

# Lecture 12: Numerical solution of the Tobin's Q model

José L. Torres

Universidad de Málaga

Advanced Macroeconomics

## 12. Tobin's Q model: Numerical solution

- Tobin's Q model:

$$\Delta K_t = (q_t - 1) \frac{K_t}{\phi} \quad (1)$$

$$\Delta q_t = \frac{(R_t + \delta)q_t - \alpha K_{t+1}^{\alpha-1} + \frac{\phi}{2} \left( \frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 - \phi \left( \frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}}}{(1 - \delta)} \quad (2)$$

where  $\Delta q_t = q_{t+1} - q_t$ , and  $\Delta K_t = K_{t+1} - K_t$ .

## 12. Tobin's Q model: Numerical solution

- Non-linear equations. For the numerical solution, we need to use a linear approximation.
- To obtain a linear approximation, we define the variables as deviations (in logarithmic) from the steady state. The log-deviation for a variable,  $x_t$ , from its steady state value,  $\bar{x}_t$ , is defined as  $\hat{x}_t$ , where  $\hat{x}_t = \ln x_t - \ln \bar{x}_t$ . We will use three simple rules.

## 12. Tobin's Q model: Numerical solution

- Log-linearization rules:

- ① A variable can be defined as:

$$x_t \approx \bar{x}_t \exp(\hat{x}_t) \approx \bar{x}_t(1 + \hat{x}_t) \quad (3)$$

- ② The product of two variables can be defined as:

$$x_t z_t \approx \bar{x}_t(1 + \hat{x}_t) \bar{z}_t(1 + \hat{z}_t) \approx \bar{x}_t \bar{z}_t(1 + \hat{x}_t + \hat{z}_t) \quad (4)$$

that is, we assume that the product of two deviations from steady state,  $\hat{x}_t \hat{z}_t$ , is a very small number and approximately equal to zero.

- ③ Finally, we use the following approximation:

$$x_t^a \approx \bar{x}_t^a(1 + \hat{x}_t)^a \approx \bar{x}_t^a(1 + a\hat{x}_t) \quad (5)$$

## 12. Tobin's Q model: Numerical solution

- Log-linearization of the dynamic equation for the capital stock.
- Initial equation:

$$K_{t+1} - K_t = (q_t - 1) \frac{K_t}{\phi} \quad (6)$$

or:

$$K_{t+1} - K_t = q_t \frac{K_t}{\phi} - \frac{K_t}{\phi} \quad (7)$$

- Using the first and second log-linearization rules, we have:

$$\bar{K}_{t+1}(1 + \hat{k}_{t+1}) - \bar{K}_t(1 + \hat{k}_t) = \frac{1}{\phi} \bar{q}_t \bar{K}_t(1 + \hat{q}_t + \hat{k}_t) - \frac{1}{\phi} \bar{K}_t(1 + \hat{k}_t) \quad (8)$$

## 12. Tobin's Q model: Numerical solution

- Equivalently, eliminating time indexes for the steady state values (given that  $\bar{K}_{t+1} = \bar{K}_t = \bar{K}$ ),

$$\bar{K} + \bar{K}\hat{k}_{t+1} - \bar{K} - \bar{K}\hat{k}_t = \frac{1}{\phi}\bar{q}\bar{K} + \frac{1}{\phi}\bar{q}\bar{K}\hat{q}_t + \frac{1}{\phi}\bar{q}\bar{K}\hat{k}_t - \frac{1}{\phi}\bar{K} - \frac{1}{\phi}\bar{K}\hat{k}_t \quad (9)$$

and given that  $\bar{q}_t = 1$ , yields:

$$\hat{k}_{t+1} - \hat{k}_t = \frac{1}{\phi} + \frac{1}{\phi}\hat{q}_t + \frac{1}{\phi}\hat{k}_t - \frac{1}{\phi} - \frac{1}{\phi}\hat{k}_t \quad (10)$$

or:

$$\hat{k}_{t+1} - \hat{k}_t = \frac{1}{\phi}\hat{q}_t \quad (11)$$

and finally, the linear approximation for the capital stock (deviations from the steady state) changes over time is given by:

$$\Delta\hat{k}_t = \frac{1}{\phi}\hat{q}_t \quad (12)$$

## 12. Tobin's Q model: Numerical solution

- Log-linearization of the dynamic equation for the ratio  $q$ .
- The dynamic equation is given by:

$$(1 - \delta)q_{t+1} = (1 + R_t)q_t - \alpha K_{t+1}^{\alpha-1} + \frac{\phi}{2} \left( \frac{I_{t+1}}{K_{t+1}} - \delta \right)^2 - \phi \left( \frac{I_{t+1}}{K_{t+1}} - \delta \right) \quad (13)$$

In steady state we have:

$$(1 - \delta)\bar{q} = (1 + \bar{R})\bar{q} - \alpha \bar{K}_t^{\alpha-1} \quad (14)$$

given that in steady state  $\bar{I}_t = \delta \bar{K}_t$ . On the other hand, the investment-capital stock ratio is given by:

$$\frac{I_t}{K_t} = \frac{1}{\phi}(q_t - 1) + \delta \quad (15)$$

## 12. Tobin's Q model: Numerical solution

- By substituting the previous expression in the dynamic equation (13), we obtain that:

$$(1 - \delta)q_{t+1} = (1 + R_t)q_t - \alpha K_{t+1}^{\alpha-1} + \frac{\phi}{2} \left( \frac{1}{\phi}(q_{t+1} - 1) \right)^2 \quad (16)$$

$$-\phi \left( \frac{1}{\phi}(q_{t+1} - 1) \right) \left( \frac{1}{\phi}(q_{t+1} - 1) + \delta \right) \quad (17)$$

Operating, we arrive to:

$$(1 - \delta)q_{t+1} = (1 + R_t)q_t - \alpha K_{t+1}^{\alpha-1} - \frac{1}{2\phi}(q_{t+1} - 1)^2 - \delta(q_{t+1} - 1) \quad (18)$$

## 12. Tobin's Q model: Numerical solution

- Using the above log-linearization rules, we have:

$$\begin{aligned}(1 - \delta)\bar{q}(1 + \hat{q}_{t+1}) &= (1 + R)\bar{q}(1 + \hat{q}_t) - \alpha\bar{K}^{\alpha-1}(1 + (\alpha - 1)\hat{k}_{t+1}) \\ &\quad - \frac{1}{2\phi}\bar{q}^2(1 + 2\hat{q}_{t+1}) - \frac{1}{2\phi} + \frac{1}{\phi}\bar{q}(1 + \hat{q}_{t+1}) \\ &\quad - \delta\bar{q}(1 + \hat{q}_{t+1}) + \delta\end{aligned}\tag{19}$$

## 12. Tobin's Q model: Numerical solution

- Given that the steady state value for the ratio  $q$  is 1, ( $\bar{q} = 1$ ), we can write:

$$\begin{aligned}(1 - \delta)(1 + \hat{q}_{t+1}) &= (1 + R)(1 + \hat{q}_t) - \alpha \bar{K}^{\alpha-1} (1 + (\alpha - 1) \hat{k}_{t+1}) \\ &\quad - \frac{1}{2\phi} (1 + 2\hat{q}_{t+1}) - \frac{1}{2\phi} + \frac{1}{\phi} (1 + \hat{q}_{t+1}) \\ &\quad - \delta(1 + \hat{q}_{t+1}) + \delta\end{aligned}\tag{20}$$

and operating:

$$1 + \hat{q}_{t+1} - \delta - \delta \hat{q}_{t+1} = (1 + \hat{q}_t + R_t + R_t \hat{q}_t)\tag{21}$$

$$- \alpha \bar{K}^{\alpha-1} (1 + (\alpha - 1) \hat{k}_{t+1}) - \delta \hat{q}_{t+1}\tag{22}$$

or equivalently,

$$(1 - \delta) + \hat{q}_{t+1} = (1 + R_t) + (1 + R_t) \hat{q}_t - \alpha \bar{K}^{\alpha-1} - \alpha \bar{K}^{\alpha-1} (\alpha - 1) \hat{k}_{t+1}\tag{23}$$

## 12. Tobin's Q model: Numerical solution

- Using the steady state expression (14) and cancelling terms, we arrive to:

$$\hat{q}_{t+1} = (1 + R_t)\hat{q}_t - \alpha \bar{K}^{\alpha-1}(\alpha - 1)\hat{k}_{t+1} \quad (24)$$

and using the definition for the capital stock in steady state:

$$\hat{q}_{t+1} = (1 + R_t)\hat{q}_t - (\alpha - 1)(R_t + \delta)\hat{k}_{t+1} \quad (25)$$

## 12. Tobin's Q model: Numerical solution

- On the other hand, from expression (11) we have that:

$$\hat{k}_{t+1} = \hat{k}_t + \frac{1}{\phi} \hat{q}_t \quad (26)$$

and substituting in the previous expression yields:

$$\hat{q}_{t+1} = (1 + R_t) \hat{q}_t - (\alpha - 1)(R_t + \delta) \left( \hat{k}_t + \frac{1}{\phi} \hat{q}_t \right) \quad (27)$$

- Given that our interest is to find an expression for the change over time of  $\hat{q}_t$ , defining  $\Delta \hat{q}_t = \hat{q}_{t+1} - \hat{q}_t$ , and operating yields:

$$\Delta \hat{q}_t = \frac{R_t \phi - (\alpha - 1)(R_t + \delta)}{\phi} \hat{q}_t - (\alpha - 1)(R_t + \delta) \hat{k}_t \quad (28)$$

## 12. Tobin's Q model: Numerical solution

- In matrix notation, the Tobin's Q model (the linear approximation), can be defined by the following linear system as deviations from the steady state:

$$\begin{bmatrix} \Delta \hat{q}_t \\ \Delta \hat{k}_t \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{R_t \phi - (\alpha - 1)(R_t + \delta)}{\phi} & -(\alpha - 1)(R_t + \delta) \\ \frac{1}{\phi} & 0 \end{bmatrix}}_A \begin{bmatrix} \hat{q}_t \\ \hat{k}_t \end{bmatrix} \quad (29)$$

## 12. Tobin's Q model: Numerical solution

- Stability of the system:

$$\text{Det} \begin{bmatrix} \frac{R_t\phi - (\alpha - 1)(R_t + \delta)}{\phi} - \lambda & -(\alpha - 1)(R_t + \delta) \\ \frac{1}{\phi} & 0 - \lambda \end{bmatrix} = 0 \quad (30)$$

The corresponding second order equation is:

$$\lambda^2 - \frac{R_t\phi - (\alpha - 1)(R_t + \delta)}{\phi}\lambda + \frac{(\alpha - 1)(R_t + \delta)}{\phi} = 0 \quad (31)$$

Solving, the two eigenvalues are:

$$\lambda_1, \lambda_2 = \frac{\frac{R_t\phi - (\alpha - 1)(R_t + \delta)}{\phi} \pm \sqrt{\left(\frac{R_t\phi - (\alpha - 1)(R_t + \delta)}{\phi}\right)^2 - 4\frac{(\alpha - 1)(R_t + \delta)}{\phi}}}{2} \quad (32)$$

## 12. Tobin's Q model: Numerical solution

- Stable saddle path.
- By assuming  $\lambda_1$  as the eigenvalue for which  $|\lambda_1 + 1| < 1$ , the stable saddle path is given by:

$$\begin{bmatrix} \Delta \hat{q}_t \\ \Delta \hat{k}_t \end{bmatrix} = \lambda_1 \begin{bmatrix} \hat{q}_t \\ \hat{k}_t \end{bmatrix} \quad (33)$$

- The Tobin's  $Q$  model is represented by two variables: the capital stock and the ratio  $q$ . Capital stock is a very sticky variable, whereas the ratio  $q$  is a full flexible variable.
- In the case of a shock, we will observe an instantaneous change in the ratio  $q$ .

## 12. Tobin's Q model: Numerical solution

- Readjustment (jump) in the ratio  $q$
- We start from the dynamic equation for the deviations of the ratio  $q$  with respect to its steady state:

$$\Delta \hat{q}_t = \frac{R_t \phi - (\alpha - 1)(R_t + \delta)}{\phi} \hat{q}_t - (\alpha - 1)(R_t + \delta) \hat{k}_t \quad (34)$$

- Additionally, we can define the following stable path:

$$\Delta \hat{q}_t = \lambda_1 \hat{q}_t \quad (35)$$

## 12. Tobin's Q model: Numerical solution

- By equating both expressions in the period of the shock ( $t = 1$ ) results:

$$\frac{R_t \phi - (\alpha - 1)(R_t + \delta)}{\phi} \hat{q}_1 - (\alpha - 1)(R_t + \delta) \hat{k}_1 = \lambda_1 \hat{q}_1 \quad (36)$$

and solving for the deviation of the ratio  $q$  from its steady state value:

$$\hat{q}_1 = \frac{(\alpha - 1)(R_t + \delta)}{\frac{R_t \phi - (\alpha - 1)(R_t + \delta)}{\phi} - \lambda_1} \hat{k}_1 \quad (37)$$