Lecture 1: Introduction to Dynamic Macroeconomics: Differential equations, difference equations and dynamic systems of equations

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Advanced Macroeconomics

1. Introduction to Dynamic Macroeconomics

- Microeconomics: Studies the individual behavior of economic agents or individual markets.
- Macroeconomics: Studies the economy at an aggregate level.

Inside Macroeconomics we have two focuses:

Economic growth (long run).

Business cycle fluctuations (short run).

1. Introduction to Dynamic Macroeconomics

- Key concepts: Microfoundation of Macroeconomics. Both "micro" and "macro" are consistent applications of the neoclassical theory.
- General Equilibrium:

  Economic agents are rationals, optimizers, given an endowment, preferences and technology.

  The economic agents decisions are mutually compatible.

1. Introduction to Dynamic Macroeconomics

Schools of economic thought:

1. **Classical (Ibn-Khaldun, Ricardo,...): No distinction between "micro" and "macro".**
2. **Ramsey (1928): Dynamic Optimization.**
3. Keynes (1936): The General Theory of Employment, Interest and Money: Breakdown between microeconomics and macroeconomics.
4. 1940-1970: Neoclassical synthesis (IS-LM).
5. **70s: Rational expectations**.
6. **1982: Real Business Cycle (RBC) theory (Dynamic Stochastic General Equilibrium, DSGE, model): Return to Ramsey (1928).**
7. Today: Extensions around the neoclassical DSGE model, introducing New Keynesian elements.

1. Introduction to Dynamic Macroeconomics

- Importance of time (t).
- Relationship among economic variables depends on the timing.
- Not all economic variables move at the same speed. For instance, output moves slowly, whereas other variables, as the exchange rate adjusts instantaneously.
- Continuous time versus discrete time.

DEFINITION OF MODEL:

- Concept of model: Street map, ...
- A macroeconomic model can be described as a system of differential equations. Those equations include a number of dynamic relationship among a set of endogenous variables $\mathbf{X}_t \in R^n$ and a set of exogenous variables $\mathbf{Z}_t \in R^m$.
- Similar to the model for a physic system, but with an important difference: The behavior of the economic variables depends on expectations about the future derived from human thought.

Key concept: Endogenous versus exogenous variables:

- Output.
- Martian attack to the Earth.
- Price level.
- Money.
- Interest rate.
- An earthquake.
- Exchange rate.
- Capital stock.
- Tax rate.

1. Differential equations, difference equations, and dynamic systems of equations

General model economy:

$$\mathbf{X}_t = E_t \left[F(\mathbf{X}_{t+1}, \mathbf{Z}_t, \mathbf{u}_t) \right]$$

$$\mathbf{Z}_t = G(\mathbf{Z}_{t-1}, \mathbf{v}_t)$$

where $\mathbf{X}_t$ is a vector of endogenous variables, $\mathbf{Z}_t$ is a vector of exogenous variables, E_t is the expectation operator, and $\mathbf{u}_t$ and $\mathbf{v}_t$ are i.i.d. random shocks.

F : Economic Theory.
G: Economic Policy.

- Solution: A sequence of probability distributions.

1. Differential equations, difference equations, and dynamic systems of equations

Simplifying the general model.

- First step: We eliminate the economic policy rules ($G = 0$):

$$\mathbf{X}_t = E_t\left[F(\mathbf{X}_{t+1}, \mathbf{Z}_t, \mathbf{u}_t)\right]$$

- Second step: We eliminate uncertainty ($u_t = 0$):

$$\mathbf{X}_t = \left[F(\mathbf{X}_{t+1}, \mathbf{Z}_t)\right]$$

1. Differential equations, difference equations, and dynamic systems of equations

Continuous time: The above model can we written as a system of differential equations:

$$\dot{\mathbf{X}}_t = F(\mathbf{X}_t, \mathbf{Z}_t) \quad (1)$$

where

$$\dot{\mathbf{X}}_t = \frac{d\mathbf{X}_t}{dt} \quad (2)$$

1. Differential equations, difference equations, and dynamic systems of equations

It is easier to work with logarithms:

$$\mathbf{x}_t = \ln \mathbf{X}_t \tag{3}$$

This way the time derivative is equivalent to the growth rate of the variable:

$$\dot{x}_t = \frac{d \ln X_t}{dt} = \frac{\frac{dX_t}{dt}}{X_t} = \frac{\dot{X}_t}{X_t} \tag{4}$$

1. Differential equations, difference equations, and dynamic systems of equations

Discrete time: The above model can we written as a system of differential equations:

$$\Delta \mathbf{X}_t = F(\mathbf{X}_t, \mathbf{Z}_t) \tag{5}$$

where

$$\Delta \mathbf{X}_t = \mathbf{X}_{t+1} - \mathbf{X}_t \tag{6}$$

1. Differential equations, difference equations, and dynamic systems of equations

It is easier to work with logarithms:

$$x_t = \ln X_t \tag{7}$$

This way the time derivative is equivalent to the growth rate of the variable:

$$\begin{aligned} \Delta x_t &= x_{t+1} - x_t = \ln X_{t+1} - \ln X_t = \ln \left(\frac{X_{t+1}}{X_t} \right) && (8) \\ &= \ln \left(1 + \frac{X_{t+1} - X_t}{X_t} \right) \simeq \frac{X_{t+1} - X_t}{X_t} && (9) \end{aligned}$$

1. Differential equations, difference equations, and dynamic systems of equations

Definition of Equilibrium: **Steady State**:

$$\bar{\mathbf{x}} \Longrightarrow \dot{\mathbf{x}}_t = f(\mathbf{x}_t, \mathbf{z}_t) = 0 \Longrightarrow f(\bar{\mathbf{x}}, \mathbf{z}_t) = 0 \tag{10}$$

Looking for a vector of zeros of dimension n. The differential equations system can we written as:

$$\dot{\mathbf{x}}_t = A\mathbf{x}_t + B\mathbf{z}_t \tag{11}$$

where A is a matrix $n \times n$, B is a matrix $n \times m$ and $\mathbf{z}_t$ is a vector of endogenous variables $m \times 1$.

In order to do graphical representations, we will work with system of two differential equations. Therefore $n = 2$.

$$\begin{bmatrix} \dot{x}_{1,t} \\ \dot{x}_{2,t} \end{bmatrix} = A \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + Bz_t \tag{12}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{13}$$

1. Differential equations, difference equations, and dynamic systems of equations

Computing the Steady State:

$$\begin{bmatrix} \dot{x}_{1,t} \\ \dot{x}_{2,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow A \begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} = -Bz_t \tag{14}$$

Or

$$\begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} = -A^{-1}Bz_t \tag{15}$$

Stability:

$$Det\left[A - \lambda I\right] = 0 \tag{16}$$

$$Det\left[\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right] = 0 \tag{17}$$

Solution:

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \tag{18}$$

Eigenvalues:

$$\lambda_{1,2} = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2} \quad (19)$$

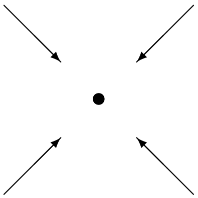
Three alternative solutions. In continuous time:

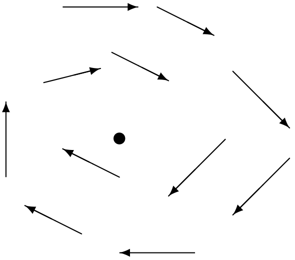
Eigenvalues	Case 1	Case 2	Case 3
λ_1	<0	<0	>0
λ_2	<0	>0	>0

1. Differential equations, difference equations, and dynamic systems of equations

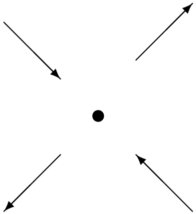
- Case 1 ($\lambda_1 < 0, \lambda_2 < 0$): Global stability. All dynamic paths tend to the Steady State.
- Case 2 ($\lambda_1 < 0, \lambda_2 > 0$): Saddle point. Some dynamic trajectories tend to the Steady State but other trajectories move away from the Steady State.
- Case 3 ($\lambda_1 > 0, \lambda_2 > 0$): Global instability. All trajectories move away from the Steady State.

Case 1: $(\lambda_1 < 0, \lambda_2 < 0)$



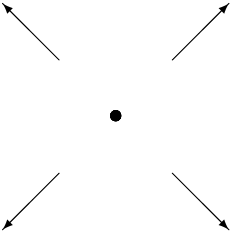


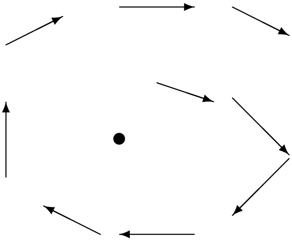
Case 2 ($\lambda_1 < 0, \lambda_2 > 0$)



1. Differential equations, difference equations, and dynamic systems of equations

Case 3: $(\lambda_1 > 0, \lambda_2 > 0)$





1. Differential equations, difference equations, and dynamic systems of equations

Eigenvalues:

$$\lambda_{1,2} = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2} \quad (20)$$

Three alternative solutions. In discrete time:

Eigenvalues (Modulus)	Case 1	Case 2	Case 3
$\lambda_1 + 1$	<1	<1	>1
$\lambda_2 + 1$	<1	>1	>1