MIXED WEAK TYPE INEQUALITIES FOR ONE-SIDED OPERATORS

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ABSTRACT. We discuss mixed weak type inequalities in weighted spaces for one-sided operators. In particular, we prove that if $T_c f(x) = (x-c)^{-1}\int_c^x f(y)\,dy$, $x > c$, is the Hardy averaging operator, $u \in A_1^-$ (one-sided Muckenhoupt A_1 class) and $v \in A_1^+$ (the another one-sided Muckenhoupt A_1 class) then there exists a constant C such that $\sup_{c\in\mathbb{R}} \int_{\{x:|T_c f(x)|>v(x)\}} uv \le C\int_{\mathbb{R}} |f|u$.

1. Introduction

Let T be a sublinear operator defined on measurable functions on $\mathbb{R}^n$, that is,

$$|T(f+g)| \le |Tf| + |Tg|, \quad |T(\lambda f)| = |\lambda||Tf|,$$

for all scalars λ and all measurable functions f. A mixed weak type (p,p) inequality for T is an inequality of the form

$$\int_{\{x:|Tf(x)|>v(x)\}} u(x)v(x)\,dx \le C\int |f(x)|^p w(x)\,dx, \tag{1.1}$$

where v, u and w are nonnegative measurable functions and C is independent of f. On one hand, this inequality contains the weighted weak type (p,p) inequality, since if $v \equiv 1$ and we take the functions f/λ, $\lambda > 0$, the above inequality becomes

$$\int_{\{x:|Tf(x)|>\lambda\}} u(x)\,dx \le \frac{C}{\lambda^p}\int |f(x)|^p w(x)\,dx, \tag{1.2}$$

that is, the weighted weak type (p,p) inequality for the operator T with respect to the weights u and w. On the other hand, mixed weak type inequalities are related to the two weighted norm inequalities [2] and, probably, that is the reason why they are more difficult to handle than the corresponding weak type inequalities.

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Let M be the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q:x\in Q} \frac{1}{|Q|}\int_Q |f|,$$

where the supremum is taken over all cubes with sides parallel to the axis such that $x \in Q$. It is known that the weighted weak type $(1,1)$ inequality

$$\int_{\{x:Mf(x)>\lambda\}} u(x)\,dx \le \frac{C}{\lambda}\int |f|(x)u(x)\,dx$$

holds if and only if the weight u satisfies the A_1 condition ($u \in A_1$), that is, there exists $C > 0$ such that

$$Mu(x) \le Cu(x) \quad \text{a.e.}$$

Andersen and Muckenhoupt [2] proved the mixed weak type $(1,1)$ inequality

$$\int_{\{x:Mf(x)>|x|^{-d}\}} |x|^{-d}u(x)\,dx \le C\int |f|(x)u(x)\,dx, \tag{1.3}$$

under the assumptions $n = 1$, $d \neq 1$ and $u \in A_1$. The same inequality was established for the Hilbert transform [2] and it was extended to singular integral operators in $\mathbb{R}^n$ [5]. Sawyer [7] proved that the mixed inequality holds for some general non-power weights v. More precisely, he established that if $n = 1$, $u \in A_1$ and $v \in A_1$ then

$$\int_{\{x:Mf(x)>v(x)\}} u(x)v(x)\,dx \le C\int |f|(x)u(x)\,dx. \tag{1.4}$$

The problem for the Hilbert transform was left open in that paper. Recently, the last inequality was proved [3] in $\mathbb{R}^n$ not only for M but also and for singular integrals including the Hilbert transform.

This paper is devoted to the study of mixed weak type $(1,1)$ inequalities for one-sided operators. In the real line, the one-sided Hardy-Littlewood maximal operators M^- and M^+ are defined by

$$M^-f(x) = \sup_{h>0}\frac{1}{h}\int_{x-h}^{x} |f(x)|\,dx \quad \text{and} \quad M^+f(x) = \sup_{h>0}\frac{1}{h}\int_{x}^{x+h} |f(x)|\,dx.$$

Weighted inequalities for M^- and M^+ were studied first in [8] (see also [6]). It was established [6] that the weighted weak type $(1,1)$ inequality

$$\int_{\{x:M^-f(x)>\lambda\}} u(x)\,dx \le \frac{C}{\lambda}\int |f|(x)u(x)\,dx$$

holds if and only if the weight u satisfies the A_1^- condition, that is, there exists $C > 0$ such that

$$M^+u(x) \le Cu(x) \quad \text{a.e.}$$

The analogous result hold for M^+ and $u \in A_1^+$ which means $M^-u(x) \leq Cu(x)$ almost everywhere. Arguing as in [7] we conjecture that the mixed weak type $(1,1)$ inequality

$$\int_{\{x:M^- f(x)>v(x)\}} u(x)v(x)\,dx \leq C \int |f|(x)u(x)\,dx \tag{1.5}$$

holds, under the assumptions $u \in A_1^-$ and $v \in A_1^+$. In other words, the conjecture says that the mixed weak type $(1,1)$ inequality for M^- holds if M^- is of weak type $(1,1)$ with respect to $u(x)\,dx$ and M^+ (the "adjoint"of M^-) is of weak type $(1,1)$ with respect to $v(x)\,dx$. So far, we have not been able to prove it. However we have found a proof of that inequality with M^- replaced by the Hardy averaging operators

$$T_c f(x) = \begin{cases} \frac{1}{x-c}\int_c^x f(y)\,dy, & \text{if } x > c; \\ 0, & \text{if } x \leq c., \end{cases}$$

where c is any fixed real number. Clearly, the operators T_c are smaller than M^- and they are closely related to M^- since

$$M^- f = \sup_{c\in\mathbb{R}} T_c|f|.$$

For these operators we prove (see Corollary 2.8) that if $u \in A_1^-$ and $v \in A_1^+$ then there exists a constant C such that

$$\sup_{c\in\mathbb{R}} \int_{\{x:|T_c f(x)|>v(x)\}} uv \leq C \int_{\mathbb{R}} |f|u$$

for all measurable functions f. We obtain this result as a consequence of Theorem 2.6, where we state that the mixed weak type inequality holds for T_c if T_c is of weak type $(1,1)$ with respect to $u(x)\,dx$ and the formal adjoint T_c^* is of weak type $(1,1)$ with respect to $v(x)\,dx$. In the next section we state and prove our results.

We shall use standard notations. In particular, if E is a measurable set $E \subset \mathbb{R}$ then $|E|$ is the lebesgue measure of E.

2. Mixed weak type inequalities for Hardy operators

We shall establish our results for the operators T_c for any number c but the proofs will be given in the case $c = 0$, since the general case is proved in a completely similar way. In what follows, the Hardy operator T_0 will be denoted by T.

We start with a characterization of the mixed weak type inequality for T_c. The next theorem is essentially contained in [5] although in that paper a more general setting is considered and the Hardy operator is the one in $\mathbb{R}^n$ given by

$$Hf(x) = \frac{1}{|x|^n} \int_{B(0,|x|)} f(y)\,dy,$$

where $B(0,|x|)$ stands for the euclidian ball of center 0 and radius $|x|$. Observe that for $n=1$ the operator H is the two-sided operator

$$Hf(x)=\frac{1}{|x|}\int_{-|x|}^{|x|}f(y)\,dy.$$

We include the proof for completeness.

Theorem 2.1. *Let u and v be nonnegative measurable functions defined on $\mathbb{R}$. Let $c\in\mathbb{R}$. The following statements are equivalent.*

(a) There exists a constant C such that

$$\int_{\{x:|T_cf(x)|>v(x)\}}uv\le C\int_{\mathbb{R}}|f|u$$

for all measurable functions.

(b) There exists a constant $\widetilde{C}$ such that for all $a>c$

$$\sup_{\lambda>0}\lambda\int_{\{x>a:\frac{1}{x-c}>\lambda v(x)\}}uv\le\widetilde{C}u(x)\quad\textit{for a.e. } x\in(c,a).$$

Further, if C and $\widetilde{C}$ are the best constants in (a) and (b), respectively, then $\widetilde{C}\le C\le 4\widetilde{C}$.

Proof. As we said above we work with $c=0$.

(a)⇒(b). Let us fix $a>0$. Let E be any measurable subset of $(0,a)$ and consider $f=\frac{1}{|E|}\chi_E$. If $x>a$ then

$$Tf(x)=\frac{1}{x}$$

Therefore

$$\int_{\{x>a:\frac{1}{x}>v(x)\}}uv\le\int_{\{x:Tf(x)>v(x)\}}uv\le\frac{C}{|E|}\int_E u,$$

where the last inequality follows from statement *(a)*. Since E is any measurable subset of $(0,a)$, we obtain

$$\int_{\{x>a:\frac{1}{x}>v(x)\}}uv\le C\operatorname{ess\,inf}\{u(x):x\in(0,a)\},$$

which is *(b)* for $\lambda=1$. The inequality for all λ follows in the same way since *(a)* holds for the pairs of functions $(u,\lambda v)$ for all $\lambda>0$ with the same constant.

(b)⇒(a). We may assume without loss of generality that f is integrable, $f\ge0$ and $\int_0^a f>0$ for all $a>0$. Let $\{x_n\}_n$ be the decreasing sequence defined by $x_0=+\infty$ and

$$\int_0^{x_{n+1}}f=\int_{x_{n+1}}^{x_n}f.$$

It is clear that $\lim_{n\to\infty} x_n = 0$. If $x \in [x_{n+1}, x_n)$ then

$$Tf(x) \le \frac{1}{x}\int_0^{x_n} f = \frac{4}{x}\int_{x_{n+2}}^{x_{n+1}} f.$$

Therefore

$$\{x : Tf(x) > v(x)\} \subset \bigcup_{n=1}^{\infty}\left\{x \in [x_{n+1}, x_n) : \frac{1}{x} > \frac{v(x)}{4\int_{x_{n+2}}^{x_{n+1}} f}\right\}.$$

If $\beta_n = \operatorname{ess\,inf}\{u(x) : x \in (0, x_{n+1})\}$ we have by *(b)*

$$\int_{\{x:Tf(x)>v(x)\}} uv \le 4\widetilde{C}\sum_{n=1}^{\infty}\beta_n\int_{x_{n+2}}^{x_{n+1}} f$$

$$\le 4\widetilde{C}\sum_{n=1}^{\infty}\int_{x_{n+2}}^{x_{n+1}} fu \le 4\widetilde{C}\int_0^{\infty} fu.$$

□

Observe that taking $v = 1$ in the theorem we obtain a characterization of the weights u such that T applies $L^1(u)$ into weak-$L^1(u)$. We state it as a corollary.

Corollary 2.2. *Let u be a nonnegative measurable functions defined on $\mathbb{R}$. Let $c \in \mathbb{R}$. The following statements are equivalent.*

(a) There exists a constant C such that

$$\int_{\{x:|T_cf(x)|>\lambda\}} u \le \frac{C}{\lambda}\int_{\mathbb{R}} |f|u$$

for all measurable functions.

(b) u satisfies $A_1(T_c)$, that is, there exists $\widetilde{C} > 0$ such that for all $a > c$

$$\sup_{y>a}\frac{1}{y-c}\int_a^y u \le \widetilde{C}u(x) \quad \text{for a.e. } x \in (c,a)., \tag{2.3}$$

Further, if C and $\widetilde{C}$ are the best constants in (a) and (b), respectively, then $\widetilde{C} \le C \le 4\widetilde{C}$.

The proof is direct from the theorem and the equality $\{x > a : \frac{1}{x-c} > \lambda\} = (a, c + \frac{1}{\lambda})$.

Remark 2.4. *Notice that Andersen and Muckenhoupt* [2] *proved that statement (a) holds if and only if there exist $\alpha > 0$ and $C(\alpha)$ such that for all $a > c$*

$$\int_a^{\infty}\left(\frac{a}{t-c}\right)^{\alpha}\frac{u(t)}{t-c}\,dt \le C(\alpha)u(x) \quad \text{for a.e. } x \in (c,a).$$

It is easy to see directly that this condition and $A_1(T_c)$ are equivalent.

It can be proved also that the formal adjoint operator T_c^* defined by

$$T_c^* f(x) = \begin{cases} \int_x^\infty \frac{f(t)}{t-c}\,dt, & \text{if } x > c; \\ 0, & \text{if } x \le c, \end{cases}$$

is of weak type $(1,1)$ with respect to the measure $v(x)dx$ if and only if $v \in A_1(T_c^*)$, that is, there exists $C > 0$ such that

$$\frac{1}{x-c}\int_c^x v \le Cv(x) \quad \text{for almost every } x > c. \tag{2.5}$$

The proof is similar to the one for T_c and we omit it (alternatively, the result can be obtained from the theorems in [2]). With the help of these conditions we can establish the mixed weak type inequality for T_c for a wide class of weights.

Theorem 2.6. *Let u and v be nonnegative measurable functions defined on $\mathbb{R}$. Let $c \in \mathbb{R}$. Assume that there exists $\varepsilon > 0$ such that $u^{1+\varepsilon} \in A_1(T_c)$ and $v^{1+\varepsilon} \in A_1(T_c^*)$, i.e., there is a constant $C > 0$ such that for all $a > c$*

$$\sup_{y>a} \frac{1}{y-c}\int_a^y u^{1+\varepsilon} \le Cu^{1+\varepsilon}(x) \quad \textit{for a.e. } x \in (c,a),$$

and

$$\frac{1}{x-c}\int_c^x v^{1+\varepsilon} \le Cv^{1+\varepsilon}(x) \quad \textit{for almost every } x > c. \tag{2.7}$$

Then there exists a constant C such that

$$\int_{\{x:|T_c f(x)|>v(x)\}} uv \le C\int_{\mathbb{R}} |f|u$$

for all measurable functions.

As a corollary we obtain our result for weights in the one-sided Muckenhoupt classes.

Corollary 2.8. *Let u and v nonnegative measurable functions defined on $\mathbb{R}$. Assume that $u \in A_1^-$ and $v \in A_1^+$. Then there exists a constant C such that*

$$\sup_{c\in\mathbb{R}} \int_{\{x:|T_c f(x)|>v(x)\}} uv \le C\int_{\mathbb{R}} |f|u$$

for all measurable functions.

The corollary follows from the theorem, the easy implications $u \in A_1^- \Rightarrow u \in A_1(T_c)$, $v \in A_1^+ \Rightarrow v \in A_1(T_c^*)$, and the well-known implications $u \in A_1^- \Rightarrow u^{1+\varepsilon} \in A_1^-$ and $v \in A_1^+ \Rightarrow v^{1+\varepsilon} \in A_1^+$ for some $\varepsilon > 0$ (see [8, 6]).

Proof of Theorem 2.6. We work with $c = 0$. By Theorem 2.1, we only have to prove that

$$\lambda \int_{\{x>a:\frac{1}{x}>\lambda v(x)\}} uv \le C \operatorname{ess\,inf}\{u(x) : x \in (0,a)\}$$

for all $a > 0$ and all $\lambda > 0$. Fix $\lambda > 0$ and $a > 0$ and set

$$E = \{x > a : \frac{1}{x} > \lambda v(x)\}.$$

We may assume that $|E| > 0$. Let us take any $z \in E$ such that

$$\frac{1}{z}\int_0^z v^{1+\varepsilon} \le Cv^{1+\varepsilon}(z). \tag{2.9}$$

We shall prove that

$$\lambda \int_{E\cap(a,z)} uv \le C\operatorname{ess\,inf}_{(0,a)} u.$$

Then letting z tend to the essential supremum of E we obtain the required inequality. Fix any number $\beta > 1$ and choose $b \in (0, a)$ such that b is a Lebesgue point of $u^{1+\varepsilon}$ and $u(b) \le \beta(\operatorname{ess\,inf}_{(0,a)} u)$. Now choose α such that $1 - \varepsilon < \alpha < \frac{1}{1+\varepsilon}$. Applying the definition of E and Hölder's inequality we obtain

$$\int_{E\cap(a,z)} uv \le \frac{1}{\lambda^\alpha}\int_{E\cap(a,z)} \frac{u(x)}{x^\alpha} v^{1-\alpha}(x)\,dx$$

$$\le \frac{1}{\lambda^\alpha}\left(\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} \left(\int_a^z v^{(1-\alpha)\frac{1+\varepsilon}{\varepsilon}}(x)\,dx\right)^{\frac{\varepsilon}{1+\varepsilon}}$$

$$\le \frac{1}{\lambda^\alpha}\left(\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} \left(\int_a^z v^{1+\varepsilon}(x)\,dx\right)^{\frac{1-\alpha}{1+\varepsilon}} (z-a)^{\frac{\varepsilon-1+\alpha}{1+\varepsilon}}$$

Using (2.9), $z - a \le z$ and $z \in E$ we obtain

$$\begin{aligned}\int_{E\cap(a,z)} uv &\le C\frac{z^{\frac{\varepsilon}{1+\varepsilon}}}{\lambda^\alpha}\left(\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} v^{1-\alpha}(z)\\ &\le C\frac{z^{\alpha-\frac{1}{1+\varepsilon}}}{\lambda}\left(\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}}\end{aligned} \tag{2.10}$$

To estimate the last integral we take $c \in (b, a)$ and $f = \chi_{(b,c)}$. It is clear that for $x > a$

$$Tf(x) = \frac{c-b}{x}.$$

Applying this equality

$$\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx = \frac{1}{(c-b)^{\alpha(1+\varepsilon)}}\int_a^z (Tf(x))^{\alpha(1+\varepsilon)} u^{1+\varepsilon}(x)\,dx \tag{2.11}$$

Since $u^{1+\varepsilon}$ satisfies (2.3) we have that T applies $L^1(u^{1+\varepsilon})$ into weak-$L^1(u^{1+\varepsilon})$. Therefore, by Kolmogorov's inequality (for instance, see [4])

$$\begin{aligned}\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx \\ \le \frac{C}{(c-b)^{\alpha(1+\varepsilon)}}\left(\int_a^z u^{1+\varepsilon}(x)\,dx\right)^{1-\alpha(1+\varepsilon)}\left(\int_b^c u^{1+\varepsilon}(x)\,dx\right)^{\alpha(1+\varepsilon)}.\end{aligned} \tag{2.12}$$

Applying again the assumption on u we have

$$\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx \le C\left(\operatorname{ess\,inf}_{(0,a)}u\right)^{(1+\varepsilon)(1-\alpha(1+\varepsilon))} z^{1-\alpha(1+\varepsilon)}\left(\frac{1}{c-b}\int_b^c u^{1+\varepsilon}\right)^{\alpha(1+\varepsilon)}.$$

Since c is any point in (b,a) and b is a Lebesgue point of $u^{1+\varepsilon}$, we get

$$\left(\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} \le C\left(\operatorname{ess\,inf}_{(0,a)}u\right)^{1-\alpha(1+\varepsilon)} z^{\frac{1}{1+\varepsilon}-\alpha}u^{\alpha(1+\varepsilon)}(b).$$

Now the property of b gives

$$\left(\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} \le C\left(\operatorname{ess\,inf}_{(0,a)}u\right) z^{\frac{1}{1+\varepsilon}-\alpha}\beta^{\alpha(1+\varepsilon)}.$$

Letting β tend to 1 we obtain

$$\left(\int_a^z \frac{u^{1+\varepsilon}(x)}{x^{\alpha(1+\varepsilon)}}\,dx\right)^{\frac{1}{1+\varepsilon}} \le C\left(\operatorname{ess\,inf}_{(0,a)}u\right) z^{\frac{1}{1+\varepsilon}-\alpha}.$$

This inequality together with (2.10) gives

$$\int_{E\cap(a,z)} uv \le \frac{C}{\lambda}\left(\operatorname{ess\,inf}_{(0,a)}u\right),$$

as we wished to prove.

□

Remark 2.13. *We point out that $v \in A_1(T_c^*)$ does not imply $v^{1+\varepsilon} \in A_1(T_c^*)$ for some $\varepsilon > 0$. We shall give an example because we have not found it in the literature.*

Example 2.1. Let $I_i = \left(2^i + \frac{1}{2^i}, 2^i + 1\right)$, for all natural number i, and $\Omega = \cup_{i=1}^{\infty} I_i$. Now, we define

$$w(x) = \chi_{\Omega^c}(x) + \sum_{i=1}^{\infty} \frac{\chi_{I_i}(x)}{(x-2^i)^2}dx.$$

We shall see that $w \in A_1(T_0^*)$ and $w^{1+\varepsilon} \notin A_1(T_0^*)$ for any $\varepsilon > 0$. Observe that $w \ge 1$. A simple computation gives

$$\int_{I_i} w^{1+\varepsilon} = \int_{I_i} \frac{dx}{(x-2^i)^{2(1+\varepsilon)}} = \frac{1}{1+2\varepsilon}\left(2^{i(1+2\varepsilon)} - 1\right) \sim 2^{i(1+2\varepsilon)} \tag{2.14}$$

We now show that w satisfies $A_1(T_0^*)$. Let $x > 2$ (since $w(y) = 1$ for $y \leq 2$, for $x \leq 2$ it is easy), we choose a natural number N such that $2^N < x \leq 2^{N+1}$. It is enough to see that $\frac{1}{x}\int_0^x w$ is uniformly bounded, because $w(x) \geq 1$ for every x. We have that

$$\frac{1}{x}\int_0^x w \leq \frac{1}{2^N}\int_{\Omega^c\cap(0,2^{N+1})} w + \frac{1}{2^N}\int_{\Omega\cap(0,2^{N+1})} w.$$

Since $w(x) = 1$ for $x \in \Omega^c$, the first summand is bounded by 2 and the second one is bounded by

$$\frac{1}{2^N}\sum_{i=1}^{N}\int_{I_i} w \leq \frac{1}{2^N}\sum_{i=1}^{N} 2^i \leq 2.$$

Now, we will see that for any $\varepsilon > 0$, $w^{1+\varepsilon}$ does not satisfy $A_1(T_0^*)$. Fix $\varepsilon > 0$. If $x = 2^N + s$ (with $1 \leq s \leq 2$) we have that $w(x) = 1$, and by (2.14) we have

$$\frac{1}{x}\int_0^x w^{1+\varepsilon} > \frac{C}{2^N}\sum_{i=1}^{N}\int_{I_i} w^{1+\varepsilon} \geq \frac{C}{2^N}\sum_{i=1}^{N} 2^{i(1+2\varepsilon)} \geq \ C\ 2^{2N\varepsilon},$$

which shows that $w^{1+\varepsilon}$ does not satisfy $A_1(T_c^*)$.

The same example shows that $u \in A_1(T_c)$ does not imply $u^{1+\varepsilon} \in A_1(T_c)$ for some $\varepsilon > 0$. Keeping in mind this example, it is clear that the assumptions in Theorem 2.6 are stronger than $u \in A_1(T_c)$ and $v \in A_1(T_c^*)$. It is an open problem to know whether the conclusions of the theorem hold under these weaker assumptions. However, the answer is affirmative in the particular case of decreasing weights.

Theorem 2.15. *Let $c \in \mathbb{R}$. Assume that u is a decreasing weight in (c,∞) and the weight $v \in A_1(T_c^*)$. Then there exists a constant C such that*

$$\int_{\{x:|T_c f(x)|>v(x)|\}} uv \leq C\int_{\mathbb{R}} |f|u$$

for all measurable functions.

Proof. Assume $c = 0$. As in the proof of Theorem 2.6, we only have to prove that

$$\lambda\int_{\{x>a:\frac{1}{x}>\lambda v(x)\}} uv \leq C\operatorname{ess\,inf}_{(0,a)} u \tag{2.16}$$

for all $a > 0$ and all $\lambda > 0$. Since $v \in A_1(T_c^*)$ we obtain that

$$\{x > a : \frac{1}{x} > \lambda v(x)\} \subset \{x > a : \frac{C}{\lambda} > \int_0^x v\} = E_\lambda.$$

Let $s_0 = \sup\{x : x \in E_\lambda\}$. We have that $E_\lambda \subset (a, s_0)$ and $\int_0^{s_0} v \leq \frac{C}{\lambda}$. Using that u is decreasing and we obtain

$$\begin{aligned}\lambda \int_{\{x>a:\frac{1}{x}>\lambda v(x)\}} uv &\leq \lambda(\operatorname{ess\,inf}_{(0,a)} u) \int_{E_\lambda} v \\ &\leq \lambda(\operatorname{ess\,inf}_{(0,a)} u) \int_0^{s_0} v \leq C \operatorname{ess\,inf}_{(0,a)} u.\end{aligned} \tag{2.17}$$

□

To finish the paper we show that for decreasing weights u, the natural condition A_1^+ on the weight v is sufficient to obtain the mixed weak type inequality for M^-.

Theorem 2.18. *Let u be decreasing in $\mathbb{R}$. Let $v \in A_1^+$. Then there exists $C > 0$ such that*

$$\int_{\{x:M^-f(x)>v(x)\}} uv \leq C \int_0^\infty |f|u$$

Proof. In fact, if $v \in A_1^+$ then

$$\big\{x : v(x) < M^- f(x)\big\} \subset \left\{x : M_v^-(fv^{-1})(x) > \frac{1}{C}\right\},$$

where

$$M_v^- g(x) = \sup_{h>0} \frac{\int_{x-h}^x |g|v}{\int_{x-h}^x v}$$

(M_v^+ is defined reversing the orientation in the real line). Now we recall [1, 6] that M_v^- applies $L^1(uv)$ into weak-$L^1(uv)$ if and only if $M_v^+ u \leq Cu$ almost everywhere. It is clear that u satisfies that condition because u decreases. Therefore,

$$\int_{\{x:M^-f(x)>v(x)\}} uv \leq \int_{\{x:M_v^-(fv^{-1})(x)>\frac{1}{C}\}} uv \leq C \int_{\mathbb{R}} |f|v^{-1}uv = C \int_{\mathbb{R}} |f|u,$$

as we wanted to prove. □

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