

Lecture 13: The canonical Dynamic (Stochastic) General Equilibrium model

José L. Torres

Universidad de Málaga

Advanced Macroeconomics

13. The Canonical Dynamic (Stochastic) General Equilibrium model

- DSGE: Dynamic Stochastic General Equilibrium modelling. This is the standard tool in modern macroeconomic analysis. This is a fully integrated framework for policy analysis.¹ Unified framework to study both business cycles and economic growth.
- **Dynamic:** Time is important. Notice that saving today implies consumption in the future. Forward-looking behavior of rational agents. Decisions today depend on expectations about the future.
- **Stochastic:** The model can be deterministic or stochastic. The difference is in the consideration of stochastic shocks (business cycles).
- **General Equilibrium:** Simultaneous determination of endogenous variables, markets clearing, and Walras law.

¹Ramsey, F. (1927): A contribution to the theory of taxation. *Economic Journal*, 37(145), 47-61.

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- Steps in DSGE modelling:
 - 1 Define the model economy environment (Agents, preferences, technology, institutions, etc.). This involves a small number of equations, both function and identities.
 - 2 Parameterize functions (Consumers' utility function, production function, adjustment cost functions, etc.).
 - 3 Obtain First Order Conditions. Together with feasibility conditions, technological restrictions and identities, we will arrive to a system of equations defining the model.
 - 4 Assign values to the parameters. Calibration and/or estimation.
 - 5 Find the steady state.
 - 6 (log)-linearize the equations.
 - 7 Write the model in state space form.
 - 8 Numerical solution of the DSGE model.
 - 9 Simulation, policy analysis, shocks and forecasting.

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- DSGE models based on four key components:
 - **Preferences** (we want to consume as much as possible, we don't like to work,...).
 - **Endowments** (who is the owner of productive factors).
 - **Technology** (inputs, returns to scale, factor productivity, elasticity of substitution, ...).
 - **Institutional environment** (perfect or imperfect competition, information, government, nominal and real frictions,...)

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- Types of agents:
 - Households
 - Firms
 - The government
 - Capitalists
 - Central Bank
 - Financial Institutions
 - Foreign sector
 - International investors

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- DSGE models varieties: From the basic canonical RBC model (7-8 equations, 5-6 equations under the central planner solution in the deterministic and stochastic cases, respectively) to large-scale huge New Keynesian model (hundreds of equations: For instance, the QUEST III model has 100 equations).
- This framework is adequate to study the effects of a stochastic shock hitting the economy.
- It can also be used to study structural changes in the economy (for instance, a tax change or the introduction of a new tax).
- Shocks can hit the economy today or in any future time (in this case they are anticipated with perfect foresight).
- Shocks can be one shot or be long lasting.
- Standard procedure: A positive shock today and no other shock in the future.

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- DSGE models do not have explicit solutions, except for some simple cases (logarithmic utility functions and full depreciation of capital). DSGE models cannot be solved directly by hand.
- DSGE modelling requires the use of numerical and computational methods to obtain approximate solutions.
- DSGE models have two key characteristics:
 - Non-linear system of dynamic equations
 - Expectations about future endogenous variables

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- The key to solve a DSGE model consists in representing functional forms for the control variables (for instance, consumption) as function of lagged state variables (for instance, capital stock). Once we have these functions, the system becomes **recursive** and then, given initial values for the state variables, the dynamic process for the control variables can be generated.
- These are the so called **decisions rules** or **policy functions**. The term decision rule or policy function refers to functional equations, that is, functions of functions, describing the dynamics of the forward-looking control variables.

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- In general, a DSGE model can be defined as a stochastic non-linear forward-looking system with rational expectations. This system (mainly first order conditions) can be represented as follows:

$$\begin{aligned}E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_{t+1}, \Omega)] &= 0 \\E_t(\varepsilon_{t+1}) &= 0 \\E_t(\varepsilon_{t+1}, \varepsilon'_{t+1}) &= \Sigma_\varepsilon\end{aligned}$$

where E_t is the expectations operator, y is the vector of all endogenous variables (including state variables), ε is a vector of exogenous stochastic shocks (structural innovations) and Ω is the set of structural or deep parameters of the model.

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- A solution to that system is a function in recursive form such as first order conditions and feasibility conditions are satisfied:

$$y_t = g(x_{t-1}, \varepsilon_t, \Omega)$$

that is, endogenous variables are calculated as a function of their past values and the contemporaneous structural shocks.

- The function $g(x_{t-1}, \varepsilon_t, \Omega)$ represents the set of so-called policy (for the endogenous) and transition (for the states) functions. In general, it is not possible to obtain a closed form solution, so we use a local approximation to the true solution.

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- Basic RBC model. Very simple model economy setup:
 - Two type of economic agents: Households and firms.
 - Each agent maximizes a given objective function subject to a given restriction.
 - A lot of implicit assumptions: perfect capital markets, utility function additively separable in time, separability between consumption and leisure, saving as a state variable, perfect competition, constant return to scale, no externalities, etc. These implicit assumptions can be relaxed resulting in a large variety of DSGE models.

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- Households maximization problem (the heart of a DSGE model):

$$\max_{(C_t, O_t)} E_t \sum_{t=0}^{\infty} \beta^t U(C_t, O_t)$$

where β is the intertemporal discount factor, $\beta \in (0, 1)$, and where $E_t(\cdot)$ is the mathematical expectation operator of future variables at time t , C_t is consumption and O_t is leisure.

- Utility function can be extended to include more arguments:

$$U(C_t, C_{g,t}, O_t, M_t, H_t, S_t, X_t, \dots)$$

where $C_{g,t}$ is consumption of goods and services provided by the government, M_t is money, H_t time devoted to home production (meals, child rearing, laundry, house cleaning, etc.), S_t , time devoted to skill activities, X_t pollution, ...

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- However, we cannot use numerical method with general functional forms. We must use a given specification for the utility function.
- We can use any particular specification that satisfy the following conditions:

$$U_C > 0$$

$$U_O > 0$$

$$U_{CC} < 0$$

$$U_{OO} < 0$$

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- We use the following specification:

$$U(C_t, O_t) = \gamma \log C_t + (1 - \gamma) \log(N_t \bar{H} - L_t),$$

- γ : weight for consumption over total income.
- N_t : Population (Working-age population between 16 and 65 years old).
- \bar{H} : Total discretionary available time in hours. Approximately 5.000 hours per year (16 hours per day x 6 days per week x 52 weeks per year=4,992 hours).
- L_t : Total working hours.
- Total available discretionary time is normalized to 1: $L_t + O_t = 1$.

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- More assumptions: Household are the owner of capital. Investment decisions are taken by the households and not by the firms.
- Saving is directly converted to investment at no cost.
- Output and investment measured in units of consumption.
- Firms rent production factors every period.

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- Households budget constraint:

$$P_t^C C_t + P_t^I I_t = W_t L_t + R_t K_t$$

where I_t is investment (under the assumption that investment is equal to saving), W_t is the wage, R_t is the capital rate of return and K_t is the physical capital stock.

- Physical capital accumulation process:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

where δ ($0 < \delta < 1$) is the capital depreciation rate.

- It is assumed that $P_t^C = P_t^I = 1$.

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- General problem for the households:

$$\max_{(C_t, I_t, O_t)} \mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t [\gamma \log C_t + (1 - \gamma) \log(1 - L_t)]$$

subject to:

$$C_t + I_t = W_t L_t + R_t K_t$$

$$K_{t+1} = (1 - \delta) K_t + I_t$$

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- One simple solution is the use of the Lagrangian auxiliary function:

$$\max_{(C_t, K_t, O_t)} \beta^t \left\{ \begin{array}{l} \gamma \log C_t + (1 - \gamma) \log(1 - L_t) \\ -\lambda_t [C_t + K_{t+1} - W_t L_t - (R_t + 1 - \delta)K_t] \end{array} \right\}$$

Household first order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_t} : \frac{\gamma}{C_t} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial L_t} : \frac{1 - \gamma}{1 - L_t} - \lambda_t W_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_t} : \beta^t \lambda_t [R_t + 1 - \delta] - \beta^{t-1} \lambda_{t-1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} : C_t + K_{t+1} - (R_t + 1 - \delta)K_t - W_t L_t = 0$$

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- Equilibrium condition that equals the marginal rate of substitution between consumption and leisure to the opportunity cost of an additional unit of leisure:

$$\frac{1 - \gamma}{\gamma} \frac{C_t}{1 - L_t} = W_t$$

- Equilibrium condition that equals the marginal rate of consumption and investment:

$$\frac{C_t}{C_{t-1}} = \beta [R_t + 1 - \delta]$$

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- The firms: Aggregate production function:

$$Y_t = A_t F(K_t, L_t)$$

- Y_t : Output.
- A_t : Total Factor Productivity. This can be considered as exogenously given (a constant) in the case of a deterministic model or an endogenous variable that follows a stochastic process in the case of a stochastic model.
- Technology must satisfy the following properties:

$$F_K > 0, F_L > 0$$

$$F_{KK} < 0, F_{LL} < 0$$

$$F_{KL} > 0$$

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- Problem for the firms: profit maximization:

$$\max \Pi_t = P_t Y_t - W_t L_t - R_t K_t$$

subject to:

$$Y_t = A_t F(K_t, L_t)$$

- Under the assumption of constant returns to scale and competitive markets: $\Pi_t = 0$.
- First order conditions:

$$A_t P_t F_K(K_t, L_t) - R_t = 0$$

$$A_t P_t F_L(K_t, L_t) - W_t = 0$$

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- Factor relative prices equals to their productivity:

$$A_t F_K(K_t, L_t) = \frac{R_t}{P_t}$$

$$A_t F_L(K_t, L_t) = \frac{W_t}{P_t}$$

- The price of the final good is normalized to 1 ($P_t = 1$) :

$$A_t F_K(K_t, L_t) = R_t$$

$$A_t F_L(K_t, L_t) = W_t$$

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- Standard specification: Cobb-Douglas production function:

$$A_t F(K_t, L_t) = A_t K_t^\alpha L_t^{1-\alpha}$$

- α : elasticity of capital relative to output.
- Main characteristic: Unitary elasticity of substitution between capital and labor. This is something between a Leontief technology and a perfect substitution among inputs technology.

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- Therefore, first order conditions are:

$$\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0$$

$$(1 - \alpha) A_t K_t^{\alpha} L_t^{-\alpha} - W_t = 0$$

- Or:

$$R_t = \frac{\alpha A_t K_t^{\alpha} L_t^{1-\alpha}}{K_t} = \alpha \frac{Y_t}{K_t}$$

$$W_t = \frac{(1 - \alpha) A_t K_t^{\alpha} L_t^{1-\alpha}}{L_t} = (1 - \alpha) \frac{Y_t}{L_t}$$

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- Equilibrium: Each type of economic agent takes its own decisions over the control variables. Their interaction determines the macroeconomic equilibrium.
- Households decide how much to consume, C_t , how much to invest (save), I_t and how much to work, L_t , with the objective of maximizing their happiness, taking as given the prices of the inputs.
- Firms produce a given amount of final goods, Y_t , depending on how much capital, K_t and labor L_t , they will hire, given the prices of the production factors.

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- The balanced path of the economy is composed of the following three sets of elements:
 - A pricing system for W and R .
 - A set of values assigned to Y , C , L and K .
 - A feasibility constraint of the economy, given by:

$$Y_t = C_t + I_t$$

- All markets (good market, labor market, capital market) are in equilibrium.

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- The competitive equilibrium for our economy is a sequence of consumption, leisure, and investment by consumers $\{C_t, L_t, I_t\}_{t=0}^{\infty}$ and a sequence of capital and labor hours used by firms $\{K_t, L_t\}_{t=0}^{\infty}$, such that given a sequence of prices $\{W_t, R_t\}_{t=0}^{\infty}$:
 - i) The consumers optimization problem is satisfied;
 - ii) Profit maximization FOCs for the firms hold;
 - and iii) The feasibility condition of the economy holds.

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- By substituting the relative price of inputs in the households equilibrium conditions:

$$\frac{1-\gamma}{\gamma} \frac{C_t}{1-L_t} = (1-\alpha) K_t^\alpha L_t^{-\alpha}$$
$$\frac{C_t}{C_{t-1}} = \beta [\alpha K_t^{\alpha-1} L_t^{1-\alpha} + 1 - \delta]$$

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- On the other hand, substituting the relative price of inputs in the households budget constraint we find that:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = C_t + K_{t+1} - (R_t + 1 - \delta)K_t - W_t L_t = 0$$

$$C_t + K_{t+1} - (\alpha K_t^{\alpha-1} L_t^{1-\alpha} + 1 - \delta)K_t - (1 - \alpha)K_t^{\alpha} L_t^{-\alpha} L_t = 0$$

$$C_t + K_{t+1} - K_t - \alpha K_t^{\alpha} L_t^{1-\alpha} + \delta K_t - K_t^{\alpha} L_t^{1-\alpha} + \alpha K_t^{\alpha} L_t^{1-\alpha} = 0$$

$$C_t + K_{t+1} - (1 - \delta)K_t - K_t^{\alpha} L_t^{1-\alpha} = 0$$

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- In summary, we have two dynamic equations driving consumption and capital:

$$C_t = \beta [\alpha K_t^{\alpha-1} L_t^{1-\alpha} + 1 - \delta] C_{t-1}$$

$$K_{t+1} = (1 - \delta)K_t + K_t^\alpha L_t^{1-\alpha} - C_t$$

plus a static equation for the labor supply:

$$\frac{1 - \gamma}{\gamma} \frac{C_t}{1 - L_t} = (1 - \alpha) K_t^\alpha L_t^{-\alpha}$$

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Finally, the equations describing the model (deterministic) economy are:

$$\begin{aligned}\frac{(1-\gamma)}{\gamma} \frac{C_t}{1-L_t} &= W_t \\ E_t \frac{C_{t+1}}{C_t} &= \beta E_t [R_{t+1} + 1 - \delta] \\ R_t &= \frac{\alpha A_t K_t^\alpha L_t^{1-\alpha}}{K_t} = \alpha \frac{Y_t}{K_t} \\ W_t &= \frac{(1-\alpha) A_t K_t^\alpha L_t^{1-\alpha}}{L_t} = (1-\alpha) \frac{Y_t}{L_t} \\ Y_t &= A_t K_t^\alpha L_t^{1-\alpha} \\ K_{t+1} &= (1-\delta) K_t + I_t \\ C_t + I_t &= Y_t\end{aligned}$$

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- Moving to a stochastic model. In the previous system of equations, Total Factor Productivity, A_t , is not defined. If we consider TFP as an exogenous and constant variable, thus the model is deterministic.
- Alternatively we can consider TFP as an endogenous variable just by assuming that it is not a constant but follows a particular stochastic process. In this case the model will be stochastic.
- We can consider a large number of stochastic shocks in a DSGE model. For instance, in this simple DSGE model we can introduce up to six stochastic shocks (an aggregate productivity shock, an aggregate shock to the utility function, a consumption shock, a labor supply shock, an investment-specific technological shock, and a labor-augmented shock).

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- We assume that the aggregate productivity shock follows a first order autoregressive process, such that:

$$\ln A_t = (1 - \rho_A) \ln \bar{A} + \rho_A \ln A_{t-1} + \varepsilon_t^A, \quad \varepsilon_t^A \sim N(0, \sigma_A^2)$$

where $|\rho_A| < 1$ for the process to be stationary and where \bar{A} is the steady state value for A_t .

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- As an example, if we are interesting is study stochastic shocks in the households utility function, we can consider the following problem:

$$\max_{(C_t, K_t, L_t)} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t B_t [\gamma D_t \log C_t + (1 - \gamma) H_t \log(1 - L_t)]$$

where B_t is a disturbance that reflects a preference shock that affects the consumer's intertemporal substitution, D_t represents a consumption shock and H_t represents a labor supply shock.