BLASCHKE PRODUCTS WITH ZEROS IN A STOLZ ANGLE

JOSÉ ÁNGEL PELÁEZ

Abstract. The derivative of a Blaschke product whose zeros lie in a Stolz domain belongs to all Hardy spaces with exponent smaller than one-half and this exponent cannot be improved even when the zeros belong to a single radius. For Bergman spaces and under the same assumption on the zeros, a striking phenomenon occurs: the derivative belongs to all such spaces with exponent less than three-halves, which is more than one may expect. Although we do not know whether or not this result is sharp, we solve the problem for interpolating Blaschke products: Indeed, if B is an interpolating Blaschke product whose zeros lie in a Stolz angle then $B' \in \cap_{0<p<1} H^p \subset \cap_{0<p<2} A^p$.

On the other hand, we prove that there exists an interpolating Blaschke product B such that $B' \notin \cup_{0<p} H^p$.

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1. Introduction and Main Results

If f is a function which is analytic in $\Delta = \{z : |z| < 1\}$ and $0 < r < 1$, we set

$$\begin{aligned} M_p(r,f) = & \quad \left(\tfrac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it})|^p\,dt\right)^{1/p}, && 0<p<\infty,\\ I_p(r,f) = & \quad M_p^p(r,f), && 0<p<\infty,\\ M_\infty(r,f) = & \quad \sup_{|z|=r}|f(z)|. \end{aligned}$$

For $0 < p \leq \infty$, the Hardy space H^p consists of all analytic functions f in the disc for which

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

The Nevanlinna characteristic $T(r,f)$ is defined by

$$T(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|f(re^{it})|\,dt, \quad 0 \leq r < 1.$$

The Nevanlinna class, denoted by N, is the space of those f analytic in Δ for which

$$\sup_{0\leq r<1} T(r,f) < \infty.$$

It is known that $H^p \subset N$, $0 < p \leq \infty$. We refer the reader to [9] for the theory of Hardy spaces and of the Nevanlinna class.

The Bergman space A^p $(0 < p < \infty)$ is the space of all analytic functions f in the Δ satisfying

$$\|f\|_{A^p} \stackrel{\text{def}}{=} \left(\int_\Delta |f(z)|^p\,dA(z)\right)^{1/p} < \infty,$$

where $dA(z)$ denotes the normalized Lebesgue measure in Δ. For the theory of these spaces, we refer the reader to [5], [16] and to the forthcoming text [10].

A sequence $\{a_k\}_{k=1}^{\infty}$ of points in the unit disc is said to be a *Blaschke sequence* if

$$\sum (1-|a_k|) < \infty.$$

The corresponding Blaschke product B is defined as

$$B(z) = \prod_k \frac{\overline{a}_k}{|a_k|} \frac{a_k - z}{1 - \overline{a}_k z}.$$

Such a product is analytic in Δ, bounded by one, and with radial limits of modulus one almost everywhere on the unit circle.

One of the central questions about Blaschke products is that of the membership of their derivatives in the classical function spaces such as Hardy spaces and Bergman spaces. This problem was studied in many papers as [3], [2], [17] and [21] in the 70's and 80's. More recently, mean growth of the derivative of Blaschke products has been investigated by Girela - González [13], Kutbi [18], and also by the author and Girela in [14].

Let us recall the following well known result of Privalov (see Theorem 3.11 of [9]).

THEOREM A. *Let f be an analytic function in Δ, f has a continuous extension to the closed unit disc $\overline{\Delta}$ whose boundary values are absolutely continuous on $\partial\Delta$ if and only if $f' \in H^1$.*

In particular,

$$f' \in H^1 \quad \Longrightarrow \quad f \in \mathcal{A},$$

where, as usual, $\mathcal{A}$ denotes the disc algebra, that is, the space of all functions f which are analytic in Δ and have a continuous extension to the closed unit disc $\overline{\Delta}$.

Since the boundary values of a Blaschke product have modulus 1 almost everywhere, it is clear that if B is an infinite Blaschke product then $B \notin \mathcal{A}$ and, hence, $B' \notin H^1$. Consequently if B is an infinite Blaschke product the best result, that we could obtain is the following

$$B' \in \cap_{0<p<1} H^p.$$

Nevertheless there exist many Blaschke products B which do not verify this condition. Indeed, Frostman in [12] proved that there exists a Blaschke product B such that $B' \notin N$ and, hence, $B' \notin \cup_{0<p<\infty} H^p$. The proof of this fact is difficult and non-constructive. In section 4 we are going to prove in a easier and constructive way the following two results:

Proposition 1. *The Blaschke product B with zeros $a_k = r_k e^{it_k}$, where*

$$r_k = 1 - \frac{1}{k \log^2 k}, \quad t_k = \frac{1}{k \log^2 k}, \quad k = 3, 4, \dots$$

verifies that $B' \notin \cup_{0<p} H^p$.

Theorem 2. *There exists an interpolating Blaschke product B such that $B' \notin \cup_{0<p} H^p$.*

We recall that a Blaschke product B is said to be an interpolating Blaschke product if its sequence of zeros $\{a_k\}$ is *uniformly separated* or *interpolating*, that is, if there exists a positive constant $\delta > 0$ such that

$$\prod_{j=1, j\neq k}^{\infty} \left| \frac{a_j - a_k}{1 - a_j a_k} \right| \geq \delta, \quad \text{for all } k.$$

The Schwarz-Pick lemma readily implies that the derivative of any Blaschke product belongs to $\cap_{0<p<1} A^p$. On the other hand, H. O. Kim proved in p. 176 of [17] the following result.

Theorem B. *If B is an infinite Blaschke product then $B' \notin A^2$.*

Consequently if B is an infinite Blaschke product the best result, that we could have is the following

$$B' \in \cap_{0<p<2} A^p.$$

However there exist a lot of Blaschke products which do not satisfy this condition. Indeed, Rudin [22] showed that there exists a Blaschke product whose derivative does not belong to A^1 and Piranian [20] gave a more explicit example.

We are mainly interested in Blaschke products B whose sequence of zeros lie in some Stolz angle.

Given $\xi \in \partial\Delta$ and $\sigma \in (1, \infty)$ we set

$$\Omega_\sigma(\xi) = \left\{ z \in \Delta : |1 - \overline{\xi} z| \leq \sigma(1 - |z|) \right\}.$$

The domains $\Omega_\sigma(\xi)$ $(1 < \sigma < \infty)$ are called Stolz angles with vertex at ξ. The domain $\Omega_\sigma(1)$ will be simply denoted by Ω_σ.

Ahern and Clark proved in [3] the following result.

Theorem C. *Let be B a Blaschke product with zeros in a Stolz angle, then $B' \in \cap_{0<p<1/2} H^p$ and the exponent one-half is sharp.*

A different and very short proof of this result is presented in [15]. The positive part is proved in a perhaps unexpected way, by generalizing a known exercise in [23]. For the negative one, we show that the Blaschke product B with zeros $a_k = 1 - 1/(k \log^2 k)$ $(k \geq 2)$ has the property that $B' \notin H^{1/2}$. So the exponent $1/2$ cannot be improved even when the zeros belong to a single radius. The proof is based on the following lemma of Ahern and Clark [3]:

Lemma D. *If a Blaschke product B has zeros $a_k = r_k e^{it_k}$ then $B' \in H^p$ if and only if the function f defined on the unit circle by*

$$f(t) = \sum_{k=1}^{\infty} \frac{1-|a_k|}{(1-|a_k|)^2 + (t-t_k)^2}$$

belongs to $L^p(0, 2\pi)$.

By a theorem of Hardy and Littlewood, $H^p \subset A^{2p}$ and the exponent $2p$ cannot be improved (see Theorem 5.6 of [9], or [25] for a simple proof). It is, thus, natural to ask whether the exponent one in the inclusion $B' \in \cap_{0<p<1} A^p$ is sharp if the zeros of B converge nontangentially to a point in the unit circle. A rather surprising phenomenon occurs here:

Theorem E. *If the zeros of a Blaschke product B all lie in some Stolz angle, then $B' \in \cap_{0<p<3/2} A^p$.*

This result which can be deduced from Theorem 6.1 of Ahern's paper [1] is proved in [15] using Theorem C and the following proposition.

Proposition F. *Let φ be an arbitrary analytic function in Δ for which $\varphi(\Delta) \subset \Delta$. If $\varphi\prime \in H^p$ then $\varphi\prime \in A^{p+1-\varepsilon}$ for any positive and sufficiently small ε.*

In section 2, we give another completely different proof of Theorem E by following in part an idea from a paper by Vinogradov [24]. We do not know whether or not the exponent $3/2$ is sharp. However Theorem 4 of [15] solves the problem for interpolating Blaschke products.

Theorem G. *If B is an interpolating Blaschke product whose sequence of zeros $\{a_k\}$ is contained in a Stolz angle, then*

(1)
$$\sum_{k=1}^{\infty}(1-|a_k|)^p < \infty, \quad \textit{for all } p>0,$$

(2)
$$B' \in \cap_{0<q<1} H^q \subset \cap_{0<q<2} A^q.$$

Consequently, if there is a Blaschke product with zeros in a Stolz angle which does not belong to $A^{3/2}$ then it is not an interpolating Blaschke product.

2. A new proof of Theorem E

The following observation can be found in [24]: if $z \in \Delta$ and $\lambda \in \Omega_\sigma$, then

$$\begin{aligned} \frac{|1-\overline{\lambda}z|}{|1-|\lambda|z|} &= \frac{|(1-|\lambda|z)+z(1-\overline{\lambda})-z(1-|\lambda|)|}{|1-|\lambda|z|} \\ &\leq 1+\frac{|1-\lambda|+(1-|\lambda|)}{|1-|\lambda|z|} \\ &\leq 1+\frac{(1+\sigma)(1-|\lambda|)}{1-|\lambda|} \\ &= 2+\sigma\,. \end{aligned}$$

By symmetry, a similar lower bound can be obtained, whence:

(1)
$$\frac{1}{2+\sigma} \leq \frac{|1-\overline{\lambda}z|}{|1-|\lambda|z|} \leq 2+\sigma\,, \quad \text{whenever} \quad z\in\Delta,\ \lambda\in\Omega_\sigma\,.$$

These inequalities will be the key for many calculations in what follows.

An analytic function f belongs to A^p if and only if $|f|^p$ is area integrable in some annulus $A_R = \{z \in \Delta : R \leq |z| < 1\}$. The following fact will be useful for integration over such annuli. It appears on p. 3814 of [24].

Lemma 3. *Given a constant $R \in (0,1)$, there exists a $C \in (0,1)$ such that*

$$C\big[(1-r)+(1-\varrho)+|t|\big] \le |1-\varrho r e^{it}| \le (1-r)+(1-\varrho)+|t|\,, \tag{2}$$

whenever $r, \varrho \in [R,1)$, $t \in [-\pi,\pi]$.

Proof. On the one hand, the elementary inequalities $1+\varrho r \ge \varrho + r$ and $\cos t \ge 1 - t^2/2$ imply

$$\begin{aligned}
\big[(1-\varrho)+(1-r)+|t|\big]^2 &\ge \big[(1-\varrho r)+|t|\big]^2 \\
&\ge (1-\varrho r)^2 + t^2 \\
&\ge (1-\varrho r)^2 + \varrho r t^2 \\
&\ge (1-\varrho r)^2 + 2\varrho r(1-\cos t) \\
&= |1-\varrho r e^{it}|^2\,.
\end{aligned}$$

On the other hand, when $R \le r,\ \varrho < 1$ we have

$$1-\varrho r = \frac{(1-\varrho)(1+r)+(1+\varrho)(1-r)}{2} \ge \frac{1+R}{2}\big[(1-\varrho)+(1-r)\big]\,.$$

By the well known Jordan inequality:

$$\sin(t/2) \ge t/\pi \qquad \text{when } 0 \le t \le \pi\,,$$

we deduce that for all $t \in [-\pi,\pi]$ and $r,\ \varrho \in [R,1)$,

$$\begin{aligned}
|1-\varrho r e^{it}|^2 &= (1-\varrho r)^2 + 4\varrho r \sin^2(t/2) \\
&\ge \left(\frac{1+R}{2}\right)^2 [(1-\varrho)+(1-r)]^2 + \frac{4R^2}{\pi^2}t^2 \\
&\ge \alpha(R)\cdot\big([(1-\varrho)+(1-r)]^2 + t^2\big) \\
&\ge \frac{1}{2}\alpha(R)\cdot[(1-\varrho)+(1-r)+|t|]^2\,,
\end{aligned}$$

where

$$\alpha(R) = \min\left(\left(\frac{1+R}{2}\right)^2, \frac{4R^2}{\pi^2}\right).$$

Hence the lemma is proved with $C = \alpha(R)/2$. $\square$

Proof of Theorem E. The A^p spaces decrease as the exponent increases, so without loss of generality we may assume that $p \geq 1$. In what follows we are guided by an idea similar to that of Theorem 2.9 of [24].

Let us assume without loss of generality that $B(0) \neq 0$ and that all the zeros a_k of B lie in the Stolz angle Ω_σ for a certain $\sigma \geq 1$. Let us agree to write

$$b_k(z) = \frac{|a_k|}{a_k}\frac{a_k - z}{1 - \overline{a}_k z}, \quad B(z) = \prod_{k=1}^{\infty} b_k(z), \quad B_k(z) = \frac{B(z)}{b_k(z)}. \tag{3}$$

Take $R \in (0,1)$ such that $|a_k| \geq R$ for all k. The elementary inequality $\log(1-x) \leq -x$, $0 < x < 1$, yields

$$\log|b_k(z)| = \frac{1}{2}\log\big(1 - (1 - |b_k(z)|^2)\big) \leq -\frac{1}{2}(1 - |b_k(z)|^2) \tag{4}$$

for all $z \in \Delta$. Summing up over all $j \neq k$ and using the well known identity

$$1 - |b_j(z)|^2 = \frac{(1-|z|^2)(1-|a_j|^2)}{|1-\overline{a}_j z|^2}, \tag{5}$$

we get from (4) that

$$\log|B_k(z)| \leq -\frac{1}{2}\sum_{j\neq k}\frac{(1-|z|^2)(1-|a_j|^2)}{|1-\overline{a}_j z|^2}. \tag{6}$$

Taking into account inequalities (6) and (1), as well as Lemma 3, we have

$$\begin{aligned}
|B'(re^{it})| &\leq \sum_{k=1}^{\infty} \frac{1-|a_k|^2}{|1-\overline{a}_k re^{it}|^2}|B_k(re^{it})| \\
&\leq \sum_{k=1}^{\infty} \frac{1-|a_k|^2}{|1-\overline{a}_k re^{it}|^2} \exp\left(-\frac{1}{2}\sum_{j\neq k}\frac{(1-r^2)(1-|a_j|^2)}{|1-\overline{a}_j re^{it}|^2}\right) \\
&\leq e^{\frac{1}{2}}\sum_{k=1}^{\infty} \frac{1-|a_k|^2}{|1-\overline{a}_k re^{it}|^2}\cdot \exp\left(-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(1-r^2)(1-|a_k|^2)}{|1-\overline{a}_k re^{it}|^2}\right) \\
&\leq e^{\frac{1}{2}}(2+\sigma)^2\sum_{k=1}^{\infty} \frac{1-|a_k|^2}{|1-|a_k|re^{it}|^2}\cdot \\
&\quad \cdot \exp\left(-\frac{1}{2(2+\sigma)^2}\sum_{k=1}^{\infty}\frac{(1-r^2)(1-|a_k|^2)}{|1-|a_k|re^{it}|^2}\right) \\
&\leq A\sum_{k=1}^{\infty} \frac{1-|a_k|^2}{\big[(1-|a_k|)+(1-r)+|t|\big]^2}\cdot \\
&\quad \cdot \exp\left(-K\sum_{k=1}^{\infty}\frac{(1-r^2)(1-|a_k|^2)}{\big[(1-|a_k|)+(1-r)+|t|\big]^2}\right) \\
&\leq A\varphi\big((1-r)+|t|\big)\exp\Big(-K(1-r)\varphi\big((1-r)+|t|\big)\Big),
\end{aligned}$$

whenever $r \in [R,1)$, $t \in [-\pi,\pi]$. Here A and K are two constants that depend only upon σ and R and

$$\varphi(u) = \sum_{k=1}^{\infty} \frac{1-|a_k|^2}{\big[u+(1-|a_k|)\big]^2}, \quad u \in (0,\infty)\,.$$

Observe that

$$\varphi(u) \leq \frac{1}{u^2}\sum_{k=1}^{\infty}(1-|a_k|^2)\,. \tag{7}$$

After two changes of variable: first $\theta = \theta(t) = 1-r+t$ and then $u = u(r) = 1-r$, some obvious estimates, and Fubini's theorem, we obtain

$$
\begin{aligned}
& \textstyle\int_{R\le|z|<1} |B'(z)|^p \, dA(z) \\
\le\ & A \textstyle\int_R^1 \left(\int_{-\pi}^{\pi} \varphi^p\big((1-r)+|t|\big) \exp\left(- Kp(1-r)\varphi\big((1-r)+|t|\big)\right) dt \right) dr \\
=\ & 2A \textstyle\int_R^1 \left(\int_0^{\pi} \varphi^p\big((1-r)+t\big) \exp\left(- Kp(1-r)\varphi\big((1-r)+t\big)\right) dt \right) dr \\
=\ & 2A \textstyle\int_R^1 \left(\int_{1-r}^{1-r+\pi} \varphi^p(\theta) \exp\big(- Kp(1-r)\varphi(\theta)\big) \, d\theta \right) dr \\
\le\ & 2A \textstyle\int_R^1 \left(\int_0^{2\pi} \varphi^p(\theta) \exp\big(- Kp(1-r)\varphi(\theta)\big) \, d\theta \right) dr \\
=\ & 2A \textstyle\int_0^{1-R} \left(\int_0^{2\pi} \varphi^p(\theta) \exp\big(- Kpu\varphi(\theta)\big) \, d\theta \right) du, \\
\le\ & 2A \textstyle\int_0^{2\pi} \left(\int_0^{1} \varphi^p(\theta) \exp\big(- Kpu\varphi(\theta)\big) \, du \right) d\theta,
\end{aligned}
$$

Making yet another change of variable $x = x(u) = u\varphi(\theta)$ and using (7), we get

(8)

$$
\begin{aligned}
\int_{R\le|z|<1} |B'(z)|^p \, dA(z) &\le 2A \int_0^{2\pi} \left(\int_0^{\varphi(\theta)} \varphi^p(\theta) \exp(-Kpx) \frac{1}{\varphi(\theta)} \, dx \right) d\theta, \\
&= 2A \left(\int_0^{2\pi} \varphi^{p-1}(\theta) \, d\theta \right) \left(\int_0^{\varphi(\theta)} \exp(-Kpx) \, dx \right) \\
&\le 2A \left(\int_0^{2\pi} \varphi^{p-1}(\theta) \, d\theta \right) \left(\int_0^{\infty} \exp(-Kpx) \, dx \right) \\
&\le \frac{2A}{Kp} \left(\int_0^{2\pi} \varphi^{p-1}(\theta) \, d\theta \right) \\
&\le \frac{2A}{Kp} \left(\sum_{k=1}^{\infty} (1-|a_k|^2) \right)^{p-1} \int_0^{2\pi} \frac{1}{u^{2p-2}} \, du < \infty,
\end{aligned}
$$

because the points a_k satisfy the Blaschke condition and $0 < p < \frac{3}{2}$. This ends the proof. $\square$

3. Interpolating Blaschke products with zeros in a Stolz angle

The following result was proved by Protas.

Theorem H. *If* $1/2 < p < 1$ *and* B *is a Blaschke product whose sequence of zeros* $\{a_k\}$ *satisfies*

$$\sum_{k=1}^{\infty}(1-|a_k|)^{1-p} < \infty, \tag{9}$$

then $B' \in H^p$.

For a generalization, see Ahern [1], Cohn [7] proved the converse of Theorem H for interpolating Blaschke products.

The following result is due to H. O. Kim [17].

Theorem I. *Let* $\{a_k\}$ *be the zero set of a Blaschke product* B *and suppose that also*

$$\sum_{k=1}^{\infty}(1-|a_k|)^{2-p} < \infty, \tag{10}$$

for a certain $p \in (1,2)$. *Then* $B' \in A^p$.

In a joint work with D. Girela and D. Vukotić [15], we have proved the converse for interpolating Blaschke products.

Theorem J. *If* $0 < p < 2$ *and* B *is an interpolating Blaschke product with zeros* $\{a_k\}$ *such that* $B' \in A^p$ *then*

$$\sum_{k=1}^{\infty}(1-|a_k|)^{2-p} < \infty.$$

In the proof of this result several results on Q_p spaces are used.

When $0 < p < \infty$, a function f, analytic in Δ, belongs to the space Q_p if

$$\sup_{a\in\Delta}\int_{\Delta}|f'(z)|^2 g(z,a)^p\,dA(z) < \infty,$$

where g denotes the Green function for the disc given by

$$g(z,a) = \log\left|\frac{1-\overline{a}z}{a-z}\right|,\ z\neq a.$$

The Q_p spaces are nested,

$$Q_p \subsetneq Q_s \subsetneq Q_1 = BMOA \subsetneq Q_t = \mathcal{B} \quad 0 < p < s < 1 < t,$$

where $\mathcal{B}$ is the Bloch space. There are various characterizations of Q_p spaces. The one that will be useful for us is expressed in terms of p-Carleson measures.

A positive Borel measure μ is said to be a p-Carleson measure on Δ if there exists a positive constant C such that for all interval $I \subset \mathbb{T}$

$$\mu\big(S(I)\big) \leq C|I|^p,$$

where $|I|$ is the length of I and $S(I)$ is the Carleson square

$$S(I) = \{re^{it} : e^{it}\in I, \quad 1-\frac{|I|}{2\pi} \leq r < 1\}.$$

The special case $p = 1$ yields the classical Carleson measures (cf.[9]). The following characterization of Q_p spaces was obtained by Aulaskari, Stegenga and Xiao [4].

Theorem K. *Let $0 < p < \infty$. A function f holomorphic in Δ is a member of Q_p if and only if the measure μ on Δ defined by*

$$d\mu(z) = (1-|z|^2)^p|f'(z)|^p\,dA(z)$$

is a p-Carleson measure.

Essén and Xiao [11] used this result to characterize the inner functions that belong to Q_p spaces $(0 < p < 1)$. In particular, their result can be stated for Blaschke products as follows.

THEOREM L. *Let $p \in (0,1)$ and let B be the Blaschke product with zeros $\{a_k\}$. Then $B \in Q_p$ if and only if the measure*

$$d\mu_p(z) = \sum (1 - |a_k|^2)^p \delta_{a_k}$$

is a p-Carleson measure. As usual, δ_{a_k} denotes the point mass at a_k.

Danikas and Mouratides [8] obtained a sufficient condition for membership of a Blaschke product in $\cap_{0<p<1} Q_p$ expressed in term of the sequence of the moduli of its zeros. They introduced the following concept: A sequence $\{\alpha_n\}_{n=1}^{\infty}$ such that $\alpha_n \geq 0$, $\alpha_n \geq \alpha_{n+1}$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \alpha_n = 0$, is said to be *asymptotically concentrated* if for each $k = 1, 2, \ldots$ there exists an increasing sequence n_p of positive integers that only depends on k, with the property that

$$\lim_{p\to\infty} \frac{\alpha_{n_p}}{\alpha_{n_p+k}} = 1.$$

Sometimes it is difficult to manage this definition, so this lemma will be very useful for us.

LEMMA M. *Suppose $\{\alpha_n\}_{n=1}^{\infty}$ such that $\alpha_n \geq 0$, $\alpha_n \geq \alpha_{n+1}$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \alpha_n = 0$. Then the sequences*

$$c_n = \alpha_n \sum_{k=1}^{n} \alpha_k^{-1}$$

and

$$d_n = \alpha_n^{-1} \sum_{k=n+1}^{\infty} \alpha_k$$

are both bounded if and only if the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is not asymptotically concentrated.

Using Theorem L, Danikas and Mouratides [8] were able to prove the following result.

THEOREM N. *Let B be a Blaschke product with ordered sequence of zeros $\{a_k\}_{k=1}^{\infty}$. If the sequence $\{1-|a_k|\}_{k=1}^{\infty}$ is not* asymptotically concentrated*, then $B \in \cap_{0<p<1} Q_p$.*

The inequalities (1), the above theorem and Lemma 1 on p. 150 of [9] are used in [15] to obtain a sufficient condition for membership of a Blaschke product in $\cap_{0<p<1} Q_p$.

THEOREM O. *Let B be an interpolating Blaschke product whose ordered sequence of zeros $\{a_k\}_{k=1}^{\infty}$ is contained in a Stolz angle. Then*

$$B \in \cap_{0<p<1} Q_p.$$

We include the proof of Theorem G, which can be found in [15], for the sake of completeness.

Proof of Theorem G. Let B be an interpolating Blaschke product whose sequence of zeros $\{a_k\}$ is contained in a Stolz angle. Using Theorem O we see that

$$B \in \cap_{0<p<1} Q_p,$$

and then Theorem L implies that for all $0<p<1$

$$d\mu_p(z) = \sum (1-|a_k|^2)^p \delta_{a_k}$$

is a p-Carleson measure.

So it is clear that $d\mu_p(z)$ is a finite measure for all $0<p<1$, that is,

$$\sum_{k=1}^{\infty} (1-|a_k|)^p < \infty,$$

for all $0<p<1$.

Consequently by Theorem H, $B' \in \cap_{0<q<1} H^q \subset \cap_{0<q<2} A^q$. $\square$

4. Two constructions of a Blaschke product whose derivative does not belong to any Hardy space

Proof of Proposition 1. Let $p > 0$, by Lemma D, it suffices to prove that

$$\begin{aligned} f(t) &= \sum_{k=3}^{\infty} \frac{(k\log^2 k)^{-1}}{(k\log^2 k)^{-2} + \left(t - \log^{-2} k\right)^2} \\ &= \sum_{k=3}^{\infty} \frac{k\log^2 k}{1 + k^2\left(t\log^2 k - 1\right)^2} \end{aligned} \tag{11}$$

does not belong $L^p(0, 2\pi)$.

For each $t \in [0, \frac{1}{4}]$, we set $x_t = e^{\frac{1}{\sqrt{t}}}$, which is the solution of the equation

$$t\log^2 x - 1 = 0,$$

and let K_t be the biggest integer $\leq x_t$.

Then there exist $C > 0$ and $A > 0$, indepedent of t, such that

$$f(t) \geq C\left(K_t \log^2 K_t\right) \geq CA\left(x_t \log^2 x_t\right) = CA\frac{e^{\frac{1}{\sqrt{t}}}}{t}.$$

Consequently

$$\int_0^{2\pi} f^p(t)\, dt = \infty,$$

and this finishes the proof. $\square$

A basic tool in the proof of Theorem 2 will be this deep result of Naftalevich.

Theorem P. *For any Blaschke sequence $\{a_k\}$, there exists an interpolating sequence $\{z_k\}$ such that $|z_k| = |a_k|$ for each k.*

The original paper [19] is not easily accessible, but a detailed and constructive proof can be found in Cochran's paper [6].

Proof of Theorem 2. Let the sequence $\{a_k\}$ be defined by

$$a_k = 1 - (k \log^2 k)^{-1}, \quad k = 2, 3 \ldots.$$

Since $\{a_k\}$ is a Blaschke sequence, using Cochran's construction, we obtain an interpolating sequence $\{z_k\}$, such that $|z_k| = a_k$, for all $k = 2, 3 \ldots.$

Now whenever $p > 1$, we have

$$\sum_{k=1}^{\infty} (1 - |z_k|)^{2-p} = \sum_{k=1}^{\infty} \frac{1}{k^{2-p} \log^{2(2-p)} k} = \infty,$$

then by Theorem J, the interpolating Blaschke product whose sequence of zeros is $\{z_k\}$ satisfies that $B' \notin \cup_{1<p} A^p$.

Consequently by Proposition F, $B' \notin \cup_{0<p} H^p$. $\square$

References

[1] P. Ahern, The mean modulus of the derivative of an inner function, *Indiana Univ. Math. J.* **28** (1979), no. 2, 311–347.

[2] P. Ahern, The Poisson integral of a singular measure, *Canad. J. Math.* **35** (1983), no. 4, 735–749.

[3] P. Ahern and D. Clark, On inner functions with H^p derivative, *Michigan Math. J.* **21** (1974), 115–127.

[4] R. Aulaskari, D. A. Stegenga and J. Xiao, Some subclasses of BMOA and their characterizacions in terms of Carleson measures, *Rocky Mountain J. Math.* **26**, (1996), 485–506.

[5] S. Axler, Bergman spaces and their operators, In *Surveys of Some Recent Results in Operator Theory, 1*, (J.B. Conway and B.B. Morrel, Eds.), Vol. **171**, pp. 1–50, Pitman Research Notes in Mathematics, Longman, Harlow, Essex 1988.

[6] G.W. Cochran, Random Blaschke products, *Trans. Amer. Math. Soc.* **322** (1990), no. 2, 731–755.

[7] W.S. Cohn, On the H^p classes of derivative of functions orthogonal to invariant subspaces, *Michigan Math. J.* **30**, (1983), 221–229.

[8] N. Danikas and C. Mouratides, Blaschke products in Q_p-spaces, *Complex Variables Theory Appl.*, **43**, 199–209.

[9] P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York-London 1970. Reprint: Dover, Mineola, New York 2000.

[10] P.L. Duren and A.P. Schuster, *Bergman Spaces*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, to appear.

[11] M. Essén and J. Xiao, Some results on Q_p spaces, $0 < p < 1$, *J. Reine Angew. Math.*, **485**, (1997), 173–195.

[12] O. Frostman, Potential d'equilibre et capacité des emsembles avec quelques applications à la thèorie des fonctions, *Medd. Lunds Univ. Mat. Sem.* **3**, (1935), 1-118.

[13] D. Girela and C. González, Mean growth of the derivative of infinite Blaschke products, *Complex Variables Theory Appl.* **45** (2001), 1–10.

[14] D. Girela and J.A. Peláez, On the derivative of infinite Blaschke products, *Illinois J. Math.*, to appear.

[15] D. Girela. J. A. Peláez and D. Vukotić, On the integrability of the derivative of a Blaschke product, *preprint*, (2003).

[16] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics, Vol. **199**, Springer, New York, Berlin, etc. 2000.

[17] H.O. Kim, Derivatives of Blaschke products, *Pacific J. Math.* **114** (1984), 175–190.

[18] M.A. Kutbi, Integral means for the first derivative of Blaschke products, *Kodai Math. J* **24** (2001), no. 1, 86–97.

[19] A. G. Naftalevich, On interpolation by functions of bounded characteristic, (In Russian), *Vilniaus Valst. Univ. Mokslu Darbai. Mat. Fiz. Chem. Mokslku Ser.*, **5**, (1956), 5-27.

[20] G. Piranian, Bounded functions with large circular variation, *Proc. Amer. Math. Soc.* **19**, no. 6 (1968), 1255–1257.

[21] D. Protas, Blaschke products with derivative in H^p and B^p, *Michigan Math. J.* **20** (1973), 393–396.

[22] W. Rudin, The radial variation of analytic functions, *Duke Math. J.* **22** (1955), 235–242.

[23] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, St. Louis, etc. 1974.

[24] S.A. Vinogradov, Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces (In Russian), *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **222** (1995), *Issled. po Linein. Oper. i Teor. Funktsii* **23**, 45–77, 308; translation in *J. Math. Sci. (New York)* **87**, no. 5 (1997), 3806–3827.

[25] D. Vukotić, The isoperimetric inequality and a theorem of Hardy and Littlewood, *Amer. Math. Monthly* **110**, no. 6 (2003), 532–536.

Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain

E-mail address: `pelaez@anamat.cie.uma.es`