NON-STABLE CLASSES OF ANALYTIC FUNCTIONS

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Abstract. If f and g are analytic functions in the unit disc $\mathbb{D}$, then f is said to be weakly subordinate to g, written $f \prec^w g$, if there exist analytic functions ϕ and $\omega : \mathbb{D} \to \mathbb{D}$, with ϕ an inner function, so that $f \circ \phi = g \circ \omega$. A class X of analytic functions in $\mathbb{D}$ is said to be stable if it is closed under weak subordination, that is, if $f \in X$ whenever f and g are analytic functions in $\mathbb{D}$ with $g \in X$ and $f \prec^w g$. For $0 < p < \infty$ and $\alpha > -1$, we let A^p_α denote the weighted Bergman space of all functions f, analytic in $\mathbb{D}$, such that $f \in L^p\left((1-|z|^2)^\alpha dxdy\right)$ and the space of Dirichlet type $\mathcal{D}^p_\alpha$ consists of those f such that $f' \in A^p_\alpha$. Among other results, we prove that all the Bergman spaces A^p_α $(0 < p < \infty,\ \alpha > -1)$ and all the $\mathcal{D}^p_\alpha$-spaces except the space $\mathcal{D}^2_1 = H^2$ are non-stable classes of analytic functions in $\mathbb{D}$.

1. Introduction and first results.

We denote by $\mathbb{D}$ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and by $\mathcal{H}ol(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. As usual, H^p $(0 < p \leq \infty)$ are the classical Hardy spaces of analytic functions in $\mathbb{D}$ (see [14] and [18]) and N is the Nevanlinna class (see [14] and [24]). A function I, analytic in $\mathbb{D}$, is said to be an inner function if $I \in H^\infty$ and I has a radial limit $I(e^{i\theta})$ of modulus one for almost every $e^{i\theta} \in \partial\mathbb{D}$. We recall that an inner function I can be factored in the form $I = BS$ where B is a Blaschke product and S is a *singular inner function*, that is, S is of the form

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right), \quad z \in \mathbb{D},$$

Date: November 10, 2004.

The authors are partially supported by a grant from "El Ministerio de Educación y Ciencia, Spain" (MTN2004-00078) and by a grant from "La Junta de Andalucía" (FQM-210).

where, μ is a finite positive Borel measure on $[0, 2\pi)$ that is singular with respect to Lebesgue measure.

K. Stephenson introduced in [26] the notion of weak subordination: if f and g are meromorphic functions in $\mathbb{D}$, then f is said to be *weakly subordinate* to g, written $f \prec^w g$, if there exist analytic functions ϕ and $\omega : \mathbb{D} \to \mathbb{D}$, with ϕ an inner function, so that $f \circ \phi = g \circ \omega$. If $\phi(z) \equiv z$ and $\omega(0) = 0$, we have the classical concept of subordination, and we shall simply write $f \prec g$. A class X of meromorphic functions in $\mathbb{D}$ is said to be *stable* if it is closed under weak subordination, that is, if $f \in X$ whenever $f \prec^w g$ and $g \in X$.

Stephenson [25] proved in that the Hardy spaces H^p, $0 < p \leq \infty$, and the Nevanlinna class N are stable classes, although this terminology was not used there. Furthermore, K. Stephenson also proved in [26] that the space $BMOA$ is stable while the the Bloch space $\mathcal{B}$ and the space $VMOA$ are not stable.

We recall that the space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbb{D}$ and, $VMOA$ is the closed subspace of $BMOA$ which consists of those $f \in H^1$ whose boundary values have vanishing mean oscillation on $\partial\mathbb{D}$. Alternatively, $VMOA$ is the closure of the polynomials in $BMOA$. We mention [8], [18] and [19] for the theory of the spaces $BMOA$ and $VMOA$.

If f is an analytic function in $\mathbb{D}$, then f is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ such that $\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0$. Alternatively, $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm. A very good reference for the theory of Bloch functions is [4].

Stephenson's arguments can be used to obtain the following improvement of his results.

Theorem 1. *Let X be a space of analytic functions in $\mathbb{D}$ which satisfies any of the two following conditions:*

(A) X contains the polynomials and there is some inner function I such that $I \notin X$.

(B) $X \neq \mathcal{H}ol(\mathbb{D})$ and $\mathcal{B} \subset X$.

Then X is a non-stable class of analytic functions in $\mathbb{D}$.

Proof. Suppose that X satisfies condition (A) and let I be an inner function which does not belong to X. Take $g(z) = \phi(z) = z$ and $\omega(z) = I(z)$ ($z \in \mathbb{D}$). Then ϕ and ω are inner functions and $I = I \circ \phi = g \circ \omega$. Hence, $I \prec^w g$, $g \in X$ and $I \notin X$. Thus X is not stable.

Suppose now that $X \neq \mathcal{H}ol(\mathbb{D})$ and $\mathcal{B} \subset X$. Let f be an analytic function in $\mathbb{D}$ such that $f \notin X$. Let $E = \{m + ni : m, n \in \mathbb{Z}\}$ and $F = \{z \in \mathbb{D} : f(z) \in E\}$. Since F is a countable subset of $\mathbb{D}$, it has capacity zero and therefore the universal covering map I from $\mathbb{D}$ onto $\mathbb{D} \setminus F$ is an inner function (see, for instance, Chapter 2 of [11]). Set $g = f \circ I$. Then the image of g is contained in $\mathbb{C} \setminus E$ and, hence, it does not contain arbitrarily large discs. Consequently, see [4], g is a Bloch function. Since $\mathcal{B} \subset X$, we have that $g = f \circ I \in X$ even though $f \notin X$. Thus, X is not stable. □

In adition to $VMOA$, among other, the following well known spaces satisfy condition (A) and, hence, are non-stable classes:

The little Bloch space $\mathcal{B}_0$, [4]; the spaces Q_p and $Q_{p,0}$, $0 < p < 1$, ([6, 7, 16, 28]; the Besov spaces and B^p, $1 \leq p < \infty$, [5, 12, 29]; the Dirichlet space $\mathcal{D}$ of those f analytic in $\mathbb{D}$ with finite Dirichlet integral.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $\int_{\mathbb{D}} (1 - |z|)^\alpha |f(z)|^p \, dA(z) < \infty$. Here, $dA(z) = \frac{1}{\pi} dx\, dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer to [15] and [21] for the theory of these spaces.

If $\varphi : [0, 1) \longrightarrow [0, \infty)$ is an increasing function with $\lim_{r \to 1} \varphi(r) = \infty$, we define

$$A(\varphi) = \{f \in \mathcal{H}ol(\mathbb{D}) : \ |f(z)| = \mathrm{O}\left(\varphi(|z|)\right), \ \text{as } |z| \to 1\}, \tag{1}$$

$$A_0(\varphi) = \{f \in \mathcal{H}ol(\mathbb{D}) : \ |f(z)| = \mathrm{o}\left(\varphi(|z|)\right), \ \text{ as } |z| \to 1\}. \tag{2}$$

If $f \in \mathcal{B}$ then

$$|f(z)| = \mathrm{O}\left(\log \frac{1}{1-|z|}\right), \quad \text{ as } |z| \to 1,$$

and then it follows that

$$\mathcal{B} \subset A^p_\alpha, \quad \alpha > -1, \ 0 < p < \infty, \tag{3}$$

$$\mathcal{B} \subset A(\varphi), \text{ if } \quad \log \frac{1}{1-r} = \mathrm{O}\left(\varphi(r)\right), \quad \text{as } r \to 1, \tag{4}$$

and,

$$\mathcal{B} \subset A_0(\varphi), \text{ if } \quad \log \frac{1}{1-r} = \mathrm{o}\left(\varphi(r)\right), \quad \text{as } r \to 1. \tag{5}$$

Using (3), (4), (5) and Theorem 1, we obtain the following result.

Corollary 1. *(i) If $\alpha > -1$ and $0 < p < \infty$ then the weighted Bergman space A^p_α is a non-stable class of analytic functions in $\mathbb{D}$.*

(ii) If $\varphi : [0,1) \longrightarrow [0,\infty)$ is an increasing function with $\log \frac{1}{1-r} = \mathrm{O}\left(\varphi(r)\right)$, as $r \to 1$, then the space $A(\varphi)$ is a non-stable class of analytic functions in $\mathbb{D}$.

(iii) If $\varphi : [0,1) \longrightarrow [0,\infty)$ is an increasing function with $\log \frac{1}{1-r} = \mathrm{o}\left(\varphi(r)\right)$, as $r \to 1$, then the space $A_0(\varphi)$ is a non-stable class of analytic functions in $\mathbb{D}$.

We remark that part (iii) of Corollary 1 improves Example II in pp. 575-576 of [26] which asserts that $A_0(\varphi)$ is not stable for $\varphi(r) = \exp\left(\frac{1+r}{1-r}\right)$, $(0 < r < 1)$.

The next two sections will be devoted to the main purpose of this paper which is finding other types of non-stable classes of analytic functions in $\mathbb{D}$. More precesily, we wish to study the possibility of finding a space X of analytic functions in $\mathbb{D}$ which is a not stable and does not satisfy either condition (A) or condition (B). In order to do so we shall consider spaces of Dirichlet type.

2. The spaces $\mathcal{D}^p_{p-1}$.

For $0 < p < \infty$, the space of Dirichlet type $\mathcal{D}^p_{p-1}$ consists of all analytic functions in $\mathbb{D}$ whose derivative belongs to A^p_{p-1}, that is,

$$\mathcal{D}^p_{p-1} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} (1-|z|^2)^{p-1} |f'(z)|^p \, dA(z) < \infty \right\}. \tag{6}$$

The spaces $\mathcal{D}^p_{p-1}$ are closely related to the Hardy spaces. Indeed, a direct calculation with power series shows that $H^2 = \mathcal{D}^2_1$. A classical result of Littlewood and Paley [22] (see also [23]) asserts that

$$H^p \subset \mathcal{D}^p_{p-1}, \quad 2 \le p < \infty. \tag{7}$$

On the other hand, we have

$$\mathcal{D}^p_{p-1} \subset H^p, \quad 0 < p \le 2. \tag{8}$$

The inclusion (7) can be proved by Riesz-Thorin interpolation. The same method gives (8) for $1 \le p \le 2$, since the inclusion $\mathcal{D}^1_0 \subset H^1$ is trivial. Vinogradov ([27], Lemma 1.4) extended (8) to the range $0 < p \le 2$. We refer to [27], [13], [9] and [20] for distinct aspects of the theory of the spaces $\mathcal{D}^p_{p-1}$. In particular, we mention that using Proposition 2.1 of [10] (see also Proposition A of [20]), we obtain the following result.

Proposition A. *If f is given by a power series with Hadamard gaps,*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \ (z \in \Delta) \text{ with } n_{k+1} \ge \lambda n_k \text{ for all } k \ (\lambda > 1),$$

then, for every $p \in (0, \infty)$,

$$f \in \mathcal{D}^p_{p-1} \iff \sum_{k=1}^{\infty} |a_k|^p < \infty.$$

Since for Hadamard gap series we have, for $0 < p < \infty$,

$$f \in H^p \iff \sum_{k=1}^{\infty} |a_k|^2 < \infty,$$

we immediately deduce that $\mathcal{D}^p_{p-1} \ne H^p$ if $p \ne 2$.

We remark also that if $p \neq q$ then there is no relation of inclusion between $\mathcal{D}^p_{p-1}$ and $\mathcal{D}^q_{q-1}$. However, it is easy to show (see Lemma 2 of [20]) that

$$\mathcal{B} \cap \mathcal{D}^p_{p-1} \subset \mathcal{B} \cap \mathcal{D}^q_{q-1}, \quad 0 < p < q < \infty. \tag{9}$$

We can now prove the following result.

Theorem 2. *If $0 < p < \infty$ and $p \neq 2$, then $\mathcal{D}^p_{p-1}$ is a non-stable class of analytic functions in $\mathbb{D}$.*

Remark 1. *Using (7), we see that if $2 < p < \infty$, then $\mathcal{D}^p_{p-1}$ contains all the inner functions. On the other hand, if we take the lacunary power series*

$$f(z) = \sum_{k=1}^{\infty} z^{2^k}, \quad z \in \mathbb{D},$$

we have that $f \in \mathcal{B}$, (see [4]), while Proposition (A) shows that $f \notin \mathcal{D}^p_{p-1}$. Consequently, if $2 < p < \infty$, the space $\mathcal{D}^p_{p-1}$ is not stable and does not satisfy either condition (A) or condition (B).

Proof of Theorem 2. Suppose first that $0 < p < 2$. Then, using Theorem 3.11 of [27], we see that there exists a Blaschke product B such that $B \notin \mathcal{D}^p_{p-1}$, and then the fact that $\mathcal{D}^p_{p-1}$ is not stable follows from Theorem 1.

Suppose now that $2 < p < \infty$. Take $f \in H^2 \setminus \mathcal{D}^p_{p-1}$, (for example, take $f(z) = \frac{1}{(1-z)^{1/p}}$, $z \in \mathbb{D}$). Now we argue as in the proof of Theorem 1: We set $E = \{m + ni : m, n \in \mathbb{Z}\}$ and $F = \{z \in \mathbb{D} : f(z) \in E\}$ and we let I be the universal covering map from $\mathbb{D}$ onto $\mathbb{D} \setminus F$. Since the set F is countable, I is an inner function. The function $g = f \circ I$ is a Bloch function because its image does not contain arbitrarily large discs. Also, $g \in H^2$ because H^2 is closed under subordination. Thus, we have that $g = f \circ I \in \mathcal{B} \cap H^2 = \mathcal{B} \cap \mathcal{D}^2_1$. Using (9), we deduce that $g = f \circ I \in \mathcal{B} \cap \mathcal{D}^p_{p-1}$ even though $f \notin \mathcal{D}^p_{p-1}$. Consequently, $\mathcal{D}^p_{p-1}$ is not stable. This finishes the proof. □

Remark 2. *If $2 < p < \infty$ and g is the function constructed in the proof of Theorem 2 then, bearing in mind that H^p is stable, we see that $g \in \mathcal{D}^p_{p-1} \setminus H^p$. For these values of p, most of the known examples of functions in $\mathcal{D}^p_{p-1} \setminus H^p$ are given by lacunary power series.*

3. The spaces $\mathcal{D}^p_\alpha$ $(0<p<\infty,\ \alpha>-1)$.

For $0<p<\infty$ and $\alpha>-1$ the space of Dirichlet type $\mathcal{D}^p_\alpha$ consists of all functions f which are analytic in $\mathbb{D}$ and satisfy

$$\int_{\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|^p\,dA(z)<\infty.$$

We have proved in Theorem 2 that if $p\neq 2$ then the spaces $\mathcal{D}^p_{p-1}$ are not stable. In this section we shall extend our work considering the spaces $\mathcal{D}^p_\alpha$ for all admissible values of p and α. We shall prove the following result.

Theorem 3. *If $0<p<\infty$, $\alpha>-1$ and $(p,\alpha)\neq(2,1)$ then $\mathcal{D}^p_\alpha$ is a non-stable class of analytic functions in $\mathbb{D}$.*

In other words, Theorem 3 asserts that all the $\mathcal{D}^p_\alpha$-spaces $(0<p<\infty$, $\alpha>-1)$ except the space $\mathcal{D}^2_1=H^2$ are non-stable classes of analytic functions in $\mathbb{D}$.

Proof of Theorem 3. We shall distinguish several cases.

(a) If $0<p<\infty$, $p\neq 2$ and $\alpha=p-1$ then the result follows from Theorem 2.

(b) If $0<p<\infty$ and $\alpha>p-1$ then it is well known that $\mathcal{D}^p_\alpha=A^p_{\alpha-p}$ (see, e. g., Theorem 6 of [17]) and then the fact that $\mathcal{D}^p_\alpha$ is not stable follows using Corollary 1.

(c) Suppose now that $0<p<2$ and $-1<\alpha<p-1$. As noticed above, Theorem 3.11 of [27] implies that there exists an inner function I with $I\notin\mathcal{D}^p_{p-1}$. Since $\alpha<p-1$, $\mathcal{D}^p_\alpha\subset\mathcal{D}^p_{p-1}$. Then we see that there is an inner function I which does not belong to $\mathcal{D}^p_\alpha$. Now Theorem 1 yields that $\mathcal{D}^p_\alpha$ is not stable.

(d) Suppose now that $1<p<\infty$ and $-1<\alpha\leq p-2$. Notice that the space $\mathcal{D}^p_{p-2}$ is the Besov space B^p. Hence $\mathcal{D}^p_\alpha\subset B^p$. Now, there are inner functions which do not belong to B^p. Indeed, Theorem 3.1 of [12] asserts that the only inner functions in the space B^p are the finite Blaschke products. Since $\mathcal{D}^p_\alpha\subset B^p$, the same is true for the space $\mathcal{D}^p_\alpha$. This implies that $\mathcal{D}^p_\alpha$ is not stable.

It only remains to prove that $\mathcal{D}^p_\alpha$ is not stable for $2 \leq p < \infty$ and $p-2 < \alpha < p-1$. Using again Theorem 1, this will follow from the following result. $\square$

THEOREM 4. *If $1 \leq p < \infty$ and $p-2 < \alpha < p-1$, then there exists an inner function I such that $I \notin \mathcal{D}^p_\alpha$.*

We need to introduce some notation and several results to prove Theorem 4.

If I is an inner function, we shall write

$$\Delta(r, I) = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - |I(re^{it})|^2\right) \, dt. \tag{10}$$

The quantity $\Delta(r, I)$ plays a very important role to study the membership of the derivative of an inner function in classical spaces of analytic functions in $\mathbb{D}$, (see [1], [2], [3]).

The following result is a special case of Theorem 6 of [2].

THEOREM A. *Suppose that $\alpha > -1$, $p > 1 + \alpha$ and I is an inner function. Then $I \in \mathcal{D}^p_\alpha$ if and only if*

$$\int_0^1 (1-r)^{\alpha - p} \frac{1}{2\pi} \int_0^{2\pi} \left(1 - |I(re^{it})|^2\right)^p \, dt dr < \infty.$$

Now, if μ is a finite positive Borel measure on $[0, 2\pi]$, the modulus of continuity ω_μ of μ is defined by

$$\omega_\mu(\delta) = \sup\{\mu\left([\theta, \theta + \delta)\right) : \theta \in [0, 2\pi]\}, \quad 0 \leq \delta.$$

If $0 < \beta < 1$ we let S_β denote the class of all inner functions I whose singular inner factor S is the singular inner function associated to a singular measure μ whose modulus of continuity ω_μ satisfies that $\omega_\mu(\delta) = \mathrm{O}(\delta^\beta)$, as $\delta \to 0$. Ahern proved in Theorem 2.5 of [1] the following result.

THEOREM B. *Suppose that $0 < \beta < 1$ and $I \in S_\beta$. Then there exist a constant $\varepsilon > 0$ and $r_0 \in (0, 1)$ such that*

$$\Delta(r, I) \geq \varepsilon (1-r)^{\frac{1-\beta}{2-\beta}}, \quad r \in (r_0, 1). \tag{11}$$

Proof of Theorem 4. Take and fix p with $1 \leq p < \infty$. Since the spaces $\mathcal{D}^p_\alpha$ grow with α, it suffices to prove the theorem for those α such that $\max\{p-2, \frac{p-2}{2}\} < \alpha < p-1$. So, take such an α and set

$$\beta = 2 - \frac{p}{1+a}. \tag{12}$$

Observe that $0 < \beta < 1$. Let I be an inner function with $I \in S_\beta$. Using Theorem B we see that there exist $\varepsilon > 0$ and $r_0 \in (0,1)$ such that

$$\Delta(r, I) \geq \varepsilon (1-r)^{\frac{1-\beta}{2-\beta}}, \quad r \in (r_0, 1). \tag{13}$$

Since $p \geq 1$, we have that

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \left(1 - |I_\alpha(re^{it})|^2\right)^p \, dt \\ &\geq \left(\frac{1}{2\pi}\int_0^{2\pi} \left(1 - |I_\alpha(re^{it})|^2\right) dt\right)^p = \Delta^p(r, I), \quad 0 < r < 1, \end{aligned}$$

which, together with (13), implies

$$\begin{aligned} &\int_{r_0}^1 (1-r)^{\alpha-p} \frac{1}{2\pi}\int_0^{2\pi} \left(1 - |I(re^{it})|^2\right)^p \, dt dr \\ &\geq \int_{r_0}^1 (1-r)^{\alpha-p} \Delta^p(r, I_\alpha) \, dr \\ &\geq \varepsilon^p \int_{r_0}^1 (1-r)^{\alpha-p} \left(1-r\right)^{\frac{p(1-\beta)}{2-\beta}} \, dr \\ &= \varepsilon^p \int_{r_0}^1 (1-r)^{-1} \, dr \\ &= \infty. \end{aligned} \tag{14}$$

Using Theorem A we deduce that $I \notin \mathcal{D}^p_\alpha$. This finishes the proof. $\square$

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