Lecture 17: The Ramsey model

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Advanced Macroeconomics

17. The Ramsey model

- Ramsey model or Ramsey-Cass-Koopmans model
- Optimal growth model. Equilibrium as a result of economic decisions taken by families and firms.
- Saving is an endogenous variable.

17. The Ramsey model

- Population size:

$$L_t = L_{t-1}(1+n) \tag{1}$$

where $n > 0$ is the population growth rate. In general:

$$L_t = L_0(1+n)^t \tag{2}$$

where L_0 is the population at the initial period 0. Initial population is normalized to 1 ($L_0 = 1$):

$$L_t = (1+n)^t \tag{3}$$

- Consumption per capita:

$$c_t = \frac{C_t}{L_t} \tag{4}$$

17. The Ramsey model

- Function to be maximized:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_t \sum_{t=0}^{T} \beta^t U(c_t) L_t \tag{5}$$

- where $\beta \in (0, 1)$

$$\beta = \frac{1}{1+\theta} \tag{6}$$

where $\theta > 0$.

- The maximization problem can be written as :

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t (1+n)^t U(c_t) \tag{7}$$

or:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \left(\frac{1}{1+\theta}\right)^t (1+n)^t U(c_t) \tag{8}$$

where the discount factor is given by:

$$\left(\frac{1+n}{1+\theta}\right)^t \tag{9}$$

- To solve the problem we need the following condition: $\theta > n$.

- Budget constraint:

$$C_t + I_t = W_t L_t + R_t K_t \tag{10}$$

where capital stock accumulation is given by:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{11}$$

The budget constraint can be defined as:

$$C_t + K_{t+1} = W_t L_t + (R_t + 1 - \delta)K_t \tag{12}$$

We define:

$$k_t = \frac{K_t}{L_t} \tag{13}$$

- Multiplying and dividing the budget constraint (12) by the population:

$$\frac{C_t}{L_t} + \frac{K_{t+1}}{L_t} = \frac{W_t L_t + (R_t + 1 - \delta) K_t}{L_t} \tag{14}$$

and we get:

$$c_t + (1+n) k_{t+1} = W_t + (R_t + 1 - \delta) k_t \tag{15}$$

given that:

$$\frac{L_{t+1}}{L_t} = (1+n) \tag{16}$$

Ramsey model	
Household utility function	$U = U(C_t)$
Budget constraint	$C_t + I_t = W_t L_t + R_t K_t$
Initial capital stock	$K_0 > 0$
Capital stock accumulation	$K_{t+1} = (1-\delta) K_t + I_t$
Production function	$Y_t = A_t F(K_t, L_t)$
Populaion growth rate	$L_t = L_{t-1}(1+n)$

- We will assume a logarithmic utility function:

$$U(c_t) = \ln c_t \tag{17}$$

- The household maximization problem is the following:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \left(\frac{1+n}{1+\theta} \right)^t \ln c_t \tag{18}$$

subject to:

$$c_t + (1+n)k_{t+1} = W_t + (R_t + 1 - \delta)k_t \tag{19}$$

given that $k_0 > 0$.

17. The Ramsey model

- The Lagrangian auxiliary function:

$$\mathcal{L} = \sum_{t=0}^{T} \left(\frac{1+n}{1+\theta} \right)^t \ln c_t \tag{20}$$

$$-\lambda_t \left[c_t + (1+n)k_{t+1} - W_t - (R_t + 1 - \delta)k_t \right] \tag{21}$$

- First order condition, for $t = 0, 1, 2, ..., T$, are the following:

$$\frac{\partial \mathcal{L}}{\partial c_t} : \left(\frac{1+n}{1+\theta} \right)^t \frac{1}{c_t} - \lambda_t = 0 \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \lambda_{t+1}(R_{t+1} + 1 - \delta) - \lambda_t(1+n) = 0 \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} : c_t + k_{t+1} - W_t - \frac{(R_t + 1 - \delta)}{(1+n)} k_t = 0 \tag{24}$$

- From the first order condition we have:

$$\lambda_t = \left(\frac{1+n}{1+\theta}\right)^t \frac{1}{c_t} \tag{25}$$

By substituting the Lagrange multiplier in t and in $t+1$, in the second first order condition:

$$\left(\frac{1+n}{1+\theta}\right)^{t+1} \frac{1}{c_{t+1}}(R_{t+1}+1-\delta) = (1+n)\left(\frac{1+n}{1+\theta}\right)^t \frac{1}{c_t} \tag{26}$$

and simplifying:

$$\frac{1}{1+\theta}\frac{1}{c_{t+1}}(R_{t+1}+1-\delta) = \frac{1}{c_t} \tag{27}$$

- Operating we arrive to the following optimal consumption path:

$$c_{t+1} = \frac{(R_{t+1} + 1 - \delta)}{1 + \theta} c_t \tag{28}$$

or:

$$c_{t+1} = \beta(R_{t+1} + 1 - \delta) c_t \tag{29}$$

- For aggregate consumption we have:

$$\frac{C_{t+1}}{L_{t+1}} = \beta(R_{t+1} + 1 - \delta)\frac{C_t}{L_t} \tag{30}$$

or:

$$C_{t+1} = \beta(R_{t+1} + 1 - \delta)(1 + n)C_t \tag{31}$$

17. The Ramsey model

- Firm profit maximizacion problem:

$$\max \Pi_t = Y_t - W_t L_t - R_t K_t \tag{32}$$

subject to the technological restriction:

$$Y_t = A_t F(K_t, L_t) \tag{33}$$

- Profit maximization problem in per capita terms:

$$\max \pi_t = y_t - w_t - (R_t + \delta) k_t \tag{34}$$

subject to the technological restriction:

$$y_t = A_t f(k_t) \tag{35}$$

- First order condition with respect to the capital stock per capita:

$$\frac{\partial \pi}{\partial k} = A_t f_k(k_t) - (R_t + \delta) = 0 \tag{36}$$

17. The Ramsey model

- From the first order condition we have that:

$$A_t f_k(k_t) = R_t + \delta \tag{37}$$

- Wage would be total income (output), less capital income:

$$w_t = A_t f(k_t) - k_t A_t f_k(k_t) \tag{38}$$

- Cobb-Douglas technology:

$$\max \pi_t = A_t k_t^{\alpha} - w_t - R_t k_t \tag{39}$$

- First order condition:

$$\frac{\partial \pi}{\partial k} = \alpha A_t k_t^{\alpha - 1} - R_t = 0 \tag{40}$$

- From that, we obtain that marginal productivity of capital is equal to the real interest rate:

$$\alpha A_t k_t^{\alpha - 1} = R_t \tag{41}$$

or equivalently:

$$R_t k_t = \alpha y_t \tag{42}$$

- Given that the firm profit maximization problem is defined in per capita terms, labor is not a variable (no first order condition). In this case, the equilibrium wage is obtained as the difference between total income (output), and capital income:

$$w_t = A_t k_t^{\alpha} - k_t \alpha A_t k_t^{\alpha-1} = A_t k_t^{\alpha} - \alpha A_t k_t^{\alpha} = (1-\alpha) y_t \tag{43}$$

- Competitive equilibrium: The dynamic equation for consumption per capita (optimal consumption path), is given by:

$$c_{t+1} = \beta(R_{t+1} + 1 - \delta)c_t \tag{44}$$

- Using the equilibrium value for the real interest rate we have:

$$c_{t+1} = \beta(\alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta)c_t \tag{45}$$

- On the other hand, the dynamic equation for the capital stock is given (using the real interest rate and wage) by:

$$c_t + (1+n)k_{t+1} = A_t k_t^\alpha - \alpha A_t k_t^\alpha + (\alpha A_t k_t^{\alpha-1} + 1 - \delta)k_t \tag{46}$$

or:

$$c_t + (1+n)k_{t+1} = A_t k_t^\alpha + (1-\alpha)k_t \tag{47}$$

- Steady state: Using the equilibrium condition for consumption per capita in steady state:

$$1 = \beta(\alpha A\overline{k}^{\alpha-1} + 1 - \delta) \tag{48}$$

from which capital stock per capita is steady state is given by:

$$\overline{k} = \left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{1}{\alpha-1}} \tag{49}$$

- From the equilibrium condition for the capital stock per capita in steady state:

$$\overline{c} + (1+n)\overline{k} = (1-\alpha)A\overline{k}^{\alpha} + \alpha A\overline{k}^{\alpha} + (1-\delta)\overline{k} \tag{50}$$

and operating:

$$\overline{c} = A\overline{k}^{\alpha} - (n+\delta)\overline{k} \tag{51}$$

- Finally, the consumption per capita is given by:

$$\overline{c} = A\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{\alpha}{\alpha-1}} - (n+\delta)\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{1}{\alpha-1}} \tag{52}$$