Lecture 11: The firm and the investment decision: The Tobin's Q model

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Avanced Macroeconomics

- Firms produce goods.
- To produce, firms need production factors. They rent the production factors to the households.
- We assume that the objective of the firms is profit maximization. They maximize profits subject to a technological restriction.
- From the maximization problem we will obtain the rent prices for the production factors (given the technological restriction) and the level of production.

11. The firm and the investment decision

- Two approaches:
  - Neoclassical model: The optimal investment path depends on the optimal capital stock path as a function of the relative prices for the production factors.
  - The Tobin "Q" theory. The optimal investment path depends on the so-called Q ratio. The Q ratio is calculated as the market value of a company divided by the replacement value of the firm's assets.

11. The firm and the investment decision

- Technology: Aggregate production function:

$$Y_t = F(K_t, L_t) \tag{1}$$

- Y_t: Output.
- Technology function has the following properties:

$$F_K > 0, F_L > 0 \tag{2}$$

$$F_{KK} < 0, F_{LL} < 0 \tag{3}$$

$$F_{KL} > 0 \tag{4}$$

- Inada conditions:

$$\lim_{K \longrightarrow 0} F_K = \infty, \lim_{K \longrightarrow \infty} F_K = 0 \tag{5}$$

$$\lim_{L \longrightarrow 0} F_L = \infty, \lim_{L \longrightarrow \infty} F_L = 0 \tag{6}$$

To produce, the economy need a combination of the two production factors (labor and capital).

11. The firm and the investment decision

- Profit maximization (The price is normalized to 1):

$$\max \Pi_t = Y_t - W_t L_t - R_t K_t \tag{7}$$

subject to:

$$Y_t = F(K_{t,} L_t) \tag{8}$$

- Assuming constant return to scale and perfect competition we have: $\Pi_t = 0$.
- First order conditions:

$$F_K(K_{t,} L_t) - R_t = 0 \tag{9}$$

$$F_L(K_{t,} L_t) - W_t = 0 \tag{10}$$

- Relative price for the production factors is equal to their marginal productivity.

11. The firm and the investment decision

- Example: Cobb-Douglas production function:

$$F(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{11}$$

- A_t: Total Factor Productivity.
- α : elasticity of output with respect to the capital.
- Important: The technological parameter also indicates the proportion of capital income over total income (capital compensation). $1 - \alpha$ would be the proportional of labor income over total income (compensation on employees or labor compensation).

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- First order conditions:

$$\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0 \tag{12}$$

$$(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} - W_t = 0 \tag{13}$$

- Or:

$$R_t = \frac{\alpha A_t K_t^{\alpha} L_t^{1-\alpha}}{K_t} = \alpha \frac{Y_t}{K_t} \tag{14}$$

$$W_t = \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{1-\alpha}}{L_t} = (1-\alpha) \frac{Y_t}{L_t} \tag{15}$$

11. The firm and the investment decision

- Decreasing marginal productivity conditions:

$$F_{KK} = (\alpha - 1)\alpha A_t K_t^{\alpha-2} L_t^{1-\alpha} < 0 \tag{16}$$

$$F_{LL} = -\alpha(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha-1} < 0 \tag{17}$$

11. The firm and the investment decision

- Profits are zero:

$$\Pi_t = A_t K_t^{\alpha} L_t^{1-\alpha} - R_t K_t - W_t L_t \tag{18}$$

$$\Pi_t = A_t K_t^{\alpha} L_t^{1-\alpha} - \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} = 0 \tag{19}$$

- Profits:

$$\Pi_t = Y_t - W_t L_t - P_t^K K_t \tag{20}$$

- Physical capital accumulation equation in continuous time:

$$I_t = \dot{K}_t + \delta K_t \tag{21}$$

- If the firm is the owner of the capital stock, the profit function must be defined as:

$$\Pi_t = Y_t - W_t L_t - P_t^K I_t \tag{22}$$

- Price of capital is normalized to 1, ($P_t^K = 1$).

11. The Tobin's Q model

- In the neoclassical model it is assumed that capital stock can be changed from one period to another without any restriction.
- In practice, capital accumulation process is subject to implicit costs: Capital adjustment costs or investment adjustment costs.
- Two types of adjustment costs:
  - External adjustment costs (the price of capital depends on the velocity of installation of the new capital.
  - Internal adjustment costs (a reduction in production and profits).

- Internal adjustment costs:
  - Perfect elastic supply of capital.
  - Acquisition of new capital can be done at different speeds.
  - Price of capital is a function of the speed. Higher speed larger the price.
  - Example: Public infrastructure.

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- Internal adjustment costs:
  - When new capital is installed, a portion of inputs already used in the production, basically labor, must be devoted to the installation process.
  - These inputs will be not available to produce during the installation process, which implies forgone output.

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- Adjustment cost specification:

$$C = C(I_t, K_t) \tag{23}$$

$$\Psi(I_t, K_t) = I_t - C(I_t, K_t) \tag{24}$$

- The capital stock accumulation equation can be defined as:

$$K_{t+1} = (1-\delta)K_t + I_t - C(I_t, K_t) = (1-\delta)K_t + \Psi(I_t, K_t) \tag{25}$$

- Alternatively, we can assume that the adjustment cost is an additional cost to investment that implies a loss in output and hence in profits. In this case, capital acumulation equaiton is the standard, and profit function would be defined as:

$$\Pi_t = Y_t - W_t L_t - I_t - C(I_t, K_t) \tag{26}$$

11. The Tobin's Q model

- Properties of the adjustment costs function:

$$C(0, K_t) = 0 \tag{27}$$

$$C(I_t, 0) = 0 \tag{28}$$

$$C_I(I_t, K_t) > 0 \tag{29}$$

$$C_K(I_t, K_t) > 0 \tag{30}$$

$$C_{II}(I_t, K_t) > 0 \tag{31}$$

$$C_{KK}(I_t, K_t) > 0 \tag{32}$$

- Definition of the ratio Q: The Q ratio is calculated as the market value of a company divided by the replacement value of the firm's assets.

$$Q = \frac{MV}{KRV} \tag{33}$$

Estructure of the Tobin's Q model	
Firm's profits	$\Pi_t = Y_t - W_t L_t - I_t - C(I_t, K_t)$
Technology	$Y_t = F(K_t, L_t)$
Capital accumulation	$K_{t+1} = (1-\delta)K_t + I_t$
Adjustment cost	$C = C(I_t, K_t)$
Initial capital stock	$K_0 > 0$
Ratio Q	$Q_t = \frac{V_t}{K_t}$

- The intertemporal profit maximization problem for the firms can be defined as:

$$\max E_t \sum_{t=0}^{T} \frac{1}{(1+R_t)^t} \Pi_t \tag{34}$$

subject to the technological restriction and to the capital accumulation equation:

$$Y_t = F(K_t, L_t) \tag{35}$$

$$K_{t+1} = (1-\delta)K_t + I_t \tag{36}$$

where the discount factor is the real interest rate, R_t, E_t is the expectation operator, and where $K_0 > 0$, and known. Additionally, the following condition must be hold:

$$\lim_{T \to \infty} K_T = \overline{K} \tag{37}$$

where $\overline{K}$ is the steady state capital stock.

- Intertemporal profit maximization problem:

$$\max \sum_{t=0}^{T} \frac{1}{(1+R_t)^t} \left[Y_t - W_t L_t - I_t - C(I_t, K_t) \right] \quad (38)$$

subject to the above restrictions. By substituting the technological restriction into the objective function (profits), we obtain the following auxiliary Lagrangian function:

$$\begin{aligned} V = & \sum_{t=0}^{T} \frac{1}{(1+R_t)^t} \left[F(K_t, L_t) - W_t L_t - I_t - C(I_t, K_t) \right] \\ & -\lambda_t (K_{t+1} - I_t - (1-\delta) K_t) \end{aligned} \quad (39)$$

- First order conditions:

$$\frac{\partial V}{\partial K_{t+1}} : \frac{1}{(1+R_{t+1})^{t+1}} \left[F_K(K_{t+1}, L_{t+1}) - C_K(I_{t+1}, K_{t+1}) \right] + \lambda_{t+1}(1-\delta) - \lambda_t = 0 \tag{40}$$

$$\frac{\partial V}{\partial I_t} : -\frac{1 + C_I(I_t, K_t)}{(1+R_t)^t} + \lambda_t = 0 \tag{41}$$

$$\frac{\partial V}{\partial L_t} : \frac{F_L(K_t, L_t) - W_t}{(1+R_t)^t} = 0 \tag{42}$$

$$\frac{\partial V}{\partial \lambda_t} : -K_{t+1} + I_t + (1-\delta)K_t = 0 \tag{43}$$

- From the first order condition (41) we obtain the Lagrange multiplier for period t:

$$\lambda_t = \frac{1 + C_I(I_t, K_t)}{(1 + R_t)^t} \tag{44}$$

and for period $t+1$:

$$\lambda_{t+1} = \frac{1 + C_I(I_{t+1}, K_{t+1})}{(1 + R_{t+1})^{t+1}} \tag{45}$$

- Given that the real interest rate (the discount factor) is a exogenous variable, we assume that $R_{t+1} = R_t$. By substituting the Lagrange multiplier in the first order condition (40), we have:

$$F_K(K_{t+1}, L_{t+1}) - C_K(I_{t+1}, K_{t+1}) = \\ (1 + R_t)\left[1 + C_I(I_t, K_t)\right] - \left[1 + C_I(I_{t+1}, K_{t+1})\right](1 - \delta) \quad (46)$$

  where the value of the marginal product of capital is equal to the cost of use (this is the investment decision by the firm).

- Finally, from the first order condition (42) we obtain the condition that equals labor marginal productivity with the salary:

$$F_L(K_t, L_t) = W_t \quad (47)$$

11. The Tobin's Q model

- Next, we define the variable q. The ratio q is defined as:

$$q_t = \lambda_t \left(1 + R_t\right)^t \tag{48}$$

Therefore, the shadow price of capital can be defined as:

$$\lambda_t = \frac{q_t}{(1 + R_t)^t} \tag{49}$$

and using the definition for the Lagrange multiplier yields,

$$q_t = 1 + C_I(I_t, K_t) \tag{50}$$

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- By substituting in the equilibrium condition for the capital stock (46), we obtain:

$$F_K(K_{t+1}, L_{t+1}) - C_K(I_{t+1}, K_{t+1}) = (1 + R_t)\, q_t - q_{t+1}(1 - \delta) \quad (51)$$

From the above expression we obtain the following dinamic equation for the marginal ratio q:

$$q_{t+1} = \frac{(1 + R_t)q_t - F_K(K_{t+1}, L_{t+1}) + C_K(I_{t+1}, K_{t+1})}{1 - \delta} \quad (52)$$

Without adjustment costs, $C_K(I_{t+1}, K_{t+1}) = 0$, the dynamic eaution for the marginal ratio q should be:

$$q_{t+1} = \frac{(1 + R_t)q_t - F_K(K_{t+1}, L_{t+1})}{1 - \delta} \quad (53)$$

- In sum, the Tobin's Q model can be defined as a two-equation dynamic system for the capital stock and for the ratio q, given by:

$$q_t = 1 + C_I(I_t, K_t) \tag{54}$$
$$(1-\delta)q_{t+1} + F_K(K_{t+1}, L_{t+1}) = (1+R_t)q_t + C_K(I_{t+1}, K_{t+1}) \tag{55}$$

plust a static equilibrium condition for labor. By assuming that labor is constant, the Tobin's Q model de Tobin lo podemos resolver en términos del stock de capital y la ratio q. El stock de capital es una variable de estado, que viene predeterminada por las decisiones tomadas en el periodo anterior, mientras que la ratio q es una variable flexible, que está sujeta a cambios en las expectativas, y que se ajusta de forma instantánea ante perturbaciones.

- Example:
- Labor is normalized to 1 ($L_t = 1$). Production function is given by:

$$Y_t = F(K_t) = K_t^{\alpha} \tag{56}$$

where $0 < \alpha < 1$.

- Adjustment cost function:

$$C(I_t, K_t) = \frac{\phi}{2}\left(\frac{I_t - \delta K_t}{K_t}\right)^2 K_t \tag{57}$$

depending on both capital stock and investment and where $\phi > 0$ is a parameter governing the elasticity of investment with respect to the ratio q.

- In this particular case, we have that:

$$C_I(I_t, K_t) = \phi \left(\frac{I_t - \delta K_t}{K_t} \right) \tag{58}$$

$$C_K(I_t, K_t) = \frac{\phi}{2} \left(\frac{I_t - \delta K_t}{K_t} \right)^2 - \phi \left(\frac{I_t - \delta K_t}{K_t} \right) \frac{I_t}{K_t} \tag{59}$$

- By substituting in the model solution:

$$q_t = 1 + \phi \left(\frac{I_t - \delta K_t}{K_t} \right) \tag{60}$$

$$\begin{aligned} (1-\delta) q_{t+1} &= (1+R_t) q_t - \alpha K_{t+1}^{\alpha-1} + \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 \\ &\quad - \phi \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} \end{aligned} \tag{61}$$

- From the above expression, we can derive the two dynamic equations of the model. From the capital stock accumulation equation we have that:

$$I_t - \delta K_t = K_{t+1} - K_t \tag{62}$$

Defining $\Delta K_t = K_{t+1} - K_t$, and substituting in expression (60), we have the following expression:

$$q_t - 1 = \phi \frac{\Delta K_t}{K_t} \tag{63}$$

and operating yields:

$$\Delta K_t = (q_t - 1)\frac{K_t}{\phi} \tag{64}$$

- On the other hand, operating in expression (61) and defining $\Delta q_t = q_{t+1} - q_t$, we have that:

$$
\begin{aligned}
q_{t+1} \quad = \quad & \frac{(1+R_t)}{(1-\delta)} q_t - \frac{\alpha}{(1-\delta)} K_{t+1}^{\alpha-1} + \frac{\phi}{2(1-\delta)} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 \\
& - \frac{\phi}{(1-\delta)} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} \qquad (65)
\end{aligned}
$$

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- Adding and substracting q_t in the lhs of the above expression:

$$
\begin{aligned}
q_{t+1} - q_t + q_t \quad = \quad & \frac{(1+R_t)}{(1-\delta)} q_t - \frac{\alpha}{(1-\delta)} K_{t+1}^{\alpha-1} + \frac{\phi}{2(1-\delta)} \left(\frac{I_{t+1} - \delta K}{K_{t+1}} \right. \\
& - \frac{\phi}{(1-\delta)} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}}
\end{aligned}
$$

and operating yields:

$$
\Delta q_t = \frac{(R_t + \delta) q_t - \alpha K_{t+1}^{\alpha-1} + \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 - \phi \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}}}{(1-\delta)} \tag{67}
$$

- Steady state
- The dynamic equation for the capital stock (the investment rule), in steady state, is given by:

$$\Delta K_t = (\overline{q} - 1)\frac{\overline{K}}{\phi} = 0 \tag{68}$$

Given that in steady state the above equation must be zero, the steady state value for the ratio q is one:

$$\overline{q} = 1 \tag{69}$$

In fact, in steady state $C_I(\cdot) = 0$, and therefore $\overline{q} = 1$.

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- On the other hand, the dynamic equation for the ratio q, in steady state, is given by:

$$\Delta q_t = \frac{(R_t + \delta)\overline{q} - \alpha \overline{K}^{\alpha-1}}{(1-\delta)} = 0 \tag{70}$$

given that in steady state $\overline{I}_t = \delta \overline{K}_t$. Therefore,:

$$\alpha \overline{K}^{\alpha-1} = R_t + \delta \tag{71}$$

and thus the capital stock in steady state is given by:

$$\overline{K} = \left(\frac{R_t + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{72}$$