-1} \bar{c} \\ &\quad + \beta\alpha A \bar{k}^{\alpha-1} \bar{c}(\alpha - 1)\hat{k}_t + \beta\alpha A \bar{k}^{\alpha-1} \bar{c}\hat{c}_t \end{aligned} \quad (24)$$

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In steady state,

$$\bar{c} = \beta(1 - \delta)\bar{c} + \beta\alpha A k^{\bar{\alpha}-1} \bar{c} \quad (25)$$

and operating:

$$\bar{c}\hat{c}_{t+1} = \beta(1 - \delta)\bar{c}\hat{c}_t + \beta\alpha A k^{\bar{\alpha}-1} \bar{c}((\alpha - 1)\hat{k}_{t+1} + \hat{c}_t) \quad (26)$$

or

$$\hat{c}_{t+1} = \beta(1 - \delta)\hat{c}_t + \beta\alpha A k^{\bar{\alpha}-1} ((\alpha - 1)\hat{k}_{t+1} + \hat{c}_t) \quad (27)$$

and using steady state values:

$$\hat{c}_{t+1} = \beta(1 - \delta)\hat{c}_t + (1 - \beta + \beta\delta)((\alpha - 1)\hat{k}_{t+1} + \hat{c}_t) \quad (28)$$

and operating

$$\hat{c}_{t+1} = \hat{c}_t + (1 - \beta + \beta\delta)(\alpha - 1)\hat{k}_{t+1} \quad (29)$$

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From expression (19), we have

$$\hat{k}_{t+1} = - \left[\frac{1 - \beta + \beta\delta - \alpha\beta(\delta + n)}{\alpha\beta(1 + n)} \right] \hat{c}_t + \frac{1}{\beta(1 + n)} \hat{k}_t \quad (30)$$

and substituting

$$\hat{c}_{t+1} = \hat{c}_t + (1 - \beta + \beta\delta)(\alpha - 1) \left(\frac{1}{\beta(1 + n)} \hat{k}_t - \left[\frac{1 - \beta + \beta\delta - \alpha\beta(\delta + n)}{\alpha\beta(1 + n)} \right] \right) \quad (31)$$

Define

$$\Omega = 1 - \beta + \beta\delta$$

and

$$\Gamma = 1 - \beta + \beta\delta - \alpha\beta(\delta + n)$$

resulting in:

$$\Delta \hat{c}_t = \frac{(\alpha - 1)\Omega}{\beta(1 + n)} \hat{k}_t - \frac{(\alpha - 1)\Omega\Gamma}{\alpha\beta(1 + n)} \hat{c}_t \quad (32)$$

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Finally, log-linearization of the feasibility condition:

$$i_t = y_t - c_t$$

using log-linearization rules:

$$\bar{i}_t(1 + \hat{i}_t) = \bar{y}_t(1 + \hat{y}_t) - \bar{c}_t(1 + \hat{c}_t) \quad (33)$$

or

$$\bar{i}_t + \bar{i}_t \hat{i}_t = \bar{y}_t + \bar{y}_t \hat{y}_t - \bar{c}_t - \bar{c}_t \hat{c}_t \quad (34)$$

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In steady state we have:

$$\bar{i}_t = \bar{y}_t - \bar{c}_t \quad (35)$$

or:

$$\bar{i}_t \hat{i}_t = \bar{y}_t \hat{y}_t - \bar{c}_t \hat{c}_t \quad (36)$$

and solving for investment:

$$\hat{i}_t = \frac{\bar{y}_t}{\bar{i}_t} \hat{y}_t - \frac{\bar{c}_t}{\bar{i}_t} \hat{c}_t \quad (37)$$

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Using steady state values:

$$\frac{\bar{y}_t}{\bar{i}_t} = \frac{1 - \beta + \beta\delta}{\alpha\beta(n + \delta)} \quad (38)$$

and

$$\frac{\bar{c}_t}{\bar{i}_t} = \frac{1 - \beta + \beta\delta - \alpha\beta\delta}{\alpha\beta(n + \delta)} \quad (39)$$

Substituting we obtain:

$$\hat{i}_t = \frac{1 - \beta + \beta\delta}{\alpha\beta(n + \delta)} \hat{y}_t - \frac{1 - \beta + \beta\delta - \alpha\beta(n + \delta)}{\alpha\beta(n + \delta)} \hat{c}_t \quad (40)$$

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Log-linear system. Define:

$$\Omega = 1 - \beta + \beta\delta \quad (41)$$

$$\Gamma = 1 - \beta + \beta\delta - \alpha\beta(\delta + n) \quad (42)$$

Then, we get:

$$\begin{bmatrix} \Delta \hat{c}_t \\ \Delta \hat{k}_t \end{bmatrix} = \begin{bmatrix} -\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} & \frac{(\alpha-1)\Omega}{\beta(1+n)} \\ -\frac{\Gamma}{\alpha\beta(1+n)} & \frac{1-\beta(1+n)}{\beta(1+n)} \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} \quad (43)$$

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Stability analysis:

$$\text{Det} \begin{bmatrix} -\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} - \lambda & \frac{(\alpha-1)\Omega}{\beta(1+n)} \\ -\frac{\Gamma}{\alpha\beta(1+n)} & \frac{1-\beta(1+n)}{\beta(1+n)} - \lambda \end{bmatrix} = 0 \quad (44)$$

resulting in:

$$\lambda^2 + \left[\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} - \frac{1-\beta(1+n)}{\beta(1+n)} \right] \lambda + \quad (45)$$

$$\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)\beta(1+n)} - \left(\frac{1-\beta(1+n)}{\beta(1+n)} \right) \frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} = 0 \quad (46)$$

or

$$\lambda^2 + \frac{(\alpha-1)\Omega\Gamma - \alpha + \alpha\beta(1+n)}{\alpha\beta(1+n)} \lambda + \frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} = 0 \quad (47)$$

18. Solución numérica del modelo de Ramsey

Roots:

$$\lambda_{1,2} = \frac{- \left[\frac{(\alpha-1)\Omega\Gamma - \alpha + \alpha\beta(1+n)}{\alpha\beta(1+n)} \right] \pm \sqrt{\left(\frac{(\alpha-1)\Omega\Gamma - \alpha + \alpha\beta(1+n)}{\alpha\beta(1+n)} \right)^2 - 4 \frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)}}}{2} \quad (48)$$

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Jump in consumption:

Dynamic equation for consumption:

$$\Delta \hat{c}_t = \frac{(\alpha - 1)\Omega}{\beta(1+n)} \hat{k}_t - \frac{(\alpha - 1)\Omega\Gamma}{\alpha\beta(1+n)} \hat{c}_t \quad (49)$$

Stable trajectory:

$$\Delta \hat{c}_t = \lambda_1 \hat{c}_t \quad (50)$$

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By equating both expression:

$$\frac{(\alpha - 1)\Omega}{\beta(1 + n)} \hat{k}_t - \frac{(\alpha - 1)\Omega\Gamma}{\alpha\beta(1 + n)} \hat{c}_t = \lambda_1 \hat{c}_t \quad (51)$$

resulting in:

$$\hat{c}_t = \frac{\alpha(\alpha - 1)\Omega}{(\alpha - 1)\Omega\Gamma + \alpha\beta(1 + n)\lambda_1} \hat{k}_t \quad (52)$$