

# Lecture 15: Numerical solution of the Dynamic General Equilibrium model

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## 15. Numerical solution of the DGE model

- Assume that  $L = 1$  (No leisure in the utility function). In this case, the model economy is given by:

$$C_{t+1} = \beta [R_{t+1} + 1 - \delta] C_t \quad (1)$$

$$R_t = \alpha \frac{Y_t}{K_t} = \frac{\alpha A_t K_t^\alpha}{K_t} = \alpha A_t K_t^{\alpha-1} \quad (2)$$

$$W_t = (1 - \alpha) Y_t = (1 - \alpha) A_t K_t^\alpha \quad (3)$$

$$Y_t = A_t K_t^\alpha \quad (4)$$

$$K_{t+1} = (1 - \delta) K_t + I_t \quad (5)$$

$$C_t + I_t = Y_t \quad (6)$$

## 15. Numerical solution of the DGE model

- Dynamica system. We reduce the above system of equations to a two differential equation system, for consumption (the jump variable) and the capital stock (the state or predetermined variable). If the model is stochastic, we need an additional equation for the dynamics of TFP. By substituting the real interest rate (2) in the equation for consumption (1) we have:

$$\frac{C_{t+1}}{C_t} = \beta [\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta] \quad (7)$$

## 15. Numerical solution of the DGE model

- On the other hand, substituting the relative price of production factors in the budgeted constraint:

$$C_t + K_{t+1} - K_t - (\alpha A_t K_t^{\alpha-1} - \delta)K_t - (1 - \alpha)A_t K_t^{\alpha} = 0$$

or:

$$C_t + K_{t+1} - K_t - \alpha A_{t+1} K_t^{\alpha} + \delta K_t - A_t K_t^{\alpha} + \alpha A_t K_t^{\alpha} = 0$$

and operating we arrive to:

$$C_t + K_{t+1} - (1 - \delta)K_t - A_t K_t^{\alpha} = 0 \quad (8)$$

## 15. Numerical solution of the DGE model

- In sum, the model economy is given by the following two difference equations:

$$C_{t+1} = \beta [\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta] C_t \quad (9)$$

$$K_{t+1} = (1 - \delta)K_t + A_t K_t^\alpha - C_t \quad (10)$$

## 15. Numerical solution of the DGE model

- Steady state. We start from the equation representing the optimal consumption path:

$$C_{t+1} = \beta [R_{t+1} + 1 - \delta] C_t \quad (11)$$

By eliminating time subscripts:

$$1 = \beta(\bar{R} + 1 - \delta) \quad (12)$$

and the real interest rate in steady state is given by:

$$\bar{R} = \frac{1 - \beta + \beta\delta}{\beta} \quad (13)$$

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- We can write (the real interest rate is the marginal productivity of capital):

$$\bar{C} = \beta(\alpha \bar{A} \bar{K}^{\alpha-1} + 1 - \delta) \bar{C} \quad (14)$$

and operating:

$$\beta(\alpha \frac{\bar{Y}}{\bar{K}} + 1 - \delta) = 1 \quad (15)$$

Solving for the capital stock in steady state we have (as a function of steady state output):

$$\bar{K} = \frac{\alpha \beta}{1 - \beta + \beta \delta} \bar{Y} \quad (16)$$

## 15. Numerical solution of the DGE model

- From the capital stock accumulation equation we have:

$$\overline{K} = (1 - \delta)\overline{K} + \overline{I} \quad (17)$$

and operating:

$$\overline{I} = \delta\overline{K} \quad (18)$$

and using previous expression (16) we arrive to:

$$\overline{I} = \frac{\alpha\beta\delta}{1 - \beta + \beta\delta} \overline{Y} \quad (19)$$



## 15. Numerical solution of the DGE model

- From the feasibility condition we have that:

$$\overline{C} = \overline{Y} - \overline{I} = \frac{1 - \beta + \beta\delta - \alpha\beta\delta}{1 - \beta + \beta\delta} \overline{Y} \quad (20)$$

- Finally, output in steady state is given by:

$$\overline{Y} = \overline{AK}^\alpha \quad (21)$$

and using expression (16) we have:

$$\overline{Y} = \overline{AK}^\alpha = \overline{A}^{\frac{1}{1-\alpha}} \left[ \frac{\alpha\beta}{(1 - \beta + \beta\delta)} \right]^{\frac{\alpha}{1-\alpha}} \quad (22)$$

## 15. Numerical solution of the DGE model

- Once we have the steady state value for output, we can recover the rest of steady state values. By substituting (22) in (16), we obtain that the steady state value for the capital stock is:

$$\bar{K} = \frac{\alpha\beta}{1 - \beta + \beta\delta} \bar{A}^{\frac{1}{1-\alpha}} \left[ \frac{\alpha\beta}{(1 - \beta + \beta\delta)} \right]^{\frac{\alpha}{1-\alpha}} = \left( \frac{(1 - \beta + \beta\delta)}{\alpha\bar{A}\beta} \right)^{\frac{1}{\alpha-1}} \quad (23)$$

Alternatively, steady state capital stock can be calculated from:

$$\bar{R} = \frac{1}{\beta} - 1 + \delta = \frac{1 - \beta + \delta\beta}{\beta} \quad (24)$$

## 15. Numerical solution of the DGE model

- Given that real interest rate is the marginal productivity of capital:

$$\alpha \bar{A} \bar{K}^{\alpha-1} = \bar{R} = \frac{1 - \beta + \delta\beta}{\beta} \quad (25)$$

Solving for the capital stock in steady state we have:

$$\bar{K} = \left( \frac{1 - \beta + \delta\beta}{\alpha \bar{A} \beta} \right)^{\frac{1}{\alpha-1}} \quad (26)$$

## 15. Numerical solution of the DGE model

- Log-linearization rules:

- 1 For one variable:

$$x_t \approx \bar{x}_t \exp(\hat{x}_t) \approx \bar{x}_t(1 + \hat{x}_t) \quad (27)$$

- 2 For the product of two variables:

$$x_t z_t \approx \bar{x}_t(1 + \hat{x}_t)\bar{z}_t(1 + \hat{z}_t) \approx \bar{x}_t\bar{z}_t(1 + \hat{x}_t + \hat{z}_t) \quad (28)$$

that is, we assume that the product of two deviations to the steady state,  $\hat{x}_t\hat{z}_t$ , is a very small number close to zero.

- 3 For a power function:

$$x_t^a \approx \bar{x}_t^a(1 + \hat{x}_t)^a \approx \bar{x}_t^a(1 + a\hat{x}_t) \quad (29)$$

## 15. Numerical solution of the DGE model

- Log-linear approximation to the production function:

$$Y_t = A_t K_t^\alpha \quad (30)$$

Using the first rule in the left-hand side:

$$Y_t \approx \bar{Y}(1 + \hat{y}_t) \quad (31)$$

Using the third rule in the right-hand side:

$$A_t K_t^\alpha \approx \bar{A} \bar{K}^\alpha (1 + \hat{k}_t)^\alpha \approx \bar{A} \bar{K}^\alpha (1 + \alpha \hat{k}_t) \quad (32)$$

## 15. Numerical solution of the DGE model

We obtain that:

$$\overline{Y}(1 + \hat{y}_t) = \overline{AK}^\alpha (1 + \alpha \hat{k}_t) \quad (33)$$

and operating:

$$\overline{Y} + \overline{Y}\hat{y}_t = \overline{AK}^\alpha + \overline{AK}^\alpha \alpha \hat{k}_t \quad (34)$$

Canceling terms:

$$\overline{Y}\hat{y}_t = \overline{AK}^\alpha \alpha \hat{k}_t \quad (35)$$

and finally:

$$\hat{y}_t = \alpha \hat{k}_t \quad (36)$$

## 15. Numerical solution of the DGE model

- Log-linear approximation to the capital stock equation:

$$C_t + K_{t+1} - (1 - \delta)K_t = Y_t \quad (37)$$

Using the log-linearization rules:

$$\bar{C}(1 + \hat{c}_t) + \bar{K}(1 + \hat{k}_{t+1}) - (1 - \delta)\bar{K}(1 + \hat{k}_t) = \bar{Y}(1 + \hat{y}_t) \quad (38)$$

In steady state:

$$\bar{C} + \bar{K} - (1 - \delta)\bar{K} = \bar{Y} \quad (39)$$

Operating,

$$\bar{C}\hat{c}_t + \bar{K}\hat{k}_{t+1} - (1 - \delta)\bar{K}\hat{k}_t = \bar{Y}\hat{y}_t \quad (40)$$

or:

$$\frac{\bar{C}}{\bar{K}}\hat{c}_t + \hat{k}_{t+1} - (1 - \delta)\hat{k}_t = \frac{\bar{Y}}{\bar{K}}\hat{y}_t \quad (41)$$

## 15. Numerical solution of the DGE model

Next, we use the steady state values for consumption, capital and output:

$$\frac{\bar{C}}{\bar{K}} = \frac{\frac{1-\beta+\beta\delta-\alpha\beta\delta}{(1-\beta+\beta\delta)}\bar{Y}}{\frac{\alpha\beta}{(1-\beta+\beta\delta)}\bar{Y}} = \frac{1-\beta+\beta\delta-\alpha\beta\delta}{\alpha\beta}$$

$$\frac{\bar{Y}}{\bar{K}} = \frac{\bar{Y}}{\frac{\alpha\beta}{(1-\beta+\beta\delta)}\bar{Y}} = \frac{(1-\beta+\beta\delta)}{\alpha\beta}$$

and substituting:

$$\frac{1-\beta+\beta\delta-\alpha\beta\delta}{\alpha\beta}\hat{c}_t + \hat{k}_{t+1} - (1-\delta)\hat{k}_t = \frac{(1-\beta+\beta\delta)}{\alpha\beta}\hat{y}_t \quad (42)$$



## 15. Numerical solution of the DGE model

Using the log-linear approximation for output ( $\hat{y}_t = \alpha \hat{k}_t$ ), we get:

$$\frac{1 - \beta + \beta\delta - \alpha\beta\delta}{\alpha\beta} \hat{c}_t + \hat{k}_{t+1} = (1 - \delta) \hat{k}_t + \frac{(1 - \beta + \beta\delta)}{\beta} \hat{k}_t \quad (43)$$

Finally, defining  $\Delta \hat{k}_t = \hat{k}_{t+1} - \hat{k}_t$ , we arrive to the following difference equation for the deviation of the capital stock:

$$\Delta \hat{k}_t = - \left[ \frac{1 - \beta + \beta\delta - \alpha\beta\delta}{\alpha\beta} \right] \hat{c}_t + \left[ \frac{1 - \beta}{\beta} \right] \hat{k}_t \quad (44)$$

## 15. Numerical solution of the DGE model

Log-linear approximation to the consumption equation:

$$\frac{C_{t+1}}{C_t} = \beta \left( \alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta \right) \quad (45)$$

Using the log-linearization rules we have:

$$\frac{\bar{C}}{\bar{C}}(1 + \hat{c}_{t+1} - \hat{c}_t) = \alpha \beta \frac{\bar{Y}}{\bar{K}}(1 + \hat{y}_{t+1} - \hat{k}_{t+1}) + \beta(1 - \delta) \quad (46)$$

## 15. Numerical solution of the DGE model

Operating,

$$1 + \hat{c}_{t+1} - \hat{c}_t = \alpha\beta\frac{\bar{Y}}{\bar{K}}(\hat{y}_{t+1} - \hat{k}_{t+1}) + \alpha\beta\frac{\bar{Y}}{\bar{K}} + \beta(1 - \delta) \quad (47)$$

Using the steady state:

$$\frac{\bar{Y}}{\bar{K}} = \frac{1 - \beta + \beta\delta}{\alpha\beta} \quad (48)$$

Substituting:

$$1 + \hat{c}_{t+1} - \hat{c}_t = (1 - \beta + \beta\delta)(\hat{y}_{t+1} - \hat{k}_{t+1}) + (1 - \beta + \beta\delta) + \beta(1 - \delta) \quad (49)$$

and operating:

$$\hat{c}_{t+1} - \hat{c}_t = (1 - \beta + \beta\delta)(\hat{y}_{t+1} - \hat{k}_{t+1}) \quad (50)$$

## 15. Numerical solution of the DGE model

Using the log-linear approximation for output ( $\hat{y}_{t+1} = \alpha \hat{k}_{t+1}$ ), we get:

$$\hat{c}_{t+1} - \hat{c}_t = (1 - \beta + \beta\delta)(\alpha - 1)\hat{k}_{t+1} \quad (51)$$

On the other hand, from expression (43) we have that:

$$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t - \frac{1 - \beta + \beta\delta - \alpha\beta\delta}{\alpha\beta}\hat{c}_t \quad (52)$$

## 15. Numerical solution of the DGE model

Substituting:

$$\hat{c}_{t+1} - \hat{c}_t = (1 - \beta + \beta\delta)(\alpha - 1) \left( \frac{1}{\beta} \hat{k}_t - \frac{1 - \beta + \beta\delta - \alpha\beta\delta}{\alpha\beta} \hat{c}_t \right) \quad (53)$$

Defining  $\Delta\hat{c}_t = \hat{c}_{t+1} - \hat{c}_t$ , we arrive to:

$$\Delta\hat{c}_t = \frac{(1 - \beta + \beta\delta)(\alpha - 1)}{\beta} \hat{k}_t - \frac{(1 - \beta + \beta\delta)(\alpha - 1)(1 - \beta + \beta\delta - \alpha\beta\delta)}{\alpha\beta} \hat{c}_t \quad (54)$$

## 15. Numerical solution of the DGE model

Log-linear approximation to the investment equation:

$$I_t = Y_t - C_t \quad (55)$$

and using the log-linearization rules:

$$\bar{I}_t(1 + \hat{i}_t) = \bar{Y}(1 + \hat{y}_t) - \bar{C}(1 + \hat{c}_t) \quad (56)$$

or,

$$\bar{I} + \bar{I}\hat{i}_t = \bar{Y} + \bar{Y}\hat{y}_t - \bar{C} - \bar{C}\hat{c}_t \quad (57)$$

## 15. Numerical solution of the DGE model

In steady state we have that:

$$\bar{I} = \bar{Y} - \bar{C}_t \quad (58)$$

and thus, previous equation can be simplified to:

$$\widehat{\bar{I}}_t = \bar{Y}\widehat{y}_t - \bar{C}\widehat{c}_t \quad (59)$$

and solving for the deviation of investment we have:

$$\widehat{i}_t = \frac{\bar{Y}}{\bar{I}}\widehat{y}_t - \frac{\bar{C}}{\bar{I}}\widehat{c}_t \quad (60)$$

## 15. Numerical solution of the DGE model

Using the steady state values:

$$\frac{\overline{Y}}{\overline{I}} = \frac{1 - \beta + \beta\delta}{\alpha\beta\delta}$$

and:

$$\frac{\overline{C}}{\overline{I}} = \frac{1 - \beta + \beta\delta - \alpha\beta\delta}{\alpha\beta\delta}$$

And substituting, we arrive to:

$$\hat{i}_t = \frac{1 - \beta + \beta\delta}{\alpha\beta\delta} \hat{y}_t - \frac{1 - \beta + \beta\delta - \alpha\beta\delta}{\alpha\beta\delta} \hat{c}_t \quad (61)$$



## 15. Numerical solution of the DGE model

Log-linear system. We define:

$$\Omega = 1 - \beta + \beta\delta \quad (62)$$

$$\Phi = 1 - \beta + (1 - \alpha)\beta\delta \quad (63)$$

Therefore, the log-linear difference equations system is given by:

$$\begin{bmatrix} \Delta \hat{c}_t \\ \Delta \hat{k}_t \end{bmatrix} = \begin{bmatrix} -\frac{(\alpha-1)\Omega\Phi}{\alpha\beta} & \frac{(\alpha-1)\Omega}{\beta} \\ -\frac{\Phi}{\alpha\beta} & \frac{(1-\beta)}{\beta} \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} \quad (64)$$

# 15. Numerical solution of the DGE model

Stability analysis:

$$\text{Det} \begin{bmatrix} -\frac{(\alpha-1)\Omega\Phi}{\alpha\beta} - \lambda & \frac{(\alpha-1)\Omega}{\beta} \\ -\frac{\Phi}{\alpha\beta} & \frac{1-\beta}{\beta} - \lambda \end{bmatrix} = 0 \quad (65)$$

Second order equation:

$$\lambda^2 + \left( \frac{(\alpha-1)\Omega\Phi}{\alpha\beta} - \frac{1-\beta}{\beta} \right) \lambda - \left( \frac{1-\beta}{\beta} \right) \left( \frac{(\alpha-1)\Omega\Phi}{\alpha\beta} \right) + \left( \frac{(\alpha-1)\Omega}{\beta} \right) \quad (66)$$

or:

$$\lambda^2 + \frac{(\alpha-1)\Omega\Phi - \alpha(1-\beta)}{\alpha\beta} \lambda + \frac{(\alpha-1)\Omega\Phi}{\alpha\beta} = 0 \quad (67)$$

## 15. Numerical solution of the DGE model

Eigenvalues:

$$\lambda_1, \lambda_2 = \frac{-\frac{(\alpha-1)\Omega\Phi-\alpha(1-\beta)}{\alpha\beta} \pm \sqrt{\left(\frac{(\alpha-1)\Omega\Phi-\alpha(1-\beta)}{\alpha\beta}\right)^2 - 4\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}}}{2} \quad (68)$$

as  $\alpha < 1$  and  $\beta < 1$ , one root is positive and the other is negative.

## 15. Numerical solution of the DGE model

Stable path: We define  $\lambda_1$  as the eigenvalue that satisfies  $|\lambda_1 + 1| < 1$ .  
Stable trayectories are:

$$\begin{bmatrix} \Delta \hat{c}_t \\ \Delta \hat{k}_t \end{bmatrix} = \lambda_1 \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} \quad (69)$$

## 15. Numerical solution of the DGE model

Jump in consumption: From the dynamic equation for consumption we have:

$$\Delta \hat{c}_t = -\frac{(\alpha - 1)\Omega\Phi}{\alpha\beta}\hat{c}_t + \frac{(\alpha - 1)\Omega}{\beta}\hat{k}_t \quad (70)$$

On the other hand, the stable path for consumption is given by:

$$\Delta \hat{c}_t = \lambda_1 \hat{c}_t \quad (71)$$

## 15. Numerical solution of the DGE model

Equating both expressions:

$$-\frac{(\alpha - 1)\Omega\Phi}{\alpha\beta}\hat{c}_t + \frac{(\alpha - 1)\Omega}{\beta}\hat{k}_t = \lambda_1\hat{c}_t \quad (72)$$

And finally, solving for consumption:

$$\hat{c}_t = \frac{\alpha(\alpha - 1)\Omega}{(\alpha - 1)\Omega\Phi + \alpha\beta\lambda_1}\hat{k}_t \quad (73)$$