A class of conformal mappings with applications to function spaces

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Dedicated to Professor Joseph Cima on the occasion of his 70th birthday

ABSTRACT. We discuss some of the properties of a class of conformal mappings introduced in a joint work with O. Blasco and M. A. Márquez and show that they can be used to prove the sharpness of a number of embedding theorems between certain spaces of analytic functions and, also, to prove that, for $1 \leq p < \infty$, the space $\mathcal{B} \cap H^p$ (where $\mathcal{B}$ is the Bloch space) does not have the so called f-property of Havin.

1. Some results on conformal mappings

1.1. Basic notation. Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. If $0 < r < 1$ and f is an analytic function in $\mathbb{D}$ (abbreviated $f \in H(\mathbb{D})$) we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad I_p(r, f) = M_p^p(r, f), \, (0 < p < \infty),$$

$$M_\infty(r, f) = \sup_{0 \leq t \leq 2\pi} |f(re^{it})|.$$

For $0 < p \leq \infty$ the Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ for which $||f||_{H^p} \overset{\text{def}}{=} \sup_{0 < r < 1} M_p(r, f) < \infty$. We refer to [**9**] for the theory of Hardy spaces. The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\mathbb{T} = \partial\mathbb{D}$ (cf. [**4**], [**11**] and [**14**]).

1.2. On the relation between $M_\infty(r, F)$ and $M_1(r, F')$ when F is a conformal mapping from the unit disk onto a certain domain. Let Ω be a simply connected domain contained in $\mathbb{C}$ with $0 \in \Omega \neq \mathbb{C}$ and let F be the conformal mapping from $\mathbb{D}$ onto Ω with $F(0) = 0$ and $F'(0) > 0$. We note that, for $0 < r < 1$, the quantity $2\pi r M_1(r, F')$ represents the length of the closed curve C_r which is the image under F of the circle $\{|z| = r\}$. Clearly, we have that $M_\infty(r, F) \leq \text{length}(C_r)$, $0 < r < 1$, and, hence,

$$(1.1) \qquad\qquad M_\infty(r, F) = \mathrm{O}\left(M_1(r, F')\right), \quad \text{as } r \to 1.$$

It is natural to look for a geometric condition on the domain Ω which is sufficient to imply that the quantities $M_1(r, F')$ and $M_\infty(r, F)$ are comparable. Keogh [**24**] and Hayman [**22**] proved that being starlike with respect to the origin is not such a sufficient condition.

1.3. Circularly symmetric domains. A domain Ω in $\mathbb{C}$ is said to be circularly symmetric if, for every r with $0 < r < \infty$, $\Omega \cap \{|z| = r\}$ is either empty, is the whole circle $|z| = r$, or is a single arc on $|z| = r$ which contains $z = r$ and is symmetric with respect to the real axis.

Let Ω be a simply connected and circularly symmetric domain in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$ and $0 \in \Omega$. Let F be the conformal mapping from $\mathbb{D}$ onto Ω with $F(0) = 0$ and $F'(0) > 0$. Then (see [**23**] or the Corollary in p. 154 of [**3**]) it is known that:
(a) $M_\infty(r, F) = F(r)$, $0 < r < 1$.
(b) For every r with $0 < r < 1$, $|F(re^{i\theta})|$ is a decreasing function of θ in $[0, \pi]$.

1.4. The geometric condition. In the following theorem, which was obtained in a joint work with O. Blasco and M. A. Márquez [**5**], we give a geometric condition on Ω which implies that $M_\infty(r, F) \asymp M_1(r, F')$, as desired.

THEOREM 1.1. ([**5**, Theorem 3.1]). Let Ω be a domain in $\mathbb{C}$ with $0 \in \Omega$ and $\Omega \neq \mathbb{C}$. Suppose that Ω is both circularly symmetric and starlike with respect to 0. Let F be the conformal mapping from $\mathbb{D}$ onto Ω with $F(0) = 0$ and $F'(0) > 0$. Then there exists a positive constant C such that

$$(1.2) \qquad\qquad 2\pi r M_1(r, F') \leq C M_\infty(r, F), \quad 0 < r < 1.$$

Consequently,

$$(1.3) \qquad\qquad M_\infty(r, F) \asymp M_1(r, F'), \quad \text{as } r \to 1.$$

To prove Theorem 1.1 we use (b) together with the well known fact that (see p. 43 of [**29**] or p. 41 of [**10**]), since Ω is starlike with respect to the origin, for every r with $0 < r < 1$, $\arg F(re^{i\theta})$ is an increasing function of θ in $[0, 2\pi]$.

1.5. A special type of domain. In this section we consider a certain family of domains which satisfy the conditions of Theorem 1.1 and which have been useful to solve a number of problems in the theory of spaces of analytic functions.

Let $\Gamma = \{\gamma_n\}_{n=1}^\infty$ be a decreasing sequence of positive numbers with $\gamma_1 < 1$. We set

$$(1.4) \qquad \Omega_\Gamma = \Omega_{\{\gamma_n\}} = \mathbb{D} \cup \left(\bigcup_{n=1}^\infty \{z : |\operatorname{Im} z| < \gamma_n, \operatorname{Re} z > 0, n \leq |z| < n+1\} \right).$$

It is clear that the domain Ω_Γ is circularly symmetric and starlike with respect to the origin. Consequently, if F_Γ is the conformal mapping from $\mathbb{D}$ onto Ω_Γ with $F_\Gamma(0) = 0$ and $F_\Gamma'(0) > 0$, we have that:

(i) $M_\infty(r, F_\Gamma) = F_\Gamma(r)$, $0 < r < 1$.

(ii) $M_\infty(r, F_\Gamma) \asymp M_1(r, F_\Gamma')$, as $r \to 1$.

Furthermore, $|\operatorname{Im} F_\Gamma|$ is bounded and, hence

(iii) $F_\Gamma \in BMOA$.

Using the Carathéodory kernel convergence theorem (cf. [**29**, pp. 28-31]), we can prove the following result (see the proof of Theorem 3.2 of [**5**] for the details).

THEOREM 1.2. *Let* $\varphi : [0, 1) \to [0, \infty)$ *be a function with* $\varphi(r) \to \infty$ *as* $r \to 1$. *Then there exists a decreasing sequence of positive numbers* $\Gamma = \{\gamma_n\}_{n=1}^\infty$ *with* $\gamma_1 < 1$ *such that, if* F_Γ *is the conformal mapping from* $\mathbb{D}$ *onto* Ω_Γ *with* $F_\Gamma(0) = 0$ *and* $F_\Gamma'(0) > 0$, *then we have that* $F_\Gamma(r) \to \infty$, *as* $r \to 1$ *and*

$$(1.5) \qquad M_\infty(r, F_\Gamma) = F_\Gamma(r) \le \varphi(r), \quad 0 < r < 1.$$

2. On the sharpness of certain embedding theorems

2.1. Bloch functions and normal functions. We recall that a function f analytic in $\mathbb{D}$ is a Bloch function if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. It is well known that

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

The little Bloch space $\mathcal{B}_0$ consists of those functions f which are analytic in $\mathbb{D}$ and satisfy

$$\lim_{|z| \to 1} (1 - |z|^2) |f'(z)| = 0.$$

The space $\mathcal{B}_0$ is a closed subspace of the Bloch space. In fact, $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm. We mention [**2**] as a general reference for the theory of Bloch functions

A function f which is meromorphic in $\mathbb{D}$ is said to be a normal function if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \frac{|f'(z)|}{1 + |f(z)|^2} < \infty.$$

We refer to [**2**] and [**29**] for the theory of normal functions. For simplicity, let $\mathcal{N}$ denote the set of all holomorphic normal functions in $\mathbb{D}$. Any Bloch function is a normal function, that is, $\mathcal{B} \subset \mathcal{N}$.

2.2. Mean Lipschitz spaces of analytic functions. If f is a function which is analytic in $\mathbb{D}$ and has a non-tangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \mathbb{T}$, we define

$$(2.1)$$

$$\omega_p(\delta, f) = \sup_{0 < |t| \le \delta} \left(\frac{1}{2\pi} \int_{-\pi}^\pi \left| f(e^{i(\theta+t)}) - f(e^{i\theta}) \right|^p d\theta \right)^{1/p}, \quad \delta > 0, \quad \text{if } 1 \le p < \infty,$$

$$\omega_\infty(\delta, f) = \sup_{0 < |t| \le \delta} \left(\operatorname*{ess.\,sup}_{\theta \in [-\pi, \pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})| \right), \quad \delta > 0.$$

Then $\omega_p(., f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f.

Throughout the paper $\omega : [0,1] \to [0,\infty)$ will be a continuous and increasing function with $\omega(0) = 0$. Then, for $1 \le p \le \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ consists of those functions $f \in H^p$ which satisfy

$$\omega_p(\delta, f) = O(\omega(\delta)), \quad \text{as } \delta \to 0.$$

If $0 < \alpha \le 1$ and $\omega(\delta) = \delta^\alpha$, we shall write Λ_α^p instead of $\Lambda(p, \omega)$, that is, we set

$$\Lambda_\alpha^p = \Lambda(p, \delta^\alpha), \quad 0 < \alpha \le 1, \quad 1 \le p \le \infty.$$

A classical result of Hardy and Littlewood [20] (see also Chapter 5 of [9]) asserts that for $1 \le p \le \infty$ and $0 < \alpha \le 1$, we have that

$$(2.2) \qquad \Lambda_\alpha^p = \left\{ f \text{ analytic in } \mathbb{D}: \ M_p(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}}\right), \quad \text{as } r \to 1 \right\}.$$

Blasco and de Souza found in [6] conditions on ω so that the result of Hardy and Littlewood can be extended to the spaces $\Lambda(p, \omega)$. Namely, they proved that if ω satisfies the so called Dini condition and the condition b_1 (see [6], [5] or [13] for the precise definitions) then, for $1 \le p \le \infty$,

$$(2.3) \qquad \Lambda(p, \omega) = \left\{ f \text{ analytic in } \mathbb{D} : \ M_p(r, f') = O\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\}.$$

Let $1 \le p \le \infty$ and let ϕ be a non-negative function defined in $[0,1)$. We define $\mathcal{L}(p, \phi)$ as the space of all functions f analytic in $\mathbb{D}$ for which

$$(2.4) \qquad\qquad\qquad M_p(r, f') = O(\phi(r)), \quad \text{as } r \to 1.$$

We remark that $\mathcal{L}(p, \phi) \subset \mathcal{L}(q, \phi)$, if $1 \le q < p$. Notice that (2.2) can be written as

$$(2.5) \qquad\qquad \Lambda_\alpha^p = \mathcal{L}\left(p, \frac{1}{(1-r)^{1-\alpha}}\right), \quad 1 \le p \le \infty, \quad 0 < \alpha \le 1,$$

and (2.3) is equivalent to

$$(2.6) \qquad\qquad\qquad \Lambda(p, \omega) = \mathcal{L}\left(p, \frac{\omega(1-r)}{1-r}\right).$$

Cima and Petersen proved in [8] that $\Lambda_{1/2}^2 \subset BMOA$. This result was extended by Bourdon, Shapiro and Sledd who proved the following result in [7]:

$$(2.7) \qquad\qquad\qquad \Lambda_{1/p}^p \subset BMOA, \quad 1 < p < \infty.$$

For $p = 1$ we have a stronger result. Indeed, a classical result of Privalov [9, Th. 3.11] asserts that a function f which is analytic in $\mathbb{D}$ has a continuous extension to the closed unit disk $\overline{\mathbb{D}}$ whose boundary values are absolutely continuous on $\partial \mathbb{D}$ if and only if $f' \in H^1$. In particular, we have

$$(2.8) \qquad\qquad\qquad f' \in H^1 \Rightarrow f \in \mathcal{A} \subset H^\infty,$$

where, $\mathcal{A}$ denotes the disk algebra. Using (2.2), we see that the the condition $f' \in H^1$ is equivalent to saying that $f \in \Lambda_1^1$. Hence, (2.8) can be written as

$$(2.9) \qquad\qquad\qquad \Lambda_1^1 \subset \mathcal{A} \subset H^\infty.$$

The author proved in [12] that (2.8) (or, equivalently, (2.9)) is sharp in a very strong sense:

THEOREM 2.1. ([**12**, Theorem 1]). Let ϕ be any positive continuous function defined in $[0,1)$ with $\phi(r) \to \infty$, as $r \to 1$. Then, there exists a function $f \in \mathcal{L}(1, \phi)$ which is not a normal function.

Hence, no condition on the growth of $M_1(r, f')$ other that its boundedness is enough to conclude that $f \in \mathcal{N}$. We recall here that $\mathcal{A} \subset H^\infty \subset BMOA \subset \mathcal{B} \subset \mathcal{N}$.

The original proof of Theorem 2.1 was constructive. The constructed function f was of the form $f = BF$ where B is a Blaschke product and F is a function given by a series of analytic functions in $\mathbb{D}$ which converges uniformly on every compact subset of $\mathbb{D}$. The constructions of B and F were very involved, and made use in an essential way of certain sequences introduced by K. I. Oskolkov in several contexts (see, e.g., [**26, 27, 28**]). But now is possible to give a much simpler proof of Theorem 2.1, independent of the Oskolkov's sequences, using Theorem 1.2 and the following result about Blaschke products proved in collaboration with C. González.

THEOREM 2.2. ([**15**, Theorem 1]). Let B be an interpolating Blaschke product with positive zeros. Then there exist two positive constants C_1 and C_2, and $\rho_0 \in (0,1)$ such that

$$(2.10) \qquad C_1 n(r, B) \le M_1(r, B') \le C_2 n(r, B), \quad \rho_0 < r < 1,$$

where, for $0 < r < 1$, $n(r, B)$ denotes the number of zeros of B in the disk $\{|z| < r\}$.

We remark that Girela and Peláez [**18**] have recently improved this result showing that the conclusion holds for any infinite Blaschke product whose sequence of zeros is exponential.

Proof of Theorem 2.1. Set $\varphi(r) = \sqrt{\phi(r)}$, $0 < r < 1$. Using Theorem 1.2, we see that there exists a decreasing sequence of positive numbers $\Gamma = \{\gamma_n\}_{n=1}^\infty$ with $\gamma_1 < 1$ such that, if F_Γ is the conformal mapping from $\mathbb{D}$ onto Ω_Γ with $F_\Gamma(0) = 0$ and $F_\Gamma'(0) > 0$, then we have that $F_\Gamma(r) \to \infty$, as $r \to 1$ and

$$(2.11) \qquad M_\infty(r, F_\Gamma) = F_\Gamma(r) \le \varphi(r), \quad 0 < r < 1.$$

Using the results of Section 1.5, it follows that we also have

$$(2.12) \qquad M_1(r, F_\Gamma') = O(\varphi(r)), \quad \text{as } r \to 1.$$

Now, using Theorem 2.2, it follows easily (see [**15**, pp. 7-8] for the details) that there exists an interpolating Blaschke product B whose zeros $\{a_n\}$ are positive and such that

$$(2.13) \qquad M_1(r, B') = O(\varphi(r)), \quad \text{as } r \to 1.$$

For simplicity set $F_\Gamma = F$ and define $f(z) = B(z)F(z)$ ($z \in \mathbb{D}$). Bearing in mind that B is bounded by 1 and using (2.12), (2.13) and (2.11), we see that

$$M_1(r, f') \le M_1(r, F') + M_1(r, B')M_\infty(r, F) = O\left(\varphi^2(r)\right), \quad \text{as } r \to 1.$$

Hence, $f \in \mathcal{L}(1, \phi)$.

On the other hand, we have

$$(2.14) \qquad (1 - |a_n|^2)\frac{|f'(a_n)|}{1 + |f(a_n)|^2} = (1 - |a_n|^2)|B'(a_n)||F(a_n)|.$$

Since B is an interpolating Blaschke product, it follows that there exists $\delta > 0$ such that $(1 - |a_n|^2)|B'(a_n)| \ge \delta$, for all n. The fact that F is unbounded together with (2.11) implies that $F(a_n) \to \infty$, as $n \to \infty$. Then (2.14) implies that $(1 - |a_n|^2)\frac{|f'(a_n)|}{1 + |f(a_n)|^2} \to \infty$, as $n \to \infty$ and, hence, f is not a normal function. $\square$

Similar arguments can be used to prove the sharpness of (2.7). Indeed, we have:

THEOREM 2.3. ([**15**, Theorems 1.4 and 1.5]). Let $1 \leq p < \infty$ and let $\omega : [0,1] \to [0,\infty)$ be a continuous and increasing function with $\omega(0) = 0$ and

$$\frac{\omega(\delta)}{\delta^{1/p}} \to \infty, \quad \text{as } \delta \to 0.$$

and set $\phi(r) = \omega(1-r)/(1-r)$ $(0 < r < 1)$, then there exists $f \in \mathcal{L}(p,\phi)$ which is not a normal function.

If we assume in addition that ω is a Dini weight and satisfies the condition b_1 then we can assert that there exists $f \in \Lambda(p,\omega)$ which is not a normal function.

The functions f constructed to prove this theorem are also of the form $f = FB$ where F is a conformal mapping from $\mathbb{D}$ onto one of the domains Ω_Γ considered in Section 1.5 and B can be taken to be the Blaschke product whose sequence of zeros is $\{1 - 2^{-n}\}_{n=1}^\infty$.

2.3. An open question. All the examples of non-normal functions f constructed in [**5, 12, 13, 15**] to prove the sharpness of (2.7) and (2.9) are of the form $f = FB$ where F is a certain analytic function and B is a Blaschke product. It is natural to ask whether or not Blaschke products are really needed. More precisely, we may raise the following questions.

QUESTION 2.4. If ϕ is a positive continuous function defined in $[0,1)$ with $\phi(r) \to \infty$, as $r \to 1$, does there exist an **outer function** $f \in \mathcal{L}(1,\phi)$ which is not a normal function?

QUESTION 2.5. Suppose that $1 \leq p < \infty$ and let $\omega : [0,1] \to [0,\infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(\delta)\delta^{-1/p} \to \infty$, as $\delta \to 0$. Set $\phi(r) = \omega(1-r)/(1-r)$ $(0 < r < 1)$. Does there exist an **outer function** $f \in \mathcal{L}(p,\phi)$ which is not a normal function?

We do not know the answer to these questions. We can just prove that there exist non-Bloch outer functions in the desired spaces $\mathcal{L}(p,\phi)$.

THEOREM 2.6. *(i) Let ϕ be a positive continuous function defined in $[0,1)$ with $\phi(r) \to \infty$, as $r \to 1$. Then, there exists an outer function $f \in \mathcal{L}(1,\phi)$ which is not a Bloch function.*
(ii) Suppose that $1 \leq p < \infty$ and let $\omega : [0,1] \to [0,\infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(\delta)\delta^{-1/p} \to \infty$, as $\delta \to 0$. Set $\phi(r) = \omega(1-r)/(1-r)$ $(0 < r < 1)$. Then there exists an outer function $f \in \mathcal{L}(p,\phi)$ which is not a Bloch function

Proof. We shall just prove (i), the same argument works for (ii). Let F and B be the functions constructed in the proof of Theorem 2.1. Set $G(z) = F(z)/z$ $(z \in \mathbb{D})$. Since F is a conformal mapping from $\mathbb{D}$ onto a certain domain Ω with $F(0) = 0$, it follows that G is an outer function (see [**9**, Theorem 3.17]). Also, it is clear that

(2.15) $M_\infty(r,G) \asymp M_\infty(r,F)$ and $M_1(r,G') \asymp M_1(r,F')$,   as $r \to 1$.

On the other hand, $\text{Re}(B(z) + 1) > 0$, for all z, which implies that $B + 1$ is also an outer function. Since the product of two outer functions is an outer function, it

follows that the function $g = G(B + 1)$ is outer. Using (2.15), it follows easily that $g \in \mathcal{L}(1, \phi)$.

Recall that B is an interpolating Blaschke product with positive zeros $\{a_n\}$ and that $F \in BMOA \subset \mathcal{B}$. Then, also $G \in \mathcal{B}$ and

$$(2.16) \qquad (1 - |a_n|)|G'(a_n)(B(a_n) + 1)| = (1 - |a_n|)|G'(a_n)| = \mathrm{O}(1).$$

Since $(1 - |a_n|)|B'(a_n)| \geq \delta$, $n \geq 1$, for a certain $\delta > 0$, and $|G(a_n)| \to \infty$, as $n \to \infty$, it follows that $(1 - |a_n|)|B'(a_n)G(a_n)| \to \infty$, as $n \to \infty$, which, together with (2.16), yields $(1 - |a_n|)|g'(a_n)| \to \infty$, as $n \to \infty$. Thus, $g \notin \mathcal{B}$. $\square$

3. Bloch functions and division by inner functions

In this section we shall consider another problem on analytic function spaces which can be solved making use of the conformal mappings introduced in Section 1.5.

We recall that a function I, analytic in $\mathbb{D}$, is said to be an inner function if $I \in H^\infty$ and I has a radial limit $I(e^{i\theta})$ of modulus one for almost every $e^{i\theta} \in \partial\mathbb{D}$.

DEFINITION 3.1. A subspace X of H^1 is said to have the f-property (or property of division by inner functions) if $h/I \in X$ whenever $h \in X$ and I is an inner function with $h/I \in H^1$.

These notion was introduced by Havin [21]. Lots of spaces including the Hardy spaces H^p ($1 \leq p \leq \infty$), the Dirichlet space $\mathcal{D}$ and several spaces of Dirichlet type, the spaces $BMOA$ and $VMOA$, the Lipschitz spaces Λ_α ($0 < \alpha \leq 1$), the disc algebra $\mathcal{A}$ and many other are known to have the f-property. We mention the recently published paper [17] as a source to find these and many other results and references about the f-property and the stronger K-property introduced by Korenblum in [25].

The first example of a space not possessing the f-property was given by Gurarii [19] who proved that the space of analytic functions in $\mathbb{D}$ with an absolutely convergent power series does not have the f-property. Later on Anderson proved in [1] that $\mathcal{B}_0 \cap H^\infty$ does not have the f-property. Consequently, the same is true for $\mathcal{B}_0 \cap H^p$ for every $p \in [1, \infty)$.

Since $H^\infty \subset \mathcal{B}$, we see that $\mathcal{B} \cap H^\infty = H^\infty$ which has the f-property. However, for $1 \leq p < \infty$ the question of whether or not the space $\mathcal{B} \cap H^p$ has the f-property remained open. In a joint work with C. González and J. A. Peláez (cf. [16]) we have solved this problem, proving the following result:

THEOREM 3.2. ([16, Theorem 1.4]). If $1 \leq p < \infty$ then the space $\mathcal{B} \cap H^p$ does not have the f-property.

Sketch of the proof. Let B be an infinite Blaschke product in $\mathcal{B}_0$, whose sequence of zeros has a subsequence accumulating at 1 (the existence of such a product was first proved by Sarason [30]). Set

$$\mu(r) = \sup_{r \leq |z| < 1} \left(1 - |z|^2\right)\left|B'(z)\right|, \quad \varphi(r) = \frac{1}{\mu(r)}, \quad 0 < r < 1.$$

Since $B \in \mathcal{B}_0$, we have that $\varphi(r) \uparrow \infty$, as $r \uparrow 1$. Let $F = F_\Gamma$ be the conformal mapping constructed in Theorem 1.2 for this φ and set $f = BF$. It is easy to see

that

$$(3.1) \qquad f \in \mathcal{B} \cap H^p, \quad \text{for all } p < \infty.$$

Let now B_1 be an interpolating Blaschke product, subproduct of B, whose sequence of zeros tends to 1 and set

$$(3.2) \qquad B_2 = \frac{B}{B_1}, \quad g = \frac{f}{B_2} = B_1 F.$$

It is clear that B_2 is a Blaschke product and that $g \in H^p$ for all $p < \infty$. On the other hand, using that B_1 is an interpolating Blaschke product, we can deduce that $g \notin \mathcal{B}$ which, together with (3.1) and (3.2), implies that $\mathcal{B} \cap H^p$ $(1 \le p < \infty)$ does not have the f-property. $\qquad \square$

Acknowledgements

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